Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**

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Homework 6

Exercise 1. Show that if $A, B \in \mathsf{M}(n \times n, \mathbb{F})$ then

$$\operatorname{rk} A + \operatorname{rk} B - n \leq \operatorname{rk} AB \leq \min(\operatorname{rk} A, \operatorname{rk} B).$$

(Hint: use the dimension formula for linear maps).

Exercise 2. Let V be a finite-dimensional vector space over $\mathbb{F}$ and $f\colon V \longrightarrow V$ an endomorphism. Show that if with respect to all bases f is represented by the same matrix A; i.e., $A = \Phi^{-1} f \Phi$ for all isomorphisms $\Phi\colon \mathbb{F}^n \longrightarrow V$, then there exists some $c \in \mathbb{F}$ with $f = c(\mathrm{Id})$.

Exercise 3. Let $A \in \mathsf{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ (i.e., if $Ax_0 = b$), then

$$\mathrm{Sol}(A, b) = (x_0 + \operatorname{Ker} A) := \{x_0 + x \mid x \in \operatorname{Ker} A\}.$$

Exercise 4. Let $A \in \mathsf{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: If $x_0 \in \mathbb{F}^n$ is a solution of $Ax = b$ and $(v_1, \ldots, v_r)$ is a basis of $\operatorname{Ker} A$, then

$$\mathrm{Sol}(A, b) = \{x_0 + c_1 v_1 + \cdots + c_r v_r \mid c_i \in \mathbb{F}\};$$

here, $r = \dim \operatorname{Ker} A = n - \operatorname{rk} A$.

Exercise 5. Let $A \in \mathsf{M}(m \times n, \mathbb{F})$ and $b \in \mathbb{F}^m$. Prove the following: Assume that $Ax = b$ is solvable. Then $Ax = b$ is uniquely solvable if and only if $\operatorname{Ker} A = 0$ (i.e., if and only if $\operatorname{rk} A = n$).

Exercise 6. Find all solutions of the system of equations $Ax = b$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & -2 \end{bmatrix}$$

and b has the following value:

$$\text{(a)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Exercise 7. Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.

Exercise 8. Use row reduction to find inverses of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Exercise 9. How much can a matrix be simplified if both row and column operations are allowed?

Exercise 10. Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

Exercise 11. Prove that if a product AB of $n \times n$ matrices is invertible then so are its factors A, B.

Exercise 12. Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A$ either is the identity or has its bottom row zero.

Exercise 13. Prove the following proposition from lecture. (Hint: it suffices to prove the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$).

Proposition 1. *Let A be a square matrix. The following conditions are equivalent:*

(a) *A can be reduced to the identity by a sequence of elementary row operations.*
(b) *A is a product of elementary matrices.*
(c) *A is invertible.*
(d) *The linear system $Ax = 0$ has only the trivial solution $x = 0$.*

Exercise 14. Let (v_1, v_2, v_3, v_4) be linearly independent elements of the real vector space V. If

$$\begin{aligned} w_1 &= \phantom{-v_1 + {}} v_2 - v_3 + 2v_4 \\ w_2 &= v_1 + 2v_2 - v_3 - v_4 \\ w_3 &= -v_1 + v_2 + v_3 + v_4, \end{aligned}$$

show that (w_1, w_2, w_3) is linearly independent. (Hint: first show that the linear independence of (w_1, w_2, w_3) is equivalent to a certain matrix having rank 3, and then use the procedure for determining rank to find the rank of this matrix).

Exercise 15. For which values of c, is the real matrix

$$A_c := \begin{bmatrix} 1 & c & 0 & 0 \\ c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

invertible? For these values of c determine the inverse matrix A_c^{-1}.

Exercise 16. Prove the following: If $U \subset \mathbb{F}^n$ is a subspace and $x \in \mathbb{F}^n$, then there exists a system of equations with coefficients in $\mathbb{F}$, having n equations and n unknowns, whose solution set equals $x + U$.