Homework 6

**Exercise 1.** Show that if $A, B \in M(n \times n, \mathbb{F})$ then
\[
\text{rk } A + \text{rk } B - n \leq \text{rk } AB \leq \text{min}(\text{rk } A, \text{rk } B).
\]
(Hint: use the dimension formula for linear maps).

**Exercise 2.** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ and $f : V \to V$ an endomorphism. Show that if with respect to all bases $f$ is represented by the same matrix $A$; i.e., $A = \Phi^{-1} f \Phi$ for all isomorphisms $\Phi : \mathbb{F}^n \to V$, then there exists some $c \in \mathbb{F}$ with $f = c (\text{Id})$.

**Exercise 3.** Let $A \in M(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ (i.e., if $Ax_0 = b$), then
\[
\text{Sol}(A, b) = (x_0 + \text{Ker } A) := \{ x_0 + x | x \in \text{Ker } A \}.
\]

**Exercise 4.** Let $A \in M(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ and $(v_1, \ldots, v_r)$ is a basis of $\text{Ker } A$, then
\[
\text{Sol}(A, b) = \{ x_0 + c_1 v_1 + \cdots + c_r v_r | c_i \in \mathbb{F} \};
\]
hence, $r = \dim \text{Ker } A = n - \text{rk } A$.

**Exercise 5.** Let $A \in M(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: Assume that $Ax = b$ is solvable. Then $Ax = b$ is uniquely solvable if and only if $\text{Ker } A = 0$ (i.e., if and only if $\text{rk } A = n$).

**Exercise 6.** Find all solutions of the system of equations $Ax = b$ when
\[
A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & -2 \end{bmatrix}
\]
and $b$ has the following value:

(a) \[
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
(b) \[
\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]
(c) \[
\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}
\]

**Exercise 7.** Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.

**Exercise 8.** Use row reduction to find inverses of the following matrices:

(a) \[
\begin{bmatrix} 1 & 2 \\ \end{bmatrix}
\]
(b) \[
\begin{bmatrix} 1 & 1 \\ \end{bmatrix}
\]
(c) \[
\begin{bmatrix} 1 & 1 \\ \end{bmatrix}
\]
(d) \[
\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}
\]

**Exercise 9.** How much can a matrix be simplified if both row and column operations are allowed?

**Exercise 10.** Prove that every invertible $2 \times 2$ matrix is a product of at most four elementary matrices.
Exercise 11. Prove that if a product \( AB \) of \( n \times n \) matrices is invertible then so are its factors \( A, B \).

Exercise 12. Let \( A \) be a square matrix. Prove that there is a set of elementary matrices \( E_1, \ldots, E_k \) such that \( E_k \cdots E_1 A \) either is the identity or has its bottom row zero.

Exercise 13. Prove the following proposition from lecture. (Hint: it suffices to prove the implications \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)\)).

Proposition 1. Let \( A \) be a square matrix. The following conditions are equivalent:

(a) \( A \) can be reduced to the identity by a sequence of elementary row operations.
(b) \( A \) is a product of elementary matrices.
(c) \( A \) is invertible.
(d) The linear system \( Ax = 0 \) has only the trivial solution \( x = 0 \).

Exercise 14. Let \( (v_1, v_2, v_3, v_4) \) be linearly independent elements of the real vector space \( V \). If

\[
\begin{align*}
w_1 &= v_2 - v_3 + 2v_4 \\
w_2 &= v_1 + 2v_2 - v_3 - v_4 \\
w_3 &= -v_1 + v_2 + v_3 + v_4,
\end{align*}
\]

show that \( (w_1, w_2, w_3) \) is linearly independent. (Hint: first show that the linear independence of \( (w_1, w_2, w_3) \) is equivalent to a certain matrix having rank 3, and then use the procedure for determining rank to find the rank of this matrix).

Exercise 15. For which values of \( c \), is the real matrix

\[
A_c := \begin{bmatrix}
1 & c & 0 & 0 \\
c & 1 & 0 & 0 \\
0 & c & 1 & 0 \\
0 & 0 & c & 1
\end{bmatrix}
\]

invertible? For these values of \( c \) determine the inverse matrix \( A_c^{-1} \).

Exercise 16. Prove the following: If \( U \subset \mathbb{F}^n \) is a subspace and \( x \in \mathbb{F}^n \), then there exists a system of equations with coefficients in \( \mathbb{F} \), having \( n \) equations and \( n \) unknowns, whose solution set equals \( x + U \).