Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Homework 8

Exercise 1. Prove Pythagoras' theorem: if the three points a, b, c in a Euclidean vector space form a right-angled triangle, that is if $(a - c) \perp (b - c)$, then

$$\|a - c\|^2 + \|b - c\|^2 = \|a - b\|^2$$

Exercise 2. Give $\mathbb{R}^3$ the inner product $\langle x, y \rangle := \sum_{i,j=1}^{3} a_{ij} x_i y_j$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

(That this is actually an inner product is not part of the exercise, and may be assumed). Calculate the cosines of the angles between the canonical unit vectors in $\mathbb{R}^3$.

Exercise 3. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, define $|x| := \max_i |x_i|$. Show that there exists no inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, for which $\langle x, x \rangle = |x|^2$ for all $x \in \mathbb{R}^n$.

Exercise 4. Let V be the real vector space of all bounded real sequences (i.e., of all bounded functions of the form $x\colon \{1, 2, 3, \cdots\} \longrightarrow \mathbb{R}$)

$$V := \{(x_i)_{i=1,2,\ldots} \mid x_i \in \mathbb{R} \text{ and there exists } c \in \mathbb{R} \text{ with } |x_i| \leq c \text{ for all } i\}.$$

Then

$$\langle x, y \rangle := \sum_{n=1}^{\infty} \frac{x_n y_n}{n^2}$$

obviously defines an inner product on V. Find a *proper* vector subspace $U \subset V$ (i.e., $U \neq V$) with $U^\perp = \{0\}$. (Remark: This in sharp contrast to what happens for finite-dimensional Euclidean vector spaces V. Recall from lecture that in this case, $V = U \oplus U^\perp$ for any subspace $U \subset V$.)

Exercise 5. Find the orthonormal basis of $\mathbb{R}^3$ (with the usual inner product) obtained by applying the Gram-Schmidt procedure to the basis $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

Exercise 6. Let $\mathcal{P}_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_i \in \mathbb{R}\}$ be the Euclidean space of polynomials in the indeterminate t of degree ≤ 3 with coefficients in $\mathbb{R}$ (see Homework 5) and with inner product $\langle\ ,\ \rangle\colon \mathcal{P}_3 \times \mathcal{P}_3 \longrightarrow \mathbb{R}$ defined by

$$\langle f(t), g(t) \rangle := \int_{-1}^{1} f(t) g(t)\, dt$$

Find the orthonormal basis of $\mathcal{P}_3$ obtained by applying the Gram-Schmidt procedure to the canonical basis $(1, t, t^2, t^3)$. (Remark: This is one way to obtain the "Legendre polynomials", up to a scalar multiple.)

Exercise 7. Prove the following proposition. (Hint: A short proof can be given using the proposition proved in Handout 7.)

Proposition 1. *A matrix* $A \in \mathsf{M}(n \times n, \mathbb{R})$ *is orthogonal if and only if its columns* $A_1, \ldots, A_n$ *(images of the unit vectors) form an orthonormal system with respect to the usual inner product in* $\mathbb{R}^n$*; i.e., if and only if*

$$A^t A = I$$

Here, we regard the matrix $A = [A_1 \ldots A_n]$ *as consisting of its column vectors.*

Exercise 8. Prove the following proposition.

Proposition 2. *For* $A \in \mathsf{M}(n \times n, \mathbb{R})$*, the following conditions are equivalent:*

(i) *A is orthogonal*
(ii) *The columns of A form an orthonormal system*
(iii) $A^t A = I$
(iv) *A is invertible and* $A^{-1} = A^t$
(v) $AA^t = I$
(vi) *The rows of A form an orthonormal system*

Exercise 9. Prove the following proposition.

Proposition 3. *If* $A \in \mathsf{M}(n \times n, \mathbb{R})$ *is orthogonal, then* $\det A = \pm 1$.

Definition 4. Let V, V' be Euclidean vector spaces. A linear map $f\colon V \longrightarrow V'$ is an *isometry* if (i) f is orthogonal and (ii) there exists an orthogonal map $g\colon V' \longrightarrow V$ such that

$$f \circ g = \mathrm{Id}, \qquad g \circ f = \mathrm{Id}.$$

We say that V, V' are *isometric* if there exists an isometry $f\colon V \longrightarrow V'$.

Exercise 10. Prove the following proposition.

Proposition 5. *Let* V, V' *be Euclidean vector spaces. Let* $f\colon V \longrightarrow V'$ *be an isomorphism. Then* f *is orthogonal if and only if* f^{-1} *is orthogonal. In particular, a linear map* $V \longrightarrow V'$ *is an isometry if and only if it is an orthogonal isomorphism.*

Exercise 11. Prove the following proposition.

Proposition 6. *Let* V *be a Euclidean vector space and let* $(v_1, \cdots, v_n)$ *be an orthonormal basis. If* $v, w \in V$ *are expressed in the form*

$$v = c_1 v_1 + \cdots + c_n v_n \qquad (c_i \in \mathbb{F}),$$
$$w = d_1 v_1 + \cdots + d_n v_n \qquad (d_i \in \mathbb{F}),$$

then $\langle v, w \rangle = c_1 d_1 + \cdots + c_n d_n$*. In other words, the inner product of* v, w *in* V *is the same as the usual inner product of their coordinate vectors* $(c_1, \cdots, c_n), (d_1, \ldots, d_n)$ *in* $\mathbb{R}^n$.

Exercise 12. Prove the following proposition.

Proposition 7. *Let* V, W *be finite-dimensional Euclidean vector spaces. If* $\dim(V) \leq \dim(W)$*, then there exists an orthogonal map* $f\colon V \longrightarrow W$.

Exercise 13. Prove the following theorem.

Theorem 8. *Let* V, W *be finite-dimensional Euclidean vector spaces. Then* V, W *are isometric if and only if* $\dim(V) = \dim(W)$.

Exercise 14. Prove the following proposition.

Proposition 9. *Let V be finite-dimensional Euclidean vector space of dimension n. Then V and $\mathbb{R}^n$ (with the usual inner product) are isometric.*

Remark 10. The upshot is that "up to orthogonal isomorphism", there is only one n-dimensional Euclidean vector space (e.g., $\mathbb{R}^n$ with the usual inner product).

Remark 11. Careful: It would be unwise to study $\mathbb{R}^n$ alone, since all sorts of other concrete Euclidean vector spaces will tumble across our path.

Exercise 15. Prove the following proposition.

Proposition 12. *Let V be a finite-dimensional Euclidean vector space of dimension n. If $(v_1, \ldots, v_k)$ is an orthonormal system in V, then it can be extended to an orthonormal basis $(v_1, \ldots, v_k, w_1, \ldots, w_{n-k})$ of V.*

Exercise 16. Prove the following theorem.

Theorem 13 (Riesz Representation Theorem)**.** *Let V be a finite-dimensional Euclidean vector space. If $f\colon V \longrightarrow \mathbb{R}$ is a linear map, then there exists a unique vector $a \in V$ such that*

$$f(v) = \langle v, a \rangle$$

for all $v \in V$.

Remark 14. It is useful to note for applications (after proving the above theorem), that the unique vector a associated to $f \neq 0$ is characterized by the following properties:

(i) $a \neq 0$
(ii) $a \perp \operatorname{Ker} f$
(iii) $f(a) = \langle a, a \rangle$

The trivial case $f = 0$ is clearly satisfied by $a = 0$.

Exercise 17. Prove the following proposition. (Hint: Study your proof of the above theorem.)

Proposition 15. *Let V be a finite-dimensional Euclidean vector space, $f\colon V \longrightarrow \mathbb{R}$ a linear map, $f \neq 0$, and b a nonzero vector such that $b \perp \operatorname{Ker} f$. If we define*

$$a := \frac{f(b)}{\langle b, b \rangle} b,$$

then $f(v) = \langle v, a \rangle$ for all $v \in V$.