Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
John E. Harper
Purdue University

Homework 2

The rule $x + (y - x) = y$, for example, does not appear among the axioms of a vector space, but can be easily deduced from them:

$$\begin{aligned} x + (y - x) &= x + (-x + y) && \text{by axiom (2)} \\ &= (x - x) + y && \text{by axiom (1)} \\ &= 0 + y && \text{by axiom (4)} \\ &= y + 0 && \text{by axiom (2)} \\ &= y && \text{by axiom (3)} \end{aligned}$$

So it would have been quite unnecessary to include $x + (y - x) = y$ among the axioms. In setting up an axiom system, one tries to choose the fewest and simplest axioms needed in order to deduce from them all the additional rules that one wants.

But this does not mean that for each page of linear algebra you must write ten pages of "reduction to the axioms". Given a little practice it can be assumed that you would be able to carry out reduction of your calculations to the axioms, and this does not need to be formally written out. The following exercise provides this practice.

Exercise 1. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then for all $x \in V$ and $a \in \mathbb{F}$, prove

(a) $0 + x = x$
(b) $-0 = 0$
(c) $a0 = 0$
(d) $0x = 0$
(e) $ax = 0$ if and only if $a = 0$ or $x = 0$
(f) $-x = (-1)x$

Remark 1. Properties (a)–(f) in the above exercise remain true for vector spaces over any field $\mathbb{F}$.

Exercise 2. Show that $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$.

The following exercise provides some practice with the field axioms.

Exercise 3. Let $\mathbb{F}$ be a field. Prove the following.

(a) The element $0 \in \mathbb{F}$ is uniquely determined.
(b) For each $a \in \mathbb{F}$, the element $-a \in \mathbb{F}$ is uniquely determined.
(c) The element $1 \in \mathbb{F} \setminus 0$ is uniquely determined.
(d) For each $a \in \mathbb{F} \setminus 0$, the element $a^{-1} \in \mathbb{F}$ is uniquely determined.
(e) $(-1)a = -a$ for each $a \in \mathbb{F}$.
(f) For each $a, b \in \mathbb{F}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.
(g) $(-1)(-1) = 1$.

Exercise 4. Let V be a vector space over a field $\mathbb{F}$. Let $x, y \in V$ and $a \in \mathbb{F}$. Prove the cancellation law in a vector space: If $ax = ay$ and $a \neq 0$, then $x = y$.

Exercise 5. Let V be a vector space over a field $\mathbb{F}$. Prove the following: If U is a subspace of V, then U together with the addition and scalar multiplication inherited from $(V, +, \cdot)$ is itself a vector space over $\mathbb{F}$.

Exercise 6. For $a \in \mathbb{F}$ we define $U_a := \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + x_2 + x_3 = a\}$. Show that U_a is a vector subspace of $\mathbb{F}^3$ if and only if $a = 0$; here, $\mathbb{F}$ is any field.

Exercise 7. Let V be a vector space over a field $\mathbb{F}$. Prove the following: If U_1, U_2 are subspaces of V, then $U_1 \cap U_2$ is also a subspace of V.

Exercise 8. Let V be a vector space over a field $\mathbb{F}$ and let U_1, U_2 be subspaces of V. Show that if $U_1 \cup U_2 = V$, then $U_1 = V$ or $U_2 = V$, or both.

Exercise 9. If all the field axioms for $(\mathbb{F}, +, \cdot)$ hold, with the possible exception of axiom (8), then one calls $(\mathbb{F}, +, \cdot)$ a "commutative ring with unit". If in addition $ab = 0$ only occurs if $a = 0$ or $b = 0$, then $\mathbb{F}$ is called a "commutative ring with unit element and no divisors of zero" or an "integral domain" for short. Prove that every finite integral domain is a field.