MA 353
Linear Algebra II
Handout 5

(This is some material between Lecture 17 and Lecture 18 that was distributed as a Handout)

Proposition : Every linear system

$$A x = 0$$

of m homogeneous equations in n unknowns (i.e., $A \in M(m \times n, \mathbb{F})$) with $m < n$, has a solution

$$x \neq 0 \text{ in } \mathbb{F}^n$$

(i.e., $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n$ such that $x \neq 0$)

Proof : Let $A' x = 0$ be the associated row echelon equation, and let r be the number of pivots of A'. Then

$r \le m$. Hence (by an earlier proposition) we may assign arbitrary values to $n-r$ variables x_i. ■

<u>Proposition</u>: Let M be a square row echelon matrix. Then either M is the identity matrix, or its bottom row is zero.

<u>Proof</u>. This is an exercise. ■

(We can use row reduction to characterize square invertible matrices.)

Proposition : Let A be a square matrix. The following conditions are equivalent:

(a) A can be reduced to the identity by a sequence of elementary row operations.

(b) A is a product of elementary matrices

(c) A is invertible

(d) The linear system $Ax = 0$ has only the trivial solution $(x=0)$.

Proof : This is an exercise. ∎

Hence we get immediately:

Proposition : If a row of a square matrix A is zero, then A is not invertible.

Row reduction provides a method of computing the inverse of an invertible matrix A:

We reduce A to the identity by row operations:

$$E_k \cdots E_1 A = I$$

for some elementary matrices E_i.

Multiplying both sides by A^{-1} gives:

$$E_k \cdots E_1 I = A^{-1}$$

<u>Proposition</u>: Let A be an invertible matrix. To compute its inverse A^{-1}, apply row operations $E_1, \dots, E_k$ to A, reducing it to the identity matrix. The same sequence of operations, when applied to I, yields A^{-1}.

Procedure for matrix inversion

Let $A \in M(n\times n, \mathbb{F})$ be invertible.

Use row reduction

$$[A \mid I] \overset{\text{row operations}}{\rightsquigarrow \rightsquigarrow \rightsquigarrow \cdots \rightsquigarrow} [I, A^{-1}]$$

Example: Find the inverse of the matrix $A = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$.

To compute it we form the block matrix

$$[A \mid I] = \left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right]$$

We perform row operations to reduce A to the identity, carrying the right side along, and hence ending up with A^{-1} on the right (due to our proposition above).

$[A \mid I]$ subtract (row 1) from (row 2)

$$\rightsquigarrow \left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right]$$

subtract 4·(row 2) from (row 1)

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & -4 \\ 1 & 1 & -1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & -4 \\ 0 & 1 & -6 & 5 \end{array}\right]$$