Algebraic Topology (topics course) **Spring 2010**

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Series 8 & 9

References for the following include [1, Sections 10-11], [3], [4, Sections 1.5 and 4.1-4.4], [5, Sections 1-3] and [6].

Homotopy colimits and limits. Let C be a model category and let D be the category $\{0 \to 1 \to 2 \to 3 \to \cdots\}$ with objects the non-negative integers and a single morphism $i \to j$ for each $i \leq j$. Then a morphism $f\colon X \longrightarrow Y$ in C^{D} is a collection of maps $f_0, f_1, f_2, f_3, \ldots$ which makes the diagram

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \cdots \end{array}$$

in C commute. Define a map $f\colon X \longrightarrow Y$ in C^{D} to be

(i) a *weak equivalence* if it is an objectwise weak equivalence; i.e., if the map f_n is a weak equivalence in C for each $n \geq 0$,
(ii) a *fibration* if it is an objectwise fibration; i.e., if the map f_n is a fibration in C for each $n \geq 0$,
(iii) a *cofibration* if the induced maps

$$X_0 \longrightarrow Y_0, \qquad Y_n \amalg_{X_n} X_{n+1} \longrightarrow Y_{n+1} \qquad (n \geq 0)$$

are cofibrations in C.

Exercise 1. Prove Proposition 1.

Proposition 1. *These three classes of maps give C^{D} the structure of a model category.*

(a) *The total left derived functor*

$$\begin{array}{ccccc} \mathsf{C}^{\mathsf{D}} & \xrightarrow{\mathrm{colim}_{\mathsf{D}}} & \mathsf{C} & \longrightarrow & \mathsf{Ho}(\mathsf{C}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{C}^{\mathsf{D}}) & & \xrightarrow[\textit{total left derived functor}]{\mathrm{hocolim}_{\mathsf{D}}} & & \mathsf{Ho}(\mathsf{C}) \end{array}$$

of the colimit functor $\mathsf{C}^{\mathsf{D}} \longrightarrow \mathsf{D}$ exists.
(b) *A diagram $Y \in \mathsf{C}^{\mathsf{D}}$ is cofibrant if and only if the maps*

$$\emptyset \longrightarrow Y_0, \qquad Y_n \longrightarrow Y_{n+1} \qquad (n \geq 0)$$

are cofibrations in C.
(c) *If $Y \in \mathsf{C}^{\mathsf{D}}$ is a diagram and $\emptyset \to Y^c \to Y$ is a cofibration followed by a weak equivalence in C^{D}, then* $\mathrm{hocolim}_{\mathsf{D}}(Y) \simeq \mathrm{colim}_{\mathsf{D}}(Y^c)$.
(d) *If $f\colon X \longrightarrow Y$ is a weak equivalence between cofibrant diagrams, then the induced map* $\mathrm{colim}_{\mathsf{D}} X \longrightarrow \mathrm{colim}_{\mathsf{D}} Y$ *is a weak equivalence.*

Exercise 2. Use duality in model categories to obtain a corresponding proposition involving the total right derived functor of the limit functor $\lim_{\mathsf{D}^{\mathrm{op}}}\colon \mathsf{C}^{\mathsf{D}^{\mathrm{op}}} \longrightarrow \mathsf{C}$; note that D^{op} is the category $\{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$. Describe the corresponding model structure on $\mathsf{C}^{\mathsf{D}^{\mathrm{op}}}$.

Exercise 3. Prove Proposition 2.

Proposition 2. *Let C be a category with all small limits and colimits.*

(a) *If X is a simplicial object in C, then its colimit is naturally isomorphic to a coequalizer of the form*

$$\operatorname{colim}_{\Delta^{\mathrm{op}}} X \cong \operatorname{colim}\Big(X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \Big)$$

in C, with d_0 and d_1 the indicated face maps of X.

(b) *If X is a cosimplicial object in C, then its limit is naturally isomorphic to an equalizer of the form*

$$\lim_{\Delta} X \cong \lim\Big(X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1 \Big)$$

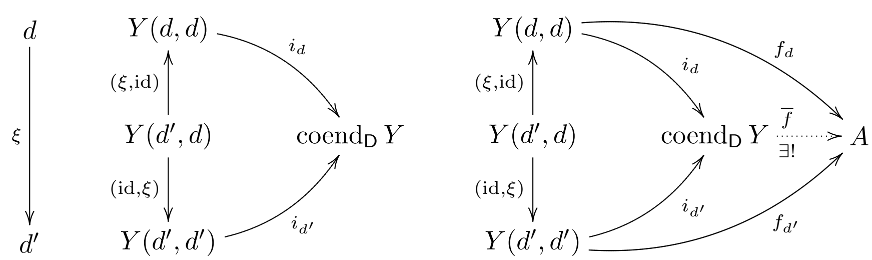
in C, with d^0 and d^1 the indicated coface maps of X.

Coends and ends.

Definition 3. Let D be a category and $Y\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \longrightarrow \mathsf{C}$ a diagram. A *coend* of Y, denoted $\operatorname{coend}_{\mathsf{D}} Y$ or Y_{D}, is an object in C with the following mapping properties: (i) there is a collection $\{i_d\}$ of maps

$$Y(d,d) \xrightarrow{i_d} \operatorname{coend}_{\mathsf{D}} Y, \qquad d \in \mathsf{D}$$

in C which make the middle diagram

(1) 

commute for each arrow ξ in D (such a collection $\{i_d\}$ is sometimes called a *wedge* out of Y) and (ii) (universal property): the wedge $\{i_d\}$ is *initial* with respect to all such wedges out of Y; i.e., for any object $A \in \mathsf{C}$ and collection $\{f_d\}$ of maps

$$Y(d,d) \xrightarrow{f_d} A, \qquad d \in \mathsf{D}$$

in C which make the right-hand outer diagram in (1) commute for each arrow ξ in D, there exists a unique map $\overline{f}$ which makes the diagram commute; i.e., $\overline{f} i_d = f_d$ for each $d \in \mathsf{D}$.

Remark 4. Property (ii) states that every wedge $\{f_d\}$ out of Y factors uniquely through the wedge $\{i_d\}$. Note that properties (i)-(ii) define the coend $\operatorname{coend}_{\mathsf{D}} Y$ up to isomorphism, provided that it exists.

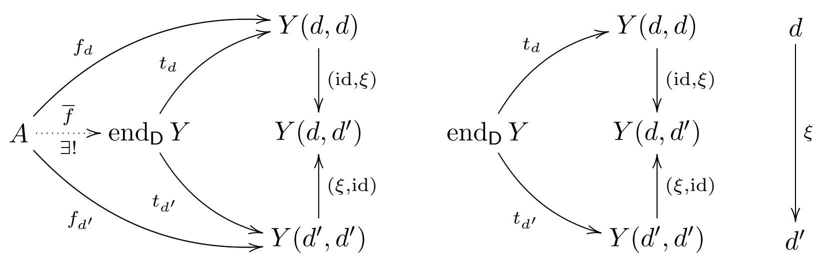
Reversing the arrows in (i)-(ii) above gives the definition of an end.

Definition 5. Let D be a category and $Y\colon \mathsf{D}^{\mathrm{op}}\times\mathsf{D}\longrightarrow\mathsf{C}$ a diagram. An *end* of Y, denoted $\mathrm{end}_{\mathsf{D}} Y$ or Y^{D}, is an object in C with the following mapping properties: (i) there is a collection $\{t_d\}$ of maps

$$\mathrm{end}_{\mathsf{D}} Y \xrightarrow{t_d} Y(d,d), \qquad d\in\mathsf{D}$$

in C which make the middle diagram

(2)



commute for each arrow ξ in D (such a collection $\{t_d\}$ is sometimes called a *wedge* into Y) and (ii) (universal property): the wedge $\{t_d\}$ is *terminal* with respect to all such wedges into Y; i.e., for any object $A\in\mathsf{C}$ and collection $\{f_d\}$ of maps

$$A \xrightarrow{f_d} Y(d,d), \qquad d\in\mathsf{D}$$

in C which make the left-hand outer diagram in (2) commute for each arrow ξ in D, there exists a unique map $\overline{f}$ which makes the diagram commute; i.e., $t_d\overline{f}=f_d$ for each $d\in\mathsf{D}$.

Remark 6. Property (ii) states that every wedge $\{f_d\}$ into Y factors uniquely through the wedge $\{t_d\}$. Note that properties (i)-(ii) define the end $\mathrm{end}_{\mathsf{D}} Y$ up to isomorphism, provided that it exists.

Exercise 4. Prove Proposition 7.

Proposition 7. *Let C be a category with all small limits and colimits. Let D be a small category and $Y\colon \mathsf{D}^{\mathrm{op}}\times\mathsf{D}\longrightarrow\mathsf{C}$ a diagram.*

(a) *The coend Y_{D} exists and is naturally isomorphic to a coequalizer of the form*

$$Y_{\mathsf{D}} \cong \operatorname{colim}\Big(\coprod_{d\in\mathsf{D}} Y(d,d) \leftleftarrows \coprod_{(\xi\colon d\to d')\in\mathsf{D}} Y(d',d) \Big).$$

(b) *The end Y^{D} exists and is naturally isomorphic to an equalizer of the form*

$$Y^{\mathsf{D}} \cong \lim\Big(\prod_{d\in\mathsf{D}} Y(d,d) \rightrightarrows \prod_{(\xi\colon d\to d')\in\mathsf{D}} Y(d,d') \Big).$$

Exercise 5. Prove Proposition 8.

Proposition 8. *Let G be a finite group, $X\in\mathsf{Top}^{G^{\mathrm{op}}}$ and $Y,Z\in\mathsf{Top}^{G}$. Consider the associated diagrams*

$$\begin{aligned} X\times Y\colon\ & G^{\mathrm{op}}\times G\longrightarrow\mathsf{Top},\\ \hom_{\mathsf{Top}}(Y,Z)\colon\ & G^{\mathrm{op}}\times G\longrightarrow\mathsf{Set},\\ \mathrm{Map}(Y,Z)\colon\ & G^{\mathrm{op}}\times G\longrightarrow\mathsf{Top}. \end{aligned}$$

(a) *The coend* $X \times_G Y$ *and ends* $\hom_{\mathsf{Top}}(Y,Z)^G$ *and* $\mathrm{Map}(Y,Z)^G$ *fit into natural isomorphisms*

$$\begin{aligned} X \times_G Y &\cong (X \times Y)/(xg, y) \sim (x, gy), \\ \hom_{\mathsf{Top}}(Y,Z)^G &\cong \hom_{\mathsf{Top}^G}(Y,Z), \\ \mathrm{Map}(Y,Z)^G &\cong \mathrm{Map}_G(Y,Z). \end{aligned}$$

Here, $\mathrm{Map}_G(Y,Z) \subset \mathrm{Map}(Y,Z)$ *is the subspace of* G*-equivariant maps* $Y \longrightarrow Z$*.*

(b) *The coend* $* \times_G Y$ *and end* $\mathrm{Map}(*,Z)^G$ *fit into natural isomorphisms*

$$* \times_G Y \cong Y/G, \qquad \mathrm{Map}(*,Z)^G \cong Z^G.$$

Here, Y/G *is the orbit space of* Y *and* Z^G *is the fixed points space of* Z*.*

Exercise 6. Prove Proposition 9.

Proposition 9. *Let* C *be a category. Let* D *be a small category and* $Y, Z\colon \mathsf{D} \longrightarrow \mathsf{C}$ *diagrams. Consider the associated diagram*

$$\hom_{\mathsf{C}}(Y,Z)\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \longrightarrow \mathsf{Set}.$$

The end $\hom_{\mathsf{C}}(Y,Z)^{\mathsf{D}}$ *fits into a natural isomorphism*

$$\hom_{\mathsf{C}}(Y,Z)^{\mathsf{D}} \cong \hom_{\mathsf{C}^{\mathsf{D}}}(Y,Z).$$

Exercise 7. Prove Proposition 10.

Proposition 10. *Let* D *be a small category,* $Y\colon \mathsf{D} \longrightarrow \mathsf{Top}$ *a diagram and* $*\colon \mathsf{D}^{\mathrm{op}} \longrightarrow \mathsf{Top}$ *the constant diagram with value the point* $*$*; i.e.,* $*(d) = *$ *for each* $d \in \mathsf{D}$*. Consider the associated diagram*

$$* \times X\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \longrightarrow \mathsf{Top}.$$

The coend $* \times_{\mathsf{D}} X$ *and end* $(* \times X)^{\mathsf{D}}$ *fit into natural isomorphisms*

$$* \times_{\mathsf{D}} X \cong \mathrm{colim}_{\mathsf{D}} X, \qquad (* \times X)^{\mathsf{D}} \cong \lim_D X.$$

Realization and singular complex functors. Recall from lecture the following.

Definition 11. Let X be a simplicial set and Y a space. Consider the diagram

$$\Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathsf{Top}, \qquad ([n],[n']) \longmapsto X_n \times \Delta^{n'}.$$

The *realization* $|X|$ of X is the coend $|X| := X \times_\Delta \Delta^{(-)}$ and the *singular complex* $\mathrm{Sing}(Y)$ of Y is the simplicial set

$$\mathrm{Sing}(Y)_{(-)} := \hom_{\mathsf{Top}}(\Delta^{(-)}, Y).$$

Exercise 8. Prove Propositions 12 and 13.

Proposition 12. *The realization and singular complex constructions define functors* $|-|\colon \mathsf{sSet} \longrightarrow \mathsf{Top}$ *and* $\mathrm{Sing}\colon \mathsf{Top} \longrightarrow \mathsf{sSet}$*, respectively.*

Proposition 13. *Let* X *be a simplicial set. The realization* $|X|$ *of* X *is naturally isomorphic to a coequalizer of the form*

$$|X| \cong \mathrm{colim}\Big(\coprod_{[n] \in \Delta} X_n \times \Delta^n \leftleftarrows \coprod_{(\xi\colon [n] \to [n']) \in \Delta} X_{n'} \times \Delta^n \Big).$$

Exercise 9. Prove Proposition 14.

Proposition 14. *Let $n \geq 0$. There are adjunctions*

$$\mathsf{Set} \underset{U}{\overset{i}{\rightleftarrows}} \mathsf{Top} \qquad \mathsf{Top} \underset{\mathrm{Map}(\Delta^n,-)}{\overset{-\times\Delta^n}{\rightleftarrows}} \mathsf{Top} \qquad \mathsf{sSet} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathsf{Top}$$

with left adjoints on top, U the forgetful functor, and i the inclusion functor which sends a set X to the discrete space X.

Decomposition of simplicial sets.

Proposition 15. *Let X be a simplicial set. For each n-simplex x of X, there is a surjection $s\colon [n] \twoheadrightarrow [k]$ in Δ and a non-degenerate k-simplex y such that $x = X(s)(y)$. Furthermore, the pair (s, y) is unique.*

Proof. A short proof is given in [2, II.3]. □

A consequence is the following decomposition of a simplicial set.

Definition 16. Let X be a simplicial set. For each $k \geq 0$, denote by $\mathrm{N}X_k \subset X_k$ the set of non-degenerate k-simplices of X; in particular, $\mathrm{N}X_0 = X_0$.

Proposition 17. *Let X be a simplicial set. There is a natural isomorphism Ψ in* sSet *defined objectwise by*

$$\Psi_n\colon \coprod_{\substack{[n]\twoheadrightarrow[k] \\ \textit{in } \Delta}} \mathrm{N}X_k \xrightarrow{\cong} X_n.$$

Here the coproduct is indexed over all surjections in Δ of the form $\xi\colon [n] \longrightarrow [k]$, and Ψ_n is the natural map induced by the corresponding maps $\mathrm{N}X_k \subset X_k \xrightarrow{\xi^*} X_n$.

By Proposition 17, each $X \in \mathsf{sSet}$ is naturally isomorphic to a simplicial set of the form

$$\mathrm{N}X_0 \leftleftarrows \mathrm{N}X_0 \amalg \mathrm{N}X_1 \leftleftarrows \mathrm{N}X_0 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_2 \leftleftarrows \cdots$$

constructed from the non-degenerate simplices of X.

Skeletal filtrations.

Definition 18. Let X be a simplicial set and $m \geq 0$. The *m-skeleton* of X, denoted $\mathrm{sk}_m X$, is the subcomplex $\mathrm{sk}_m X \subset X$ generated by the k-simplices of X of degree $k \leq m$. Define $\mathrm{sk}_{-1}X := \emptyset$.

In other words, $(\mathrm{sk}_m X)_n \subset X_n$ is the image of the restriction of Ψ_n

$$(\mathrm{sk}_m X)_n = \mathrm{image}\Big(\coprod_{\substack{[n]\twoheadrightarrow[k] \\ \text{in } \Delta,\ k\leq m}} \mathrm{N}X_k \xrightarrow{\subset} \coprod_{\substack{[n]\twoheadrightarrow[k] \\ \text{in } \Delta}} \mathrm{N}X_k \underset{\cong}{\xrightarrow{\Psi_n}} X_n \Big).$$

It follows that every simplicial set X has a *skeletal filtration* of the form

$$\emptyset = \mathrm{sk}_{-1}X \subset \mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \mathrm{sk}_2 X \subset \cdots \subset \bigcup_{m\geq 0} \mathrm{sk}_m X = X,$$

and hence $X \cong \mathrm{colim}_m\, \mathrm{sk}_m X$. Note that the 0-skeleton $\mathrm{sk}_0 X$ is the constant simplicial set with value X_0.

Exercise 10. Prove Proposition 19.

Proposition 19. *Let X be a simplicial set and $m \geq 0$. There are pushout diagrams of the form*

$$\begin{array}{ccc}
\coprod_{x \in \mathrm{N}X_m} \partial\Delta[m] & \longrightarrow & \mathrm{sk}_{m-1}X \\
{\scriptstyle \amalg j_m}\downarrow & & \downarrow{\scriptstyle \subset} \\
\coprod_{x \in \mathrm{N}X_m} \Delta[m] & \longrightarrow & \mathrm{sk}_m X
\end{array}$$

in sSet*; i.e.,* $\mathrm{sk}_m X$ *is obtained from* $\mathrm{sk}_{m-1}X$ *by attaching m-cells* $\Delta[m]$.

Exercise 11. Prove Proposition 20.

The following proposition shows that the realization $|X|$ of a simplicial set X has the structure of a CW complex. In particular, $|X|$ is a compactly generated Hausdorff space.

Proposition 20. *Let X be a simplicial set and $m \geq 0$.*

(a) *There are pushout diagrams of the form*

$$\begin{array}{ccc}
\coprod_{x \in \mathrm{N}X_m} \partial\Delta^m & \longrightarrow & |\mathrm{sk}_{m-1}X| \\
{\scriptstyle \amalg j_m}\downarrow & & \downarrow{\scriptstyle \subset} \\
\coprod_{x \in \mathrm{N}X_m} \Delta^m & \longrightarrow & |\mathrm{sk}_m X|
\end{array}$$

in Top*; i.e.,* $|\mathrm{sk}_m X|$ *is obtained from* $|\mathrm{sk}_{m-1}X|$ *by attaching m-cells* Δ^m.

(b) *There is a sequence of closed inclusions*

$$\emptyset = |\mathrm{sk}_{-1}X| \subset |\mathrm{sk}_0 X| \subset |\mathrm{sk}_1 X| \subset |\mathrm{sk}_2 X| \subset \cdots \subset \bigcup_{m \geq 0} |\mathrm{sk}_m X| = |X|,$$

and hence $|X| \cong \operatorname{colim}_m |\mathrm{sk}_m X|$.

Decomposition of simplicial R-modules and the Dold-Kan theorem. Let R be a ring and denote by sMod_R the category of simplicial left R-modules. If X is a simplicial R-module

$$X: \quad X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \;\cdots\; X_2 \;\cdots\; X_3 \cdots \qquad (s_0\colon X_0 \to X_1)$$

recall from lecture that the *associated complex* of X, also denoted X, is the chain complex in Ch_R^+ of the form

$$X: \qquad \cdots \leftarrow 0 \leftarrow 0 \leftarrow X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} X_2 \xleftarrow{\partial} X_3 \leftarrow \cdots$$

defined degreewise by

$$X_k := \begin{cases} X_k, & \text{for } k \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and with differential $\partial\colon X_k \longrightarrow X_{k-1}$ defined by $\partial := \sum_{i=0}^{k} d_i(-1)^i$ summing all of the face maps $d_i\colon X_k \longrightarrow X_{k-1}$ with alternating signs.

Exercise 12. Prove Proposition 21.

Proposition 21. *Let X be a simplicial R-module.*

(a) *The associated complex of X is a well-defined chain complex; i.e., $\partial^2 = 0$.*
(b) *The associated complex construction defines a functor $\mathsf{sMod}_R \longrightarrow \mathsf{Ch}^+_R$.*

Definition 22. Let X be a simplicial R-module and $k \geq 0$. Define the submodule $\mathrm{N}X_k \subset X_k$ by

$$\mathrm{N}X_0 := X_0, \qquad \mathrm{N}X_k := \bigcap_{0 \leq i \leq k-1} \ker(d_i) \subset X_k, \qquad (k \geq 1).$$

Recall from lecture the following decomposition of a simplicial R-module.

Proposition 23. *Let X be a simplicial R-module. There is a natural isomorphism Ψ in sMod_R defined objectwise by*

$$\Psi_n \colon \coprod_{\substack{[n] \twoheadrightarrow [k] \\ \text{in } \Delta}} \mathrm{N}X_k \xrightarrow{\cong} X_n.$$

Here the coproduct is indexed over all surjections in Δ of the form $\xi \colon [n] \longrightarrow [k]$, and Ψ_n is the natural map induced by the corresponding maps $\mathrm{N}X_k \subset X_k \xrightarrow{\xi^*} X_n$.

By Proposition 23, each $X \in \mathsf{sMod}_R$ is naturally isomorphic to a simplicial R-module of the form

$$\mathrm{N}X_0 \leftleftarrows \mathrm{N}X_0 \amalg \mathrm{N}X_1 \leftleftarrows \mathrm{N}X_0 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_2 \leftleftarrows \cdots$$

Let Y be a chain complex in Ch^+_R

$$Y: \qquad \cdots \leftarrow 0 \leftarrow 0 \leftarrow Y_0 \xleftarrow{\partial} Y_1 \xleftarrow{\partial} Y_2 \xleftarrow{\partial} Y_3 \leftarrow \cdots$$

and recall from lecture that the *denormalization* of Y, denoted $\Gamma(Y)$, is the simplicial R-module of the form

$$\Gamma(Y): \quad Y_0 \leftleftarrows Y_0 \amalg Y_1 \leftleftarrows Y_0 \amalg Y_1 \amalg Y_1 \amalg Y_2 \leftleftarrows \cdots$$

defined levelwise by

$$\Gamma(Y)_n := \coprod_{\substack{[n] \twoheadrightarrow [k] \\ \text{in } \Delta}} Y_k.$$

Exercise 13. Prove Theorem 24.

Theorem 24 (Dold-Kan)**.** *The normalization N and denormalization Γ functors fit into the following*

$$\mathsf{sMod}_R \underset{\Gamma}{\overset{\mathrm{N}}{\rightleftarrows}} \mathsf{Ch}^+_R$$

equivalence of categories; i.e., there exist natural isomorphisms

$$\mathrm{id} \xrightarrow{\cong} \Gamma\mathrm{N}, \qquad \mathrm{N}\Gamma \xrightarrow{\cong} \mathrm{id}.$$

Exercise 14. Prove Proposition 25.

Proposition 25. *The adjunction*

$$\mathsf{Ch}_R^+ \underset{\mathrm{N}}{\overset{\Gamma}{\rightleftarrows}} \mathsf{sMod}_R$$

with left adjoint on top is a Quillen equivalence.

Homotopy theory of simplicial R-modules. Recall from lecture the following lifting characterization of (acyclic) fibrations in sSet.

Proposition 26. *Let $f\colon X \longrightarrow Y$ be a map in* sSet.

(a) *f is an acyclic fibration if and only if it has the right lifting property with respect to the set of maps*

$$j_n\colon \partial\Delta[n] \longrightarrow \Delta[n], \qquad n \geq 0,$$

(b) *f is a fibration if and only if it has the right lifting property with respect to the set of maps*

$$j_{n,k}\colon \Lambda[n,k] \longrightarrow \Delta[n], \qquad n \geq 1, \qquad 0 \leq k \leq n.$$

Consider the adjunction

$$\mathsf{sSet} \underset{U}{\overset{R}{\rightleftarrows}} \mathsf{sMod}_R$$

with left adjoint on top and U the forgetful functor. Define a map $f\colon X \longrightarrow Y$ in sMod_R to be

(i) a *weak equivalence* if $Uf\colon UX \longrightarrow UY$ is a weak equivalence in sSet,
(ii) a *fibration* if $Uf\colon UX \longrightarrow UY$ is a fibration in sSet,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

Exercise 15. Prove Proposition 27.

Proposition 27. *Let $f\colon X \longrightarrow Y$ be a map in* sMod_R.

(a) *f is a fibration if and only if it has the right lifting property with respect to the set of maps*

$$Rj_{n,k}\colon R\Lambda[n,k] \longrightarrow R\Delta[n], \qquad n \geq 1, \qquad 0 \leq k \leq n,$$

(b) *f is an acyclic fibration if and only if it has the right lifting property with respect to the set of maps*

$$Rj_n\colon R\partial\Delta[n] \longrightarrow R\Delta[n], \qquad n \geq 0.$$

Exercise 16. Use small object arguments to prove Proposition 28.

Proposition 28. *These three classes of maps give* sMod_R *the structure of a model category.*

Homotopy theory of simplicial commutative rings. Denote by sCRng the category of simplicial commutative rings. Consider the adjunctions

$$\mathsf{sSet} \underset{U}{\overset{\mathbb{Z}}{\rightleftarrows}} \mathsf{sMod}_{\mathbb{Z}} \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathsf{sCRng}\ , \qquad \mathsf{sSet} \underset{U}{\overset{\mathrm{Sym}\mathbb{Z}}{\rightleftarrows}} \mathsf{sCRng}$$

with left adjoints on top and U the forgetful functor; here, the right-hand adjunction is the composition of the left-hand adjunctions. Define a map $f\colon X \longrightarrow Y$ in sCRng to be

(i) a *weak equivalence* if $Uf\colon UX \longrightarrow UY$ is a weak equivalence in sSet,
(ii) a *fibration* if $Uf\colon UX \longrightarrow UY$ is a fibration in sSet,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

Exercise 17. Prove Proposition 29.

Proposition 29. *Let $f\colon X \longrightarrow Y$ be a map in* sCRng.

(a) *f is a fibration if and only if it has the right lifting property with respect to the set of maps*

$$\mathrm{Sym}\mathbb{Z}(j_{n,k})\colon \mathrm{Sym}\mathbb{Z}\Lambda[n,k] \longrightarrow \mathrm{Sym}\mathbb{Z}\Delta[n], \qquad n \geq 1, \qquad 0 \leq k \leq n,$$

(b) *f is an acyclic fibration if and only if it has the right lifting property with respect to the set of maps*

$$\mathrm{Sym}\mathbb{Z}(j_n)\colon \mathrm{Sym}\mathbb{Z}\partial\Delta[n] \longrightarrow \mathrm{Sym}\mathbb{Z}\Delta[n], \qquad n \geq 0.$$

Exercise 18. Use small object arguments to prove Proposition 30.

Proposition 30. *These three classes of maps give* sCRng *the structure of a model category.*

Exercise 19. Please read [1, Sections 10-11], [4, Sections 1.5 and 4.1-4.4], and [5, Sections 1-3].

References

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