Math 35300: Section 001. Linear algebra II **Fall 2012**
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Homework 7

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. The determinant is a map

(a) $\mathrm{M}(n \times n, \mathbb{F}) \longrightarrow \mathbb{F}$ given by the product of the diagonal elements.
(b) $\mathrm{M}(n \times n, \mathbb{F}) \longrightarrow \mathbb{F}$, which is linear in the rows, vanishes on matrices with less than maximal rank, and takes the value 1 on I.
(c) $\mathrm{M}(n \times n, \mathbb{F}) \longrightarrow \mathbb{F}^n$, which is given by a linear combination of rows, vanishes on matrices with less than maximal rank, and takes the value 1 on I.

Exercise 2. Let $A, A' \in \mathrm{M}(n \times n, \mathbb{F})$ and let A' be obtained from A by elementary row transformations. Which of the following statements are correct?

(a) $\det A = 0 \Longleftrightarrow \det A' = 0$
(b) $\det A = \det A'$.
(c) $\det A = c \det A'$ for some $c \in \mathbb{F}$, $c \neq 0$.

Exercise 3. Which of the following assertions is correct? For $A \in \mathrm{M}(n \times n, \mathbb{F})$ we have

(a) $\det A = 0 \Longrightarrow \operatorname{rk} A = 0$.
(b) $\det A = 0 \Longleftrightarrow \operatorname{rk} A \leq n - 1$.
(c) $\det A = 0 \Longrightarrow \operatorname{rk} A = n$.

Exercise 4. Which of the following statements holds for all $A, B, C \in \mathrm{M}(n \times n, \mathbb{F})$ and all $c \in \mathbb{F}$?

(a) $\det(A + B) = \det A + \det B$.
(b) $\det(cA) = c \det A$.
(c) $\det(ABC) = (\det A)(\det B)(\det C)$.

Exercise 5. Which of the formulas below is called "expansion of the determinant by minors on the i-th row"?

(a) $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.
(b) $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ji}$.
(c) $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

Exercise 6. $\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} =$

(a) 2 (b) 4 (c) 6

Exercise 7. Let $I \in \mathrm{M}(n \times n, \mathbb{F})$ be the unit matrix. Then the transposed matrix $I^t =$

$$\text{(a)} \quad \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} t & & \\ & \ddots & \\ & & t \end{bmatrix}$$

Exercise 8. $\det \begin{bmatrix} c & c & c \\ c & c & c \\ c & c & c \end{bmatrix} =$

$$\text{(a)} \quad 0 \qquad \text{(b)} \quad c \qquad \text{(c)} \quad c^3$$

Exercise 9. $\det \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} =$

$$\text{(a)} \quad \cos 2\varphi \qquad \text{(b)} \quad 0 \qquad \text{(c)} \quad 1$$

Exercise 10. Which of the following is (or are) false?
(a) $\det A = 1 \Longrightarrow A = I$.
(b) $\det A = 1 \Longrightarrow A$ is injective as a map $\mathbb{F}^n \longrightarrow \mathbb{F}^n$.
(c) $\det A = 1 \Longrightarrow A$ is surjective as a map $\mathbb{F}^n \longrightarrow \mathbb{F}^n$.

We would like to define the determinant not only for an $n \times n$ matrix, but also for an endomorphism $f\colon V \longrightarrow V$ of an n-dimensional vector space over $\mathbb{F}$. One possibility is to choose some basis $(v_1, \ldots, v_n)$ of V, consider the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] & V \\ \mathbb{F}^n \arrow[u, "\Phi", "\cong"'] \arrow[r, "A"] & \mathbb{F}^n \arrow[u, "\cong", "\Phi"'] \end{tikzcd}$$

(i.e., A is the matrix associated to f relative to the basis $(v_1, \ldots, v_n)$ and Φ is the canonical basis isomorphism), and declare $\det f := \det A$. The only potential problem with this definition is that it appears to depend upon the choice of basis. In other words, could another choice of basis, and hence another matrix, mean another determinant for f? The following exercise settles this question.

Exercise 11. Prove the following: If $f\colon V \longrightarrow V$ is an endomorphism of an n-dimensional vector space, and if f is represented by the matrix A relative to a basis $(v_1, \ldots, v_n)$ and by the matrix A' relative to a basis $(v'_1, \ldots, v'_n)$ then $\det A = \det A'$.

Exercise 12. Compute $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}^n$

Exercise 13. Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Exercise 14. Let A, B be square matrices.
(a) When is $(A+B)(A-B) = A^2 - B^2$?
(b) Expand $(A+B)^3$.

Exercise 15. Let D be the diagonal matrix

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

and let $A = (a_{ij})$ be any $n \times n$ matrix.

(a) Compute the products DA and AD.
(b) Compute the product of two diagonal matrices.
(c) When is a diagonal matrix invertible?

Exercise 16. A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$.

(a) Prove that if A is nilpotent then $\det A = 0$.
(b) Prove that if A is nilpotent then $I + A$ is invertible.

Exercise 17. A matrix A is called *symmetric* if $A = A^t$. Prove that for any matrix A, the matrix AA^t is symmetric and that if A is a square matrix then $A + A^t$ is symmetric.

Exercise 18.

(a) Prove that $(AB)^t = B^t A^t$ and that $A^{tt} = A$.
(b) Prove that if A is invertible then $(A^{-1})^t = (A^t)^{-1}$.

Exercise 19. Prove that the inverse of an invertible symmetric matrix is also symmetric.

Exercise 20. Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

Exercise 21. Let A be an $n \times n$ matrix. What is $\det(-A)$?

Exercise 22. Prove that $\det A^t = \det A$.

Exercise 23. Derive the formula $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ from the special properties (i)–(iii),(v) (proved in lecture) of the determinant.

Exercise 24. Let A and B be $n \times n$ matrices. Prove that $\det(AB) = \det(BA)$.

Exercise 25. Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks.

Exercise 26. Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold when $AC \neq CA$.