**Math 35300: Section 002. Linear algebra II** **Fall 2012**
John E. Harper
Purdue University

## Homework 1

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

**Exercise 1.** Let $A, B$ be sets. If for each $a \in A$ we have $a \in B$, then one writes which of the following?

$$\text{(a)} \quad A \subset B \qquad \text{(b)} \quad A = B \qquad \text{(c)} \quad A \cup B$$

**Exercise 2.** For each set $A$, which of the following sets is empty?

$$\text{(a)} \quad M \cup M \qquad \text{(b)} \quad M \cap M \qquad \text{(c)} \quad M \setminus M$$

**Exercise 3.** Let $A, B$ be sets and let $a \in A$. As in lecture, we can draw a picture (or cartoon) of $A \times B$ by a rectangle (here, we are thinking of $A$ and $B$ as intervals, even though in general this will not be true). How would one draw a corresponding picture of the subset $\{a\} \times B$? (Please draw a picture).

**Exercise 4.** Which of the following statements is false? The map

$$\mathrm{Id}_A\colon A \longrightarrow A, \qquad a \longmapsto a,$$

is always

$$\text{(a)} \quad \text{surjective} \qquad \text{(b)} \quad \text{bijective} \qquad \text{(c)} \quad \text{constant}$$

**Exercise 5.** Let $A, B$ be sets and $A \times B$ the Cartesian product. By projection onto the second factor, one understands the map $\pi_2$ as which of the following?

$$\text{(a)} \quad \begin{aligned} A \times B &\longrightarrow A \\ (a,b) &\longmapsto b \end{aligned} \qquad \text{(b)} \quad \begin{aligned} A \times B &\longrightarrow B \\ (a,b) &\longmapsto b \end{aligned} \qquad \text{(c)} \quad \begin{aligned} B &\longrightarrow A \times B \\ b &\longmapsto (a,b) \end{aligned}$$

**Exercise 6.** Let $f\colon X \longrightarrow Y$ be a map. Which of the following statements implies that $f$ is surjective?

$$\text{(a)} \quad f^{-1}(Y) = X \qquad \text{(b)} \quad f(X) = Y \qquad \text{(c)} \quad f^{-1}(X) = Y$$

**Exercise 7.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps. Then the map $gf\colon X \longrightarrow Z$ is defined by which of the following?

$$\text{(a)} \quad x \longmapsto g(f(x)) \qquad \text{(b)} \quad x \longmapsto f(g(x)) \qquad \text{(c)} \quad x \longmapsto g(x)(f)$$

**Exercise 8.** Consider any commutative diagram of sets of the form

$$\begin{tikzcd} & Y \\ X \arrow[ur, bend left, "f"] \arrow[r, "g"'] & Z \arrow[u, "h"'] \end{tikzcd}$$

Then we have which of the following?

$$\text{(a)} \quad h = gf \qquad\qquad \text{(b)} \quad f = hg \qquad\qquad \text{(c)} \quad g = fh$$

**Exercise 9.** The map $f\colon\ \mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$, $x\longmapsto\frac{1}{x}$ is bijective. The inverse map $f^{-1}\colon\ \mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{0\}$ is defined by which of the following?

$$\text{(a)} \quad x\longmapsto\tfrac{1}{x} \qquad\qquad \text{(b)} \quad x\longmapsto x \qquad\qquad \text{(c)} \quad x\longmapsto-\tfrac{1}{x}$$

**Exercise 10.** Consider the map $\mathbb{R}\longrightarrow\mathbb{R}$, $x\longmapsto x^2$. Which of the following is true?

(a) The map is surjective but not injective.
(b) The map is injective but not surjective.
(c) The map is neither surjective nor injective.

Recall from lecture that if $f\colon\ X\longrightarrow Y$ is a map, then the set

$$\Gamma_f \;:=\; \{(x, f(x)) \mid x\in X\}$$

is called the *graph* of $f$. The graph is a subset of the Cartesian product $X\times Y$.

**Exercise 11.** Let $X, Y$ be sets. As in lecture, we can draw a picture (or cartoon) of the Cartesian product $X\times Y$ by a rectangle (here, we are thinking of $X$ and $Y$ as intervals, although in general this will not be true). In this way, give examples of graphs of maps $f$ with the following properties (please draw a picture for each):

(a) $f$ is surjective, but not injective
(b) $f$ is injective, but not surjective
(c) $f$ is bijective
(d) $f$ is constant
(e) $f$ is neither injective nor surjective
(f) $X = Y$ and $f = \mathrm{Id}_X$
(g) $f(X)$ consists of only two elements

(Careful: not all subsets of the Cartesian product $X\times Y$ are graphs of a map $f\colon\ X\longrightarrow Y$; i.e., many subsets of $X\times Y$ are nongraphs.)

The inverse map $f^{-1}$ of a bijective map $f\colon\ X\longrightarrow Y$ clearly has the properties

$$f\circ f^{-1} = \mathrm{Id}_Y, \qquad f^{-1}\circ f = \mathrm{Id}_X,$$

since in the first case each element $f(x)\in Y$ is mapped by $f(x)\mapsto x\mapsto f(x)$ onto $f(x)$, and in the second case each $x\in X$ is mapped by $x\mapsto f(x)\mapsto x$ onto $x$. Conversely, one has the following property.

**Proposition 1.** *Let $f\colon\ X\longrightarrow Y$ and $g\colon\ Y\longrightarrow X$ be maps such that*

$$f\circ g = \mathrm{Id}_Y, \qquad g\circ f = \mathrm{Id}_X.$$

*Then $f$ is bijective and $f^{-1} = g$.*

**Exercise 12.** Prove Proposition 1. (To get started: An injectivity proof runs like this: "Let $x, x'\in X$ and $f(x) = f(x')$, then $\dots$ . Therefore $x = x'$, and $f$ is proved to be injective." On the other hand, the pattern for a surjectivity proof is: "Let $y\in Y$. Choose $x = \dots$ . Then we have $\dots$ , therefore $f(x) = y$, and $f$ is proved to be surjective.")

**Exercise 13.** Consider any commutative diagram of sets of the form

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "\alpha", "\cong"'] \arrow[r, "g"'] & B \arrow[u, "\cong", "\beta"'] \end{tikzcd}$$

with $\alpha, \beta$ bijective.

(a) Show that $g$ is injective if and only if $f$ is injective.
(b) Show that $g$ is surjective if and only if $f$ is surjective.

(We will frequently meet this kind of diagram in the course. The situation is then mostly: $f$ is the object of our interest, $\alpha$ and $\beta$ are subsidiary constructions, means to an end, and we already know something about $g$. This information about $g$ then tells us something about $f$. In solving this exercise, you will see the mechanism of this information transfer.)