Exercise 1. Prove Proposition 1.

Proposition 1. Let $C, D$ be model categories and consider any adjunction

$$C \xrightarrow{G} D$$

with left adjoint on top. Then

(a) $F$ preserves cofibrations if and only if $G$ preserves acyclic fibrations, and
(b) $F$ preserves acyclic cofibrations if and only if $G$ preserves fibrations.

Exercise 2. Prove Theorem 2.

Theorem 2. Let $C, D$ be model categories and consider any adjunction

$$C \xrightarrow{G} D$$

with left adjoint on top. Suppose that

(i) $F$ preserves cofibrations and $G$ preserves fibrations.

Then the total derived functors $LF$ and $RG$ exist and fit into an adjunction

$$\text{Ho}(C) \xrightarrow{LF} \text{Ho}(D)$$

with left adjoint on top. If in addition we have

(ii) for each cofibrant object $A \in C$ and fibrant object $X \in D$, a map $f : A \to G(X)$

is a weak equivalence in $C$ if and only if its adjoint $F(A) \to X$ is a weak equivalence in $D$,

then the adjunction (1) is an equivalence of categories; i.e., the natural maps

$$A \xrightarrow{\eta_A} RG(LF(A)) \quad LF(RG(X)) \xrightarrow{\epsilon_X} X$$

of the adjunction (1) are isomorphisms for each $A \in C$ and $X \in D$.

Definition 3. Let $C, D$ be model categories and consider any adjunction

$$C \xrightarrow{G} D$$

with left adjoint on top. If the conditions in Theorem 2(i) are satisfied, then $F$ is a left Quillen functor, $G$ is a right Quillen functor, and the adjunction (2) is a Quillen adjunction. If in addition the conditions in Theorem 2(ii) are satisfied, then the adjunction (2) is a Quillen equivalence.

Let $R$ be a ring and denote by $\text{Ch}_R$ the category of unbounded chain complexes over $R$.

Exercise 3. Use a 5-lemma argument to prove Proposition 4.
Proposition 4. Consider any commutative diagram of the form

\[
\begin{array}{c}
0 \to A \to B \to C \\
\downarrow f \downarrow g \downarrow h \\
0 \to A' \to B' \to C' \to 0
\end{array}
\]

in \(Ch_R\) with exact rows. If any two of the three vertical maps is a homology isomorphism, then so is the third.

Recall the following right-exactness property of the tensor product functors.

Proposition 5. If \(X\) is a right \(R\)-module and \(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0\) is an exact sequence of left \(R\)-modules, then

\[
X \otimes_R A \xrightarrow{id \otimes_R \alpha} X \otimes_R B \xrightarrow{id \otimes_R \beta} X \otimes_R C \to 0
\]

is an exact sequence of abelian groups.

Definition 6. A right \(R\)-module \(X\) is flat if the functor \(X \otimes_R -\) preserves monomorphisms; i.e., if for any monomorphism \(\alpha: A \to B\) of left \(R\)-modules, the induced map \(id \otimes_R \alpha: X \otimes_R A \to X \otimes_R B\) of abelian groups is a monomorphism.


Proposition 7. The following properties of a left \(R\)-module \(X\) are equivalent.

(a) \(X\) is flat.

(b) The functor \(X \otimes_R -: \text{Mod}_R \to \text{Mod}_\mathbb{Z}\) preserves short exact sequences.

Exercise 5. Prove Proposition 8.

Proposition 8.

(a) The free right \(R\)-module \(R\) is flat.

(b) Every free right \(R\)-module is flat.

(c) Every projective right \(R\)-module is flat.


The following proposition indicates how the \(\text{Tor}^R_n(X, -)\) functors provide a measure of the inexactitude of \(X \otimes_R -\).

Proposition 9. Let \(X\) be a right \(R\)-module and \(\varepsilon: P \to X\) in \(Ch^+_R\) a projective resolution of \(X\). Consider any short exact sequence \(0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0\) of left \(R\)-modules. Then there is a short exact sequence

\[
0 \to P \otimes_R A \xrightarrow{id \otimes_R \alpha} P \otimes_R B \xrightarrow{id \otimes_R \beta} P \otimes_R C \to 0
\]

in \(Ch^+_\mathbb{Z}\), and hence a natural corresponding long exact sequence

\[
\cdots \to \text{Tor}^R_n(X, C) \xrightarrow{\partial} \text{Tor}^R_n(X, A) \to \text{Tor}^R_n(X, B) \to \text{Tor}^R_n(X, C) \to \cdots
\]

\[
\cdots \to \text{Tor}^F_n(X, A) \to \text{Tor}^F_n(X, B) \to \text{Tor}^F_n(X, C)
\]

\[
\xrightarrow{\partial} X \otimes_R A \to X \otimes_R B \to X \otimes_R C \to 0
\]

of abelian groups.
Exercise 7. Prove Propositions 10 and 11.

Proposition 10. Let $X$ be a right $R$-module. A map $\varepsilon: P \rightarrow X$ in $\text{Ch}_{R^\text{op}}^+$ is a projective resolution of $X$ if and only if it is a cofibrant replacement of $X$ in $\text{Ch}_{R^\text{op}}^+$.

Proposition 11. Let $Y$ be a left $R$-module. Then the total left derived functor

$$
\text{Ch}_{R^\text{op}}^+ \xrightarrow{- \otimes_R Y} \text{Ch}_{Z}^+ \xrightarrow{- \otimes_R Y} \text{Ho}(\text{Ch}_{Z}^+)
$$

of the tensor product functor $\text{Ch}_{R^\text{op}}^+ \rightarrow \text{Ch}_{Z}^+$ exists, and there are natural isomorphisms

$$
H_n(X \otimes_R Y) \cong \text{Tor}_n^R(X, Y) \quad (n \in \mathbb{Z})
$$

of abelian groups for each $X \in \text{Mod}_{R^\text{op}} \subset \text{Ch}_{R^\text{op}}^+$.


Proposition 12. The total left derived functor

$$
\text{Ch}_{R^\text{op}}^+ \times \text{Ch}_{R^\text{op}}^+ \xrightarrow{- \otimes_R -} \text{Ch}_{Z}^+ \xrightarrow{- \otimes_R -} \text{Ho}(\text{Ch}_{Z}^+)
$$

of the tensor product functor $\text{Ch}_{R^\text{op}}^+ \rightarrow \text{Ch}_{Z}^+$ exists, and there are natural isomorphisms

$$
H_n(X \otimes_R Y) \cong \text{Tor}_n^R(X, Y) \quad (n \in \mathbb{Z})
$$

of abelian groups for each $X \in \text{Mod}_{R^\text{op}} \subset \text{Ch}_{R^\text{op}}^+$ and $Y \in \text{Mod}_{R} \subset \text{Ch}_{R}^+$.

Recall the following left-exactness property of the hom object functors.

Proposition 13. If $X$ is a right $R$-module and $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence of right $R$-modules, then

$$
0 \rightarrow \text{Hom}_R(X, A) \xrightarrow{\alpha} \text{Hom}_R(X, B) \xrightarrow{\beta} \text{Hom}_R(X, C)
$$

is an exact sequence of abelian groups.


Proposition 14. The following properties of a right $R$-module $X$ are equivalent.

(a) $X$ is projective.

(b) The functor $\text{Hom}_R(X, -): \text{Mod}_{R^\text{op}} \rightarrow \text{Mod}_{R}$ preserves short exact sequences.


The following proposition indicates how the $\text{Ext}_R^n(X, -)$ functors provide a measure of the inexactitude of $\text{Hom}_R(X, -)$.
Proposition 15. Let $X$ be a right $R$-module and $\varepsilon : P \to X$ in $\text{Ch}^+_R$ a projective resolution of $X$. Consider any short exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules. Then there is a short exact sequence

$$0 \to \text{Hom}_R(P, A) \xrightarrow{\alpha} \text{Hom}_R(P, B) \xrightarrow{\beta} \text{Hom}_R(P, C) \to 0$$

of cochain complexes, and hence a natural corresponding long exact sequence

$$0 \to \text{Hom}_R(X, A) \to \text{Hom}_R(X, B) \to \text{Hom}_R(X, C)$$

$$\delta : \text{Ext}^1_R(X, A) \to \text{Ext}^1_R(X, B) \to \text{Ext}^1_R(X, C)$$

$$\cdots \to \text{Ext}^n_R(X, A) \to \text{Ext}^n_R(X, B) \to \text{Ext}^n_R(X, C)$$

of abelian groups.

Define a map $f : M \to N$ in $\text{Ch}_R$ to be

(i) a weak equivalence if it is a homology isomorphism,
(ii) a fibration if the map $f_k : M_k \to N_k$ is an epimorphism for each $k \in \mathbb{Z}$,
(iii) a cofibration if it has the left lifting property with respect to all acyclic fibrations.

The following proposition can be proved using similar arguments as in the case of non-negative chain complexes $\text{Ch}^+_R$.

Proposition 16. These three classes of maps give $\text{Ch}_R$ the structure of a model category.

(a) A map $i : A \to B$ in $\text{Ch}_R$ between bounded below chain complexes is a cofibration if and only if the map $i_k : A_k \to B_k$ is a monomorphism with $\text{coker}(i_k)$ a projective $R$-module for each $k \in \mathbb{Z}$.

(b) A bounded below chain complex $B \in \text{Ch}_R$ is cofibrant if and only if $B_k$ is a projective $R$-module for each $k \geq 0$.

(c) Every chain complex $B \in \text{Ch}_R$ is fibrant.

Proof. A proof is given in [3, 2.3]. □


Proposition 17. Let $Y$ be a right $R$-module. Then the total right derived functor

$$\text{Hom}_R(\text{Ch}_R^+, -) : \text{Ch}_R^+ \to \text{Ch}_Z \to \text{Ho}(\text{Ch}_Z)$$

$$\text{Ho}(\text{Ch}_R^+)^{\text{op}} \xrightarrow{\text{RHom}_R(-, Y)} \text{Ho}(\text{Ch}_Z)$$

of the hom object functor $(\text{Ch}_R^+)^{\text{op}} \to \text{Ch}_Z$ exists, and there are natural isomorphisms

$$H^n(\text{RHom}_R(X, Y)) \cong \text{Ext}^n_R(X, Y) \quad (n \in \mathbb{Z})$$

of abelian groups for each $X \in \text{Mod}_{R^{\text{op}}} \subset \text{Ch}_R^+$. 

Definition 18. Let $l \in \mathbb{Z}$. Denote by $\text{Ch}_R^l \subset \text{Ch}_R$ the full subcategory of chain complexes $M$ such that $M_k = 0$ for each $k < l$; for instance, $\text{Ch}_R^0 = \text{Ch}_R^+$. 

Proposition 19. Let \( l \in \mathbb{Z} \). There is an isomorphism of categories \( \text{Ch}^{\geq l}_R \cong \text{Ch}^+_R \).

For each \( l \in \mathbb{Z} \), define a map \( f : M \rightarrow N \) in \( \text{Ch}^{\geq l}_R \) to be

(i) a weak equivalence if it is a homology isomorphism,

(ii) a fibration if the map \( f_k : M_k \rightarrow N_k \) is an epimorphism for each \( k \geq l + 1 \),

(iii) a cofibration if it has the left lifting property with respect to all acyclic fibrations.

Proposition 20. These three classes of maps give \( \text{Ch}^{\geq l}_R \) the structure of a model category.

(a) A map \( i : A \rightarrow B \) in \( \text{Ch}^{\geq l}_R \) is a cofibration if and only if the map \( i_k : A_k \rightarrow B_k \) is a monomorphism with \( \text{coker}(i_k) \) a projective \( R \)-module for each \( k \geq l \).

(b) A chain complex \( B \in \text{Ch}^{\geq l}_R \) is fibrant if and only if \( B_k \) is a projective \( R \)-module for each \( k \geq l \).

(c) Every chain complex \( B \in \text{Ch}^{\geq l}_R \) is fibrant.

Definition 21. Let \( n \in \mathbb{Z} \). A chain complex \( M \) in \( \text{Ch}_R \) is \( n \)-connected if \( H_k(M) = 0 \) for each \( k \leq n \), and is \( \text{connective} \) if it is \(-1\)-connected.

Definition 22. Let \( l \in \mathbb{Z} \) and \( X \in \text{Ch}_R \). The chain complex \( \tau_{\geq l}(X) \) in \( \text{Ch}^{\geq l}_R \), called a good truncation of \( X \), has the form

\[
\tau_{\geq l}(X) : \cdots \leftarrow 0 \leftarrow 0 \leftarrow \ker \partial_l \leftarrow X_{l+1} \leftarrow X_{l+2} \leftarrow X_{l+3} \leftarrow \cdots
\]

and is defined degreewise by

\[
\tau_{\geq l}(X)_k := \begin{cases} 
X_k, & \text{for } k \geq l + 1, \\
\ker \partial_l, & \text{for } k = l, \\
0, & \text{otherwise.}
\end{cases}
\]


Proposition 23. Let \( l \in \mathbb{Z} \). The good truncation functor \( \tau_{\geq l} \) fits into an adjunction

\[
\begin{array}{ccc}
\text{Ch}^{\geq l}_R & \xrightarrow{i} & \text{Ch}_R \\
\tau_{\geq l} & \cong & \cong
\end{array}
\]

with left adjoint on top and \( i \) the inclusion functor. The natural inclusion of chain complexes \( j : \tau_{\geq l}(X) \rightarrow X \) in \( \text{Ch}_R \) induces isomorphisms

\[
H_k(\tau_{\geq l}(X)) \xrightarrow{j^*} H_k(X) \quad (k \geq l).
\]

Proposition 24. Let \( l, m, n \in \mathbb{Z} \) and \( X, Y \in \text{Ch}^{\geq l}_R \). The total left derived functor

\[
\begin{array}{cccc}
\text{Ch}^{\geq l}_{R^p} \times \text{Ch}^{\geq l}_R & \xrightarrow{- \otimes_R -} & \text{Ch}_Z & \xrightarrow{\text{Ho}(\text{Ch}_Z)} \\
& \xrightarrow{\text{total left derived functor}} & & \text{Ho}(\text{Ch}_Z)
\end{array}
\]

of the tensor product functor \( \text{Ch}^{\geq l}_{R^p} \times \text{Ch}^{\geq l}_R \rightarrow \text{Ch}_Z \) exists.

(a) If \( X \) is \( m \)-connected and \( Y \) is \( n \)-connected, then the derived tensor product \( X \otimes_R Y \) is \( (m + n + 1) \)-connected.

(b) If \( X \) is \( m \)-connected and cofibrant and \( Y \) is \( n \)-connected, then \( X \otimes_R Y \) is \( (m + n + 1) \)-connected.
Let $C$ be a model category and let $D = \{a \leftarrow b \rightarrow c\}$. Then a morphism $f : X \rightarrow Y$ in $C^D$ is a collection of maps $f_a, f_b, f_c$ which makes the diagram

\[
\begin{array}{ccc}
X_a & \leftarrow & X_b \\
\downarrow f_a & & \downarrow f_b \\
Y_a & \leftarrow & Y_b
\end{array}
\quad
\begin{array}{ccc}
X_b & \rightarrow & X_c \\
\downarrow f_b & & \downarrow f_c \\
Y_b & \rightarrow & Y_c
\end{array}
\]

in $C$ commute. Define a map $f : X \rightarrow Y$ in $C^D$ to be

(i) a weak equivalence if it is an objectwise weak equivalence; i.e., if the maps $f_a, f_b, f_c$ are weak equivalences in $C$,
(ii) a fibration if it is an objectwise fibration; i.e., if the maps $f_a, f_b, f_c$ are fibrations in $C$,
(iii) a cofibration if the induced maps

\[\begin{align*}
X_a \amalg_{X_b} Y_b &\rightarrow Y_a, \\
X_b &\rightarrow Y_b, \\
X_c \amalg_{X_b} Y_b &\rightarrow Y_c
\end{align*}\]

are cofibrations in $C$.

**Exercise 14.** Prove Proposition 25.

**Proposition 25.** These three classes of maps give $C^D$ the structure of a model category.

(a) The total left derived functor

\[C^D \xrightarrow{\text{colim}} C \xrightarrow{\text{Ho}} \text{Ho}(C)\]

(b) A diagram $Y \in C^D$ is cofibrant if and only if the maps

\[Y_b \rightarrow Y_a, \quad \emptyset \rightarrow Y_b, \quad Y_b \rightarrow Y_c\]

are cofibrations in $C$.

(c) If $Y \in C^D$ is a diagram and $\emptyset \rightarrow Y^c \rightarrow Y$ is a cofibration followed by a weak equivalence in $C^D$, then $\text{hocolim}_D(Y^c) \simeq \text{colim}_D(Y^c)$.

(d) If $f : X \rightarrow Y$ is a weak equivalence between cofibrant diagrams, then the induced map $\text{colim}_D X \rightarrow \text{colim}_D Y$ is a weak equivalence.

Sometimes $\text{hocolim}_D(X)$ is called the homotopy pushout of the diagram $X$.

**Exercise 15.** Use duality in model categories to obtain a corresponding proposition involving the total right derived functor of the limit functor $\text{lim}_{D^\text{op}} : C^{D^\text{op}} \rightarrow C$; note that $D^\text{op}$ is the category $\{a \rightarrow b \leftarrow c\}$. Describe the corresponding model structure on $C^{D^\text{op}}$.

**Exercise 16.** Please read [1, Sections 9-10] and [2, Section 2].
References


