Exercise 1. Prove Proposition 1 below.

**Proposition 1.** Let $V$ be a vector space over a field $\mathbb{F}$. If $V$ is finite-dimensional and $U \subset V$ is a subspace, then $U$ is also finite-dimensional and $\dim U \leq \dim V$, with equality if and only if $U = V$.

Exercise 2. Prove Proposition 2 below.

**Proposition 2.** Let $V$ be a vector space over a field $\mathbb{F}$. If $W_1, \ldots, W_n$ are subspaces of $V$, then the sum $W_1 + \cdots + W_n$ is the smallest subspace of $V$ containing $W_1, \ldots, W_n$.

Exercise 3. Prove Proposition 3 below.

**Proposition 3.** Let $V$ be a vector space over a field $\mathbb{F}$.

(a) A single subspace $W_1$ is independent.

(b) Two subspaces $W_1, W_2$ are independent if and only if $W_1 \cap W_2 = \{0\}$.

Exercise 4. Prove Proposition 4 below.

**Proposition 4.** Let $V$ be a vector space over a field $\mathbb{F}$. If $W_1, \ldots, W_n$ are subspaces of $V$, then $V = W_1 \oplus \cdots \oplus W_n$ if and only if every vector $v \in V$ can be written in the form

$$v = w_1 + \cdots + w_n,$$

(where $w_i$ is a vector in $W_i$) in exactly one way.

Exercise 5. Prove Proposition 5 below.

**Proposition 5.** Let $W_1, \ldots, W_n$ be subspaces of a finite-dimensional vector space $V$, and let $B_i$ be a basis for $W_i$.

(a) The ordered set $B$ obtained by listing the bases $B_1, \ldots, B_n$ in order is a basis of $V$ if and only if $V = W_1 \oplus \cdots \oplus W_n$.

(b) $\dim(W_1 + \cdots + W_n) \leq \dim(W_1) + \cdots + \dim(W_n)$, with equality if and only if the subspaces $W_1, \ldots, W_n$ are independent.

Exercise 6. Let $V$ be a vector space over a field $\mathbb{F}$. Show that $V = V \oplus \{0\}$.

Exercise 7. Prove Proposition 6 below.

**Proposition 6.** Let $V, W$ be vector spaces over a field $\mathbb{F}$. Let $f : V \rightarrow W$ be a linear map. Then $f$ is injective if and only if $\ker f = 0$.

Exercise 8. Prove Proposition 7 below.

**Proposition 7.** Let $V, W$ be vector spaces over a field $\mathbb{F}$. Let $f : V \rightarrow W$ be an isomorphism. If $(v_1, \ldots, v_r)$ is a linearly independent $r$-tuple of vectors in $V$, then the $r$-tuple of vectors $(f(v_1), \ldots, f(v_r))$ in $W$ is also linearly independent.

Exercise 9. Prove Proposition 8 below.
Proposition 8. Let $V, W$ be vector spaces over a field $\mathbb{F}$. If $(v_1, \ldots, v_n)$ is a basis of $V$, then a linear map $f: V \to W$ is an isomorphism if and only if $(f(v_1), \ldots, f(v_n))$ is a basis of $W$.


Proposition 9. Let $V, W$ be finite-dimensional vector spaces over a field $\mathbb{F}$. If $\dim(V) = \dim(W)$, then a linear map $f: V \to W$ is surjective if and only if it is injective.