Math 35300: Section 162. Linear algebra II **Spring 2013**

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Homework 1

Recall from lecture that if $f\colon X\longrightarrow Y$ is a map, then the set

$$\Gamma_f \ := \ \{(x, f(x)) \mid x \in X\}$$

is called the *graph* of f. The graph is a subset of the Cartesian product $X \times Y$.

Exercise 1. Let X, Y be sets. As in lecture, we can draw a picture (or cartoon) of the Cartesian product $X \times Y$ by a rectangle (here, we are thinking of X and Y as intervals, although in general this will not be true). In this way, give examples of graphs of maps f with the following properties (please draw a picture for each):

(a) f is surjective, but not injective
(b) f is injective, but not surjective
(c) f is bijective
(d) f is constant
(e) f is neither injective nor surjective
(f) $X = Y$ and $f = \mathrm{Id}_X$
(g) $f(X)$ consists of only two elements

(Careful: not all subsets of the Cartesian product $X \times Y$ are graphs of a map $f\colon X\longrightarrow Y$; i.e., many subsets of $X \times Y$ are nongraphs.)

The inverse map f^{-1} of a bijective map $f\colon X\longrightarrow Y$ clearly has the properties

$$f \circ f^{-1} = \mathrm{Id}_Y, \qquad f^{-1} \circ f = \mathrm{Id}_X,$$

since in the first case each element $f(x) \in Y$ is mapped by $f(x) \mapsto x \mapsto f(x)$ onto $f(x)$, and in the second case each $x \in X$ is mapped by $x \mapsto f(x) \mapsto x$ onto x. Conversely, one has the following property.

Proposition 1. *Let $f\colon X\longrightarrow Y$ and $g\colon Y\longrightarrow X$ be maps such that*

$$f \circ g = \mathrm{Id}_Y, \qquad g \circ f = \mathrm{Id}_X.$$

Then f is bijective and $f^{-1} = g$.

Exercise 2. Prove Proposition 1. (To get started: An injectivity proof runs like this: "Let $x, x' \in X$ and $f(x) = f(x')$, then $\dots$. Therefore $x = x'$, and f is proved to be injective." On the other hand, the pattern for a surjectivity proof is: "Let $y \in Y$. Choose $x = \dots$. Then we have $\dots$, therefore $f(x) = y$, and f is proved to be surjective.")

Exercise 3. Consider any commutative diagram of sets of the form

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "\alpha", "\cong"'] \arrow[r, "g"'] & B \arrow[u, "\beta"', "\cong"] \end{tikzcd}$$

with α, β bijective.

(a) Show that g is injective if and only if f is injective.

(b) Show that g is surjective if and only if f is surjective.

(We will frequently meet this kind of diagram in the course. The situation is then mostly: f is the object of our interest, α and β are subsidiary constructions, means to an end, and we already know something about g. This information about g then tells us something about f. In solving this exercise, you will see the mechanism of this information transfer.)