Recall from lecture that if $f: X \rightarrow Y$ is a map, then the set
\[ \Gamma_f := \{(x, f(x)) \mid x \in X\} \]
is called the graph of $f$. The graph is a subset of the Cartesian product $X \times Y$.

**Exercise 1.** Let $X, Y$ be sets. As in lecture, we can draw a picture (or cartoon) of the Cartesian product $X \times Y$ by a rectangle (here, we are thinking of $X$ and $Y$ as intervals, although in general this will not be true). In this way, give examples of graphs of maps $f$ with the following properties (please draw a picture for each):

- (a) $f$ is surjective, but not injective
- (b) $f$ is injective, but not surjective
- (c) $f$ is bijective
- (d) $f$ is constant
- (e) $f$ is neither injective nor surjective
- (f) $X = Y$ and $f = \text{Id}_X$
- (g) $f(X)$ consists of only two elements

(Careful: not all subsets of the Cartesian product $X \times Y$ are graphs of a map $f: X \rightarrow Y$; i.e., many subsets of $X \times Y$ are nongraphs.)

The inverse map $f^{-1}$ of a bijective map $f: X \rightarrow Y$ clearly has the properties
\[ f \circ f^{-1} = \text{Id}_Y, \quad f^{-1} \circ f = \text{Id}_X, \]
since in the first case each element $f(x) \in Y$ is mapped by $f(x) \mapsto x \mapsto f(x)$ onto $f(x)$, and in the second case each $x \in X$ is mapped by $x \mapsto f(x) \mapsto x$ onto $x$. Conversely, one has the following property.

**Proposition 1.** Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be maps such that
\[ f \circ g = \text{Id}_Y, \quad g \circ f = \text{Id}_X. \]
Then $f$ is bijective and $f^{-1} = g$.

**Exercise 2.** Prove Proposition 1. (To get started: An injectivity proof runs like this: “Let $x, x' \in X$ and $f(x) = f(x')$, then . . . . Therefore $x = x'$, and $f$ is proved to be injective.” On the other hand, the pattern for a surjectivity proof is: “Let $y \in Y$. Choose $x = . . .$. Then we have . . . , therefore $f(x) = y$, and $f$ is proved to be surjective.”)

**Exercise 3.** Consider any commutative diagram of sets of the form
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\alpha} & \Leftrightarrow & \downarrow{\beta} \\
A & \xrightarrow{g} & B
\end{array}
\]
with $\alpha, \beta$ bijective.

(a) Show that $g$ is injective if and only if $f$ is injective.
(b) Show that \( g \) is surjective if and only if \( f \) is surjective.

(We will frequently meet this kind of diagram in the course. The situation is then mostly: \( f \) is the object of our interest, \( \alpha \) and \( \beta \) are subsidiary constructions, means to an end, and we already know something about \( g \). This information about \( g \) then tells us something about \( f \). In solving this exercise, you will see the mechanism of this information transfer.)