**Math 35300: Section 002. Linear algebra II** **Fall 2012**
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**Homework 6**

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

**Exercise 1.** Which of the following operations cannot be made as a sequence of row and column operations?

(a) $\begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 2 & 7 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 7 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} -11 & 2 \\ 1 & 1 \end{bmatrix}$

**Exercise 2.** The rank of the real matrix

$$\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$$

is

(a) 1 (b) 3 (c) 5

**Exercise 3.** A system of linear equations with coefficients in $\mathbb{F}$ is a system of equations of the following kind:

(a) $$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_1 = b_1 \\ \vdots \qquad\qquad \vdots \quad \vdots \\ a_{n1}x_n + \cdots + a_{nn}x_n = b_n \end{matrix} \qquad \text{with } a_{ij} \in \mathbb{F},\ b_i \in \mathbb{F}$$

(b) $$\begin{matrix} a_{11}x_{11} + \cdots + a_{1n}x_{1n} = b_1 \\ \vdots \qquad\qquad \vdots \quad \vdots \\ a_{n1}x_{n1} + \cdots + a_{nn}x_{nn} = b_n \end{matrix} \qquad \text{with } a_{ij} \in \mathbb{F},\ b_i \in \mathbb{F}$$

(c) $$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \quad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{matrix} \qquad \text{with } a_{ij} \in \mathbb{F},\ b_i \in \mathbb{F}$$

**Exercise 4.** If one abbreviates a system of linear equations as $Ax = b$, then

(a) $A \in \mathrm{M}(m \times n, \mathbb{F})$, $b \in \mathbb{F}^n$.
(b) $A \in \mathrm{M}(m \times n, \mathbb{F})$, $b \in \mathbb{F}^m$.
(c) $A \in \mathrm{M}(m \times n, \mathbb{F})$, $b \in \mathbb{F}^n$ or $b \in \mathbb{F}^m$ (not fixed).

**Exercise 5.** A system of linear equations $Ax = b$ is called solvable if

(a) $Ax = b$ for all $x \in \mathbb{F}^n$.
(b) $Ax = b$ for precisely one $x \in \mathbb{F}^n$.

(c) $Ax = b$ for at least one $x \in \mathbb{F}^n$.

**Exercise 6.** If $b$ is one of the columns of $A$, then $Ax = b$ is

(a) solvable in all cases
(b) unsolvable in all cases
(c) sometimes solvable, sometimes unsolvable, depending on $A$ and $b$

**Exercise 7.** Let $Ax = b$ be a system of equations with square matrix $A$ ($n$ equations in $n$ unknowns). Then $Ax = b$ is

(a) uniquely solvable
(b) solvable or unsolvable, depending on $A, b$
(c) solvable, but perhaps not uniquely, depending on $A, b$

**Exercise 8.** Suppose once more that $A \in \mathrm{M}(n \times n, \mathbb{F})$, that is, $A$ is square. Which of the following conditions is (or are) equivalent to the unique solvability of $Ax = b$:

(a) $\dim \operatorname{Ker} A = 0$
(b) $\dim \operatorname{Ker} A = n$
(c) $\operatorname{rk} A = n$

**Exercise 9.** Let $A \in \mathrm{M}(n \times n, \mathbb{F})$ and $\operatorname{Ker} A \neq 0$. Then $Ax = b$ is

(a) solvable only for $b = 0$
(b) solvable for all $b$, possibly nonuniquely
(c) solvable only for some $b$, and then never uniquely

**Exercise 10.** Let $A$ be an $n \times n$ matrix and let $Ax = b$ have two *linearly independent* solutions. Then

(a) $\operatorname{rk} A \leq n$, and the case $\operatorname{rk} A = n$ can occur.
(b) $\operatorname{rk} A \leq n - 1$, and the case $\operatorname{rk} A = n - 1$ can occur.
(c) $\operatorname{rk} A \leq n - 2$, and the case $\operatorname{rk} A = n - 2$ can occur.

(Hint: use the dimension formula for linear maps).

**Exercise 11.** Let $A \in \mathrm{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^n$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ (i.e., if $Ax_0 = b$), then

$$\operatorname{Sol}(A, b) = (x_0 + \operatorname{Ker} A) := \{x_0 + x \mid x \in \operatorname{Ker} A\}.$$

**Exercise 12.** Let $A \in \mathrm{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^n$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ and $(v_1, \ldots, v_r)$ is a basis of $\operatorname{Ker} A$, then

$$\operatorname{Sol}(A, b) = \{x_0 + c_1 v_1 + \cdots + c_r v_r \mid c_i \in \mathbb{F}\};$$

here, $r = \dim \operatorname{Ker} A = n - \operatorname{rk} A$.

**Exercise 13.** Let $A \in \mathrm{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^n$. Prove the following: Assume that $Ax = b$ is solvable. Then $Ax = b$ is uniquely solvable if and only if $\operatorname{Ker} A = 0$ (i.e., if and only if $\operatorname{rk} A = n$).

**Exercise 14.** Find all solutions of the system of equations $Ax = b$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & -2 \end{bmatrix}$$

and $b$ has the following value:

$$\text{(a)} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

**Exercise 15.** Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.

**Exercise 16.** Use row reduction to find inverses of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad \text{(d)} \quad \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

**Exercise 17.** How much can a matrix be simplified if both row and column operations are allowed?

**Exercise 18.** Prove that every invertible $2 \times 2$ matrix is a product of at most four elementary matrices.

**Exercise 19.** Prove that if a product $AB$ of $n \times n$ matrices is invertible then so are its factors $A, B$.

**Exercise 20.** Let $A$ be a square matrix. Prove that there is a set of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A$ either is the identity or has its bottom row zero.

**Exercise 21.** Prove the following proposition from lecture. (Hint: it suffices to prove the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$).

**Proposition 1.** *Let $A$ be a square matrix. The following conditions are equivalent:*

(a) *$A$ can be reduced to the identity by a sequence of elementary row operations.*
(b) *$A$ is a product of elementary matrices.*
(c) *$A$ is invertible.*
(d) *The linear system $Ax = 0$ has only the trivial solution $x = 0$.*

**Exercise 22.** Let $(v_1, v_2, v_3, v_4)$ be linearly independent elements of the real vector space $V$. If

$$\begin{aligned} w_1 &= \phantom{-v_1 +{}} v_2 - v_3 + 2v_4 \\ w_2 &= v_1 + 2v_2 - v_3 - v_4 \\ w_3 &= -v_1 + v_2 + v_3 + v_4, \end{aligned}$$

show that $(w_1, w_2, w_3)$ is linearly independent. (Hint: first show that the linear independence of $(w_1, w_2, w_3)$ is equivalent to a certain matrix having rank 3, and then use the procedure for determining rank to find the rank of this matrix).

**Exercise 23.** For which values of $c$, is the real matrix

$$A_c := \begin{bmatrix} 1 & c & 0 & 0 \\ c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

invertible? For these values of $c$ determine the inverse matrix $A_c^{-1}$.

**Exercise 24.** Prove the following: If $U \subset \mathbb{F}^n$ is a subspace and $x \in \mathbb{F}^n$, then there exists a system of equations with coefficients in $\mathbb{F}$, having $n$ equations and $n$ unknowns, whose solution set equals $x + U$.