Jordan Normal Form

The classification of nxn matrices up to similarity is not so simple. We will merely state (rather than prove) the result, and then only for the case $F := \mathbb{C}$. That is, only for complex nxn matrices.

(\text{The theorem will at least give you})
\begin{align*}
\text{(a feel for the nature of complex endomorphisms)}.
\end{align*}

\textbf{Definition:} Let $c$ be a complex number and $m \geq 1$. The main matrix\[ J_m(c) := \begin{bmatrix} c & 1 & & \\ & c & 1 & \\ & & \ddots & 1 \\ & & & c \end{bmatrix} \]
is called the \underline{Jordan block} of degree $m$ for the eigenvalue $c$. 
Let $f$ be an endomorphism of $\mathbb{C}^n$

\[ \mathbb{C}^n \xrightarrow{f} \mathbb{C}^n \]

The Jordan block $J_m(c)$ has only one eigenvalue $c$, and the dimension of the eigenspace is the smallest that an eigenspace can have (i.e., $\dim E_c = 1$).

For this reason, $J_m(c)$ is regarded as being as non-diagonalizable as a complex $m \times m$ matrix can be.

**Theorem:** (Jordan Normal Form).

If $A$ is a complex $m \times m$ matrix, and if $c_1, \ldots, c_r \in \mathbb{C}$ are its distinct eigenvalues, then for each $k = 1, \ldots, r$ there exist uniquely determined positive natural numbers $n_k$ and

\[ m_1 \leq m_2 \leq \ldots \leq m_{n_k} \]
with the property that there exists an invertible complex non-zero matrix \( P \) such that \( P^{-1}AP \) is the "block matrix" obtained by adjunction of the Jordan blocks

\[
\begin{bmatrix}
J_{m_1(\lambda)}(c_1) & \cdots & \cdots \\
& \ddots & \\
& & J_{m_n(\lambda)}(c_n)
\end{bmatrix}
\]

along the diagonal.

Thus the \( k \)-th eigenvalue \( \lambda_k \) contributes a smaller block matrix \( B_k \)

\[
B_k :=
\begin{bmatrix}
\begin{array}{ccc}
\lambda_k & \cdots & 1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_k
\end{array}
\end{bmatrix}
\]

built up from \( n_k \) single Jordan blocks:
The whole Jordan normal form of \( A \) is then:

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{B}_y \\
\mathcal{B}_x \\
\vdots \\
\mathcal{B}_r
\end{array}
\end{array}
\]

\[
\sum_{r \geq 1} \sum_{i=1}^{m_r(\alpha)} \frac{x_i^{m_r(\alpha)}}{i!}
\]

(\text{Note: Only if each Jordan block has dimension 1 is } A \text{ diagonalizable.})