Math 35300: Section 001. Linear algebra II **Fall 2012**
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Homework 2

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. Let $n \geq 1$. Then $\mathbb{R}^n$ consists of

(a) n real numbers
(b) n-tuples of real numbers
(c) n-tuples of vectors

Exercise 2. Which of the following statements is not an axiom for real vector spaces?

(a) For all $x, y \in V$ we have $x + y = y + x$.
(b) For all $x, y, z \in V$ we have $(x + y) + z = x + (y + z)$.
(c) For all $x, y, z \in V$ we have $(xy)z = x(yz)$.

Exercise 3. For the multiplication of complex numbers we have $(x + yi)(a + bi) =$

(a) $xa + ybi$
(b) $xy + yb + (xb - ya)i$
(c) $xa - yb + (xb + ya)i$

Exercise 4. In a vector space V over a field $\mathbb{F}$ scalar multiplication is given by a map

(a) $V \times V \longrightarrow \mathbb{F}$
(b) $\mathbb{F} \times V \longrightarrow V$
(c) $\mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$

Exercise 5. Which formulation below can be completed correctly to give the definition of the concept of a real vector space?

(a) A set V is called a *real vector space* if there exist two maps $+ : \mathbb{R} \times V \longrightarrow V$ and $\cdot : \mathbb{R} \times V \longrightarrow V$ so that the following eight axioms are satisfied ...
(b) A set of real vectors is called a *real vector space* if the following eight axioms are satisfied ...
(c) A triple $(V, +, \cdot)$ in which V is a set, and $+$ and $\cdot$ are maps $V \times V \longrightarrow V$ and $\mathbb{R} \times V \longrightarrow V$, respectively, is called a *real vector space* if the following eight axioms are satisfied ...

Exercise 6. Which of the following statements is true? If V is a vector space over a field $\mathbb{F}$ then

(a) $\{x + y \mid x \in V, y \in V\} = V$.
(b) $\{x + y \mid x \in V, y \in V\} = V \times V$.
(c) $\{ax \mid a \in \mathbb{F}, x \in V\} = \mathbb{F} \times V$.

Exercise 7. Which of the following statements is true?

(a) If U is a subspace of V, then $V \setminus U$ is also a subspace of V.

(b) There exists a subspace U of V for which $V \setminus U$ is also a subspace, but $V \setminus U$ is not a subspace for all subspaces U.
(c) If U is a subspace of V, then $V \setminus U$ is never a subspace of V.

Exercise 8. Which of the following subsets $U \subset \mathbb{R}^n$ is a vector subspace?

(a) $U = \{x \in \mathbb{R}^n \mid x_1 = \cdots = x_n\}$
(b) $U = \{x \in \mathbb{R}^n \mid x_1^2 = x_2^2\}$
(c) $U = \{x \in \mathbb{R}^n \mid x_1 = 1\}$

Exercise 9. On restriction of scalar multiplication to the scalar domain $\mathbb{R} \subset \mathbb{C}$, a complex vector space $(V, +, \cdot)$ becomes a real vector space $(V, +, \cdot|\mathbb{R} \times V)$. In particular, $V := \mathbb{C}$ can itself be regarded as a real vector space in this way. Do the imaginary numbers $U = \{yi \in \mathbb{C} \mid y \in \mathbb{R}\}$ then form a vector subspace?

(a) Yes, because then $U = \mathbb{C}$.
(b) Yes, because $0 \in U$ and when $a \in \mathbb{R}$ and $xi, yi \in U$, we also have $(x+y)i \in U$ and $axi \in U$.
(c) No, because ayi does not need to be imaginary, since for example $i^2 = -1$.

Exercise 10. How many vector subspaces does $\mathbb{R}^2$ have?

(a) two: $\{0\}$ and $\mathbb{R}^2$
(b) four: $\{0\}$, the "axes" $\mathbb{R} \times 0$ and $0 \times \mathbb{R}$, and $\mathbb{R}^2$ itself.
(c) infinitely many

The rule $x + (y - x) = y$, for example, does not appear among the axioms of a vector space, but can be easily deduced from them:

$$\begin{aligned} x + (y - x) &= x + (-x + y) && \text{by axiom (2)} \\ &= (x - x) + y && \text{by axiom (1)} \\ &= 0 + y && \text{by axiom (4)} \\ &= y + 0 && \text{by axiom (2)} \\ &= y && \text{by axiom (3)} \end{aligned}$$

So it would have been quite unnecessary to include $x + (y - x) = y$ among the axioms. In setting up an axiom system, one tries to choose the fewest and simplest axioms needed in order to deduce from them all the additional rules that one wants.

But this does not mean that for each page of linear algebra you must write ten pages of "reduction to the axioms". Given a little practice it can be assumed that you would be able to carry out reduction of your calculations to the axioms, and this does not need to be formally written out. The following exercise provides this practice.

Exercise 11. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then for all $x \in V$ and $a \in \mathbb{F}$, prove

(a) $0 + x = x$
(b) $-0 = 0$
(c) $a0 = 0$
(d) $0x = 0$
(e) $ax = 0$ if and only if $a = 0$ or $x = 0$
(f) $-x = (-1)x$

Remark 1. Properties (a)–(f) in the above exercise remain true for vector spaces over any field $\mathbb{F}$.

Exercise 12. Show that $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$.

The following exercise provides some practice with the field axioms.

Exercise 13. Let $\mathbb{F}$ be a field. Prove the following.

(a) The element $0 \in \mathbb{F}$ is uniquely determined.
(b) For each $a \in \mathbb{F}$, the element $-a \in \mathbb{F}$ is uniquely determined.
(c) The element $1 \in \mathbb{F} \setminus 0$ is uniquely determined.
(d) For each $a \in \mathbb{F} \setminus 0$, the element $a^{-1} \in \mathbb{F}$ is uniquely determined.
(e) $(-1)a = -a$ for each $a \in \mathbb{F}$.
(f) For each $a, b \in \mathbb{F}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.
(g) $(-1)(-1) = 1$.

Exercise 14. Let V be a vector space over a field $\mathbb{F}$. Let $x, y \in V$ and $a \in \mathbb{F}$. Prove the cancellation law in a vector space: If $ax = ay$ and $a \neq 0$, then $x = y$.

Exercise 15. Let V be a vector space over a field $\mathbb{F}$. Prove the following: If U is a subspace of V, then U together with the addition and scalar multiplication inherited from $(V, +, \cdot)$ is itself a vector space over $\mathbb{F}$.

Exercise 16. For $a \in \mathbb{F}$ we define $U_a := \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + x_2 + x_3 = a\}$. Show that U_a is a vector subspace of $\mathbb{F}^3$ if and only if $a = 0$; here, $\mathbb{F}$ is any field.

Exercise 17. Let V be a vector space over a field $\mathbb{F}$. Prove the following: If U_1, U_2 are subspaces of V, then $U_1 \cap U_2$ is also a subspace of V.

The following is a neat exercise. One can reduce the proof to just a few lines.

Exercise 18. Let V be a vector space over a field $\mathbb{F}$ and let U_1, U_2 be subspaces of V. Show that if $U_1 \cup U_2 = V$, then $U_1 = V$ or $U_2 = V$, or both.

Exercise 19. If all the field axioms for $(\mathbb{F}, +, \cdot)$ hold, with the possible exception of axiom (8), then one calls $(\mathbb{F}, +, \cdot)$ a "commutative ring with unit". If in addition $ab = 0$ only occurs if $a = 0$ or $b = 0$, then $\mathbb{F}$ is called a "commutative ring with unit element and no divisors of zero" or an "integral domain" for short. Prove that every finite integral domain is a field.