MA 353
Linear Algebra II
Handout 7

(This is some material from the very end of Lecture 23 that was distributed as a Handout)

Proposition: Let V, V' be Euclidean vector spaces, and let $(v_1, \dots, v_n)$ be an orthonormal basis of V. Then a linear map

$$f : V \longrightarrow V'$$

is orthogonal if and only if

$$(f(v_1), \dots, f(v_n))$$

is an orthonormal system in V'.

Proof: If f is orthogonal, then

$$\langle f(v_i), f(v_j) \rangle = \langle v_i, v_j \rangle = \delta_{ij} .$$

Conversely, if we assume that

$\langle f(v_i), f(v_j) \rangle = \delta_{ij}$, then it follows for

$$\begin{cases} v := \sum c_i v_i \\ w := \sum d_j v_j \end{cases} \quad \text{in } V$$

that

$$\begin{aligned} \langle f(v), f(w) \rangle &= \langle f(\textstyle\sum c_i v_i), f(\textstyle\sum d_j v_j) \rangle \\ &= \langle \textstyle\sum c_i f(v_i), \sum d_j f(v_j) \rangle \\ &= \sum_i \sum_j c_i d_j \delta_{ij} \\ &= \sum_i \sum_j c_i d_j \langle v_i, v_j \rangle \\ &= \langle \textstyle\sum c_i v_i, \sum d_j v_j \rangle \\ &= \langle v, w \rangle . \quad \blacksquare \end{aligned}$$