The Spectral Theorem

Definition: Let $(V, \langle , \rangle)$ be a Euclidean vector space. An endomorphism $f: V \to V$ is called <u>self-adjoint</u> (or <u>symmetric</u>) if $\langle f(v), w \rangle = \langle v, f(w) \rangle$ for all $v, w \in V$

Proposition: Let $(V, \langle - \rangle)$ be a Euclidean vector space.

(a) If $(v_1, \ldots, v_n)$ is an orthonormal basis of V, then the matrix A of an endomorphism $f: V \to V$ is given by

$$a_{ij} = \langle f(v_j), v_i \rangle$$

(b) If $(v_1, \ldots, v_n)$ is an orthonormal basis of V, then an endomorphism $f: V \to V$ is self-adjoint $\Longleftrightarrow$ the corresponding matrix A is <u>symmetric</u> (i.e., $A^t = A$)

Hence in the case $V := \mathbb{R}^n$ with the usual $\langle , \rangle$, self-adjoint endomorphisms $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$ are the same as symmetric matrices

$$A = A^t \in M(n \times n, \mathbb{R}) .$$

Proposition (#): For $n \geq 1$ each self-adjoint endomorphism of an n-dimensional Euclidean vector space has at least one eigenvalue, and hence it has an eigenvector.

Proof:

$$\left(\begin{tikzcd} V \arrow[r, "f"] & V \\ \mathbb{R}^n \arrow[u, "\Phi", "\cong"'] \arrow[r, "A"] & \mathbb{R}^n \arrow[u, "\Phi"', "\cong"] \end{tikzcd} \right.$$

Choose an orthonormal basis $\underline{\underline{B}}$ of V. $(\Phi := \Phi_{\underline{\underline{B}}})$

$$\left. A := \Phi^{-1} f \, \Phi \right)$$

It suffices to prove the theorem for symmetric real $n \times n$ matrices $A : \mathbb{R}^n \to \mathbb{R}^n$.

(Idea is: Use existence of roots for complex polynomials of degree ≥ 1.)

We want: to show there exists $c \in \mathbb{R}$ such that $P_A(c) = 0$.

Regard $P_A(c)$ as a complex polynomial.
Then there is a complex number

$$c = a + ib \in \mathbb{C} \qquad (a, b \in \mathbb{R})$$

with $P_A(c) = 0$. Hence $c \in \mathbb{C}$ is
an eigenvalue of the endomorphism

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\ \cup & & \cup \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \end{array}$$

of the complex vector space $\mathbb{C}^n$ given by the same matrix A. Hence there is some $x + iy \in \mathbb{C}^n$ nonzero such that

$$A(x+iy) = c(x+iy) = (a+ib)(x+iy)$$
$$= (ax - by) + i(ay + bx)$$

separating real and imaginary parts gives

$$\begin{cases} Ax = ax - by \\ Ay = ay + bx \end{cases} \qquad (x, y \in \mathbb{R}^n)$$

(We want: to verify that $b = 0$)

By symmetry of A we have

$$\begin{array}{ccc}
\langle Ax, y\rangle & = & \langle x, Ay\rangle \\
\| & & \| \\
\langle ax-by, y\rangle & & \langle x, ay+bx\rangle \\
\| & & \| \\
a\langle x,y\rangle - b\langle y,y\rangle & & a\langle x,y\rangle + b\langle x,x\rangle \\
\| & & \| \\
a\langle x,y\rangle - b\|y\|^2 & & a\langle x,y\rangle + b\|x\|^2
\end{array}$$

$$\therefore \quad b\left(\|x\|^2 + \|y\|^2\right) = 0$$

But $c = a + ib \neq 0 \implies \|x\|^2 + \|y\|^2 \neq 0$

and hence $\therefore \boxed{b = 0}$

$$\therefore \quad c = a + i0 = a \in \mathbb{R}.$$

Hence $A : \mathbb{R}^n \to \mathbb{R}^n$ has an eigenvalue.

$\blacksquare$

<u>Theorem</u> (Spectral Theorem)

(a) (Vector space form): Let $f: V \longrightarrow V$ be a self-adjoint endomorphism on a finite-dimensional Euclidean vector space. Then there is an orthonormal basis of eigenvectors of f.

(b) (Matrix form): Let M be a real symmetric $n \times n$ matrix. Then there is an orthogonal matrix $P \in O(n)$ such that PMP^t is diagonal.

<u>Proof</u>: Since (a) and (b) are equivalent, it suffices to prove (a).

We know (by the previous proposition) that f has at least one eigenvector v_1. Normalize its length to 1:

Define $w_1 := \dfrac{v_1}{\|v_1\|}$. Extend to a basis $(w_1, v_2, \ldots, v_n)$ of V, and use the Gram-Schmidt procedure to get an orthonormal basis $(w_1, \ldots, w_n)$ of V.

Then the matrix of f becomes

$$\underline{M} = \left[\begin{array}{c|ccc} c & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & N & \\ 0 & & & \end{array}\right]$$

where c is the eigenvalue of w_1.

Since f is self-adjoint, the matrix $\underline{M}$ is symmetric. This implies that $(* \cdots *) = (0 \cdots 0)$ and that N is symmetric.

Note that N is an $(n-1) \times (n-1)$ symmetric matrix.

Hence induction on n finishes the proof. ∎

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