Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Homework 5

We can define the notion of a *polynomial* with coefficients in a field $\mathbb{F}$ to mean a linear combination of powers of the variable (or indeterminate):

$$f(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0, \tag{1}$$

where $a_i \in \mathbb{F}$. Such expressions are sometimes called *formal polynomials*, to distinguish them from polynomial functions. Every formal polynomial with coefficients in $\mathbb{F}$ determines a polynomial function on $\mathbb{F}$. The variable appearing in (1) is an arbitrary symbol, and the monomials t^i are considered linearly independent. This means that if

$$g(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0$$

is a polynomial with coefficients in $\mathbb{F}$, then $f(t)$ and $g(t)$ are equal if and only if $a_i = b_i$ for all $i = 0, 1, 2, \dots$. Sometimes it is useful to write a polynomial in the standard form

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots , \tag{2}$$

where the coefficients a_i are all in the field $\mathbb{F}$ and *only finitely many of the coefficients are different from zero*. Formally, the polynomial (2) is determined by its sequence of coefficients a_i:

$$a = (a_0, a_1, a_2, \dots),$$

where $a_i \in \mathbb{F}$ and all but a finite number of a_i are zero. Every such sequence corresponds to a polynomial.

Addition and multiplication of polynomials mimic the familiar operations on polynomial functions. Let $f(t)$ be as in (2), and let

$$g(t) = b_0 + b_1 t + b_2 t^2 + \cdots , \tag{3}$$

be a polynomial with coefficients in the same field $\mathbb{F}$, determined by the sequence $b = (b_0, b_1, b_2, \dots)$. The *sum* of f and g is

$$\begin{aligned} f(t) + g(t) :=& (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots \\ =& \sum_k (a_k + b_k)t^k, \end{aligned}$$

which corresponds to addition of sequences: $a + b = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots)$. The *product* of f and g is computed by multiplying term by term and collecting coefficients of the same degree in t. If we expand the product using the distributive law, but without collecting terms, we obtain

$$f(t)g(t) = \sum_{i,j} a_i b_j t^{i+j}.$$

Note that there are only finitely many nonzero coefficients $a_i b_j$. The right-hand side is not in standard form since the same monomial t^n appears many times—once for each pair (i, j) of indices such that $i + j = n$. Putting the right-hand side back into standard form (by collecting terms) leads to the definition

$$f(t)g(t) := p_0 + p_1 t + p_2 t^2 + \cdots ,$$

where

$$p_k := a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 = \sum_{i+j=k} a_i b_j .$$

Exercise 1. Let $\mathbb{F}$ be a field and $\mathcal{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{F}\}$ be the vector space of polynomials in the indeterminate t of degree $\leq n$ with coefficients in $\mathbb{F}$. If $f(t) \in \mathcal{P}_m$ and $g(t) \in \mathcal{P}_n$, the product $f(t)g(t) \in \mathcal{P}_{m+n}$ is defined as above. We call $(1, t, \dots, t^m)$ the canonical basis of $\mathcal{P}_m$. Determine the matrix of the linear map

$$\mathcal{P}_3 \longrightarrow \mathcal{P}_4, \qquad f(t) \longmapsto (2-t)f(t)$$

relative to the canonical bases.

Exercise 2. By a *finite chain complex* C of vector spaces over a field $\mathbb{F}$ one understands a sequence of homomorphisms

$$0 \xrightarrow{f_{n+1}} V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} 0$$

of vector spaces over $\mathbb{F}$ with the property that $f_i f_{i+1} = 0$ for each i; i.e., such that $\operatorname{Ker} f_i \supset \operatorname{Im} f_{i+1}$. The quotient vector space $H_i(C) := \operatorname{Ker} f_i / \operatorname{Im} f_{i+1}$ is called the i-th *homology group* of the complex. Show that if all the V_i are finite-dimensional, then

$$\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim H_i(C).$$

Exercise 3. Consider the following commutative diagram of homomorphisms of vector spaces over a field $\mathbb{F}$.

$$\begin{tikzcd}
V_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4", "\text{epi.}"'] & V_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3", "\cong"'] & V_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & V_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1", "\cong"'] & V_0 \arrow[d, "\varphi_0", "\text{mono.}"'] \\
W_4 \arrow[r, "g_4"'] & W_3 \arrow[r, "g_3"'] & W_2 \arrow[r, "g_2"'] & W_1 \arrow[r, "g_1"'] & W_0
\end{tikzcd}$$

Assume that the rows are *exact*, i.e., $\operatorname{Ker} f_i = \operatorname{Im} f_{i+1}$ and $\operatorname{Ker} g_i = \operatorname{Im} g_{i+1}$ for $i = 1, 2, 3$, and suppose furthermore that the vertical homomorphisms have the indicated properties; i.e., φ_4 is an epimorphism, φ_3 and φ_1 are isomorphisms, and φ_0 is a monomorphism. Show that under these conditions φ_2 is an isomorphism.

Basic Assumption. From now on in this section, assume that V, W are vector spaces over a field $\mathbb{F}$, unless otherwise specified.

Recall from lecture that if $U \subset V$ is a subspace, then a *coset* of U is a subset of the form $x + U := \{x + u \mid u \in U\}$. The following exercise motivates this definition.

Exercise 4. Let $\varphi \colon V \longrightarrow W$ be a linear map. Let $U := \operatorname{Ker} \varphi$ and let $x, y \in V$.

(a) Prove that $\varphi(x) = \varphi(y)$ if and only if $y = x + u$ for some element $u \in U$, or equivalently, if and only if $y - x \in U$.
(b) Conclude that the cosets of U partition V.

Recall from lecture the following definition.

Definition 1. Let $U \subset V$ be a subspace. The *quotient space* V/U of V modulo U is the set

$$V/U := \{x + U \mid x \in V\}$$

of all cosets of U, with addition and scalar multiplication defined by

$$\begin{aligned}(x+U)+(y+U) &:= (x+y)+U \qquad \text{“addition”}\\ a(x+U) &:= ax + U \qquad \text{“scalar multiplication”}\end{aligned}$$

for every $x, y \in V$, $a \in \mathbb{F}$. The *projection* map is defined by

$$V \longrightarrow V/U =: \overline{V}, \qquad x \longmapsto x + U := \overline{x}.$$

Exercise 5. Let $U \subset V$ be a subspace.

(a) Prove that the operations “addition” and “scalar multiplication” in Definition 1 determine well-defined maps

$$\begin{aligned} V/U \times V/U &\xrightarrow{+} V/U \qquad \text{“addition”}\\ \mathbb{F} \times V/U &\xrightarrow{\cdot} V/U \qquad \text{“scalar multiplication”}\end{aligned}$$

(b) Prove that $(V/U, +, \cdot)$ is a vector space over $\mathbb{F}$.

Exercise 6. Let $U \subset V$ be a subspace. Prove that if V is finite-dimensional, then

$$\dim V/U = \dim V - \dim U.$$

Exercise 7. Consider the subspaces $V \subset V$ and $0 \subset V$. Prove that $V/V = 0$ and $V/0 \cong V$.

Exercise 8. Prove Proposition 2.

Proposition 2. *Let $f\colon V \longrightarrow V'$ be an epimorphism, and let $U := \operatorname{Ker} f$.*

(a) *Then the induced map*

$$\begin{aligned} V/U &\xrightarrow{\overline{f}} V',\\ \overline{x} = x + U &\longmapsto \overline{f}(\overline{x}) = f(x)\end{aligned}$$

is an isomorphism.

(b) *The set of subspaces $A' \subset V'$ is in bijective correspondence with the set of subspaces $A \subset V$ which contain U, the correspondence being defined by the maps $A \longmapsto f(A)$ and $A' \longmapsto f^{-1}(A')$.*

Exercise 9. Prove the following.

(a) If $A, B \subset V$ are subspaces, then there is an isomorphism of the form

$$A/(A \cap B) \cong (A+B)/B.$$

(b) If $A \subset A' \subset B' \subset B \subset V$ are subspaces, then there is an isomorphism of the form

$$B'/A' \cong (B'/A)/(A'/A).$$

Hint: For part (a), consider the inclusion map $A \longrightarrow A + B$. For part (b), note that $A'/A \subset B'/A$ is a subspace and consider the projection map $B' \longrightarrow B'/A$.

Exercise 10. Prove Proposition 3.

Proposition 3. *Let $f\colon V \longrightarrow W$ be a linear map.*

(a) *If g is a linear map which makes the solid diagram*

$$\begin{tikzcd} & & V' \arrow[dll, dotted, "\overline{g}"', "\exists!"] \arrow[d, "g"] \arrow[dr, bend left, "0"] & \\ \operatorname{Ker} f \arrow[rr, "\subset"] & & V \arrow[r, "f"] & W \end{tikzcd}$$

commute, then there exists a unique linear map $\overline{g}$ which makes the diagram commute.

(b) *If h is a linear map which makes the solid diagram*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, bend right, "0"'] & W \arrow[r, "\pi"] \arrow[d, "h"] & W/f(V) \arrow[r, equal] \arrow[dl, dotted, "\exists!"', "\overline{h}"] & \operatorname{Coker} f \\ & W' & & \end{tikzcd}$$

commute, then there exists a unique linear map $\overline{h}$ which makes the diagram commute.