MATH 151.01: Calculus and Analytic Geometry I

Domains and Polynomial Inequalities

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1 Polynomial Inequalities

We're often confronted with the problem of solving an inequality—e.g., *when is this function* > 0*?* These come up often when we're dealing with domains. When we're dealing with polynomials, we can reduce the process to a series of steps to carry out and arrive at an answer.

1.1 The Technique

Suppose we have a polynomial $p(x)$ and we'd like to solve the inequality $p(x) \geq 0$; that is, we'd like to know all possible inputs x such that the output $p(x) \geq 0$. The following steps produce the solution.

1. Factor the polynomial completely as $p(x) = (x - a_1) \cdots (x - a_r)$. It is possible that some of the factors will be irreducible quadratic factors—i.e., quadratic factors which have no root, like $x^2 + 1$ or $3x^2 + 2x + 1$.

2. Draw a number line, and mark the zeros. If we have $p(x) = (x - a_1) \cdots (x - a_r)$, then the zeros are $a_1, \ldots, a_r$. Plot each of these on the number line and mark a "0" above it.

3. Pick the leftmost subinterval, and decide whether each factor is positive or negative. If it helps, you can mark the factor with "$+$" or "$-$" appropriately.

4. Determine the overall sign of $p(x)$ in the subinterval. This will be the product of all the signs, using your usual sign rules. For example, $(+)(+)(-) = (-)$, and $(-)(-)(+) = (+)$. Mark the sign on the number line.

5. Repeat the previous two steps for each subinterval.

6. Pick the intervals with the appropriate sign to form your solution. Also include zeros as necessary.

As you're working with polynomials, you should ask yourself why this process arrives at the correct answer. I'd be happy to explain it a bit more during my office hours.

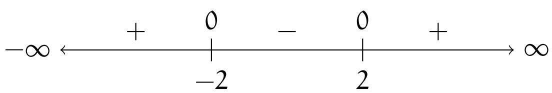
1.2 Examples

The following examples illustrate the technique. Each example has a few explanatory notes, but otherwise everything shown belongs as part of a complete solution.

Example 1. Solve the inequality $x^2 - 4 \geq 0$.

$$x^2 - 4 \geq 0$$

$$(x-2)(x+2) \geq 0$$



$$(-\infty, -2] \cup [2, \infty)$$

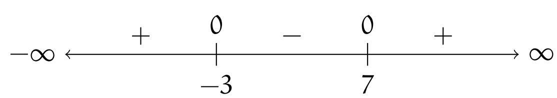
Notice that I didn't simply move 4 to the other side and take a square root—you can't take a square root over an inequality. I also didn't show the signs of each factor, which is step 3 above. You can write that in pencil and erase, or do it in your head. It doesn't need to be part of your work. I'm solving for when $x^2 - 4 \geq 0$, so I choose the intervals where the polynomial is positive. Also, $x^2 - 4 = 0$ is allowed, so I included the endpoints of the interval.

Example 2. Solve the inequality $x^2 - 5x - 21 < 0$.

This time I was solving for when the polynomial is *less* than zero, so I took the negative part. I also had a strict inequality, which means the $x^2 - 4x - 21 = 0$ is not allowed, so I didn't include the endpoints.

$$x^2 - 4x - 21 < 0$$

$$(x-7)(x+3) < 0$$

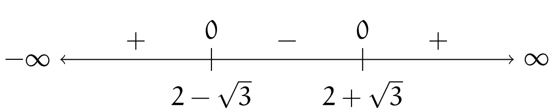


$$(-3, 7)$$

Example 3. Solve the inequality $x^2 - 4x + 1 > 0$.

$$x^2 - 4x + 1 > 0$$

$$(x - (2 - \sqrt{3}))(x - (2 + \sqrt{3})) > 0$$



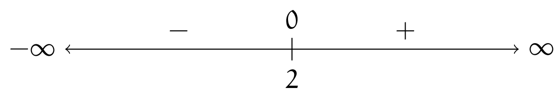
$$(-\infty, 2 - \sqrt{3})(2 + \sqrt{3})$$

I used the quadratic formula to factor the polynomial, but otherwise nothing is new.

Example 4. Solve the inequality $x^3 - 8 > 0$.

$$x^3 - 8 > 0$$

$$(x-2)(x^2+2x+4) > 0$$



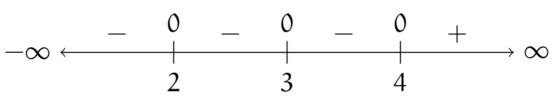
$$(2, \infty)$$

The factor $x^2 + 2x + 4$ is an irreducible quadratic, as you can verify using the discriminant. Thus, it always has the same sign, either positive or negative. In this case, it's always positive.

Example 5. Solve the inequality $(x^2 - 6x + 9)(x^2 - 4x + 4) > 0$.

$$(x^2 - 6x + 9)(x^2 - 4x + 4)(x-4) < 0$$

$$(x-3)^2(x-2)^2(x-4) > 0$$



$$(-\infty, 2) \cup (2,3) \cup (3,4)$$

Here's an example with three zeroes, but that doesn't change much. Notice that the sign didn't change when we moved past 2 and past 3. That's because of the multiplicity of the factors. Even though the factor $(x-3)$ changes from $+$ to $-$ when we move past 3, the factor is squared. Because $(-)^2 = (+)^2 = +$, the overall sign doesn't change.

You should make up some examples of your own and try to solve them. I'll happily tell you whether you've done the problem correctly or not.

2 Domains

The *domain* of a function is the collection of all allowable inputs. The domains of basic functions, such as polynomials, root functions, trigonometric functions, exponential functions, and logarithmic functions must be memorized. The domains of more complicated functions built out of these basic pieces can be computed. Rather than a series of steps, I'll just provide the idea.

When the basic functions appear in the formula for your function, they may give a constraint; this is a condition which must be satisfied, usually an inequality, and it comes from the domain of the basic function involved. For example, if $\sqrt{x-3}$ occurs anywhere in your formula, then you know we must have $x - 3 \geq 0$. All of these constraints must be satisfied in order for the function to be defined. Thus, we find the simultaneous solution of the constraints, and that is our answer.

The most common constraints that occur are the following:

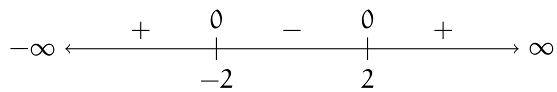
$$\sqrt{f(x)} \Rightarrow f(x) \geq 0 \qquad \frac{g(x)}{f(x)} \Rightarrow f(x) \neq 0 \qquad \log(f(x)) \Rightarrow f(x) > 0$$

Example 6. Find the domain of $f(x) = \sqrt{x^2 - 4}$.

In this case, the only constraint is $x^2 - 4 \geq 0$, because it occurs under a square root. The solution to this inequality is the domain of f. Notice that the solution of this inequality involves material from the previous section.

$$x^2 - 4 \geq 0$$

$$(x-2)(x+2) \geq 0$$



$$\text{dom}(f) = (-\infty, -2] \cup [2, \infty)$$

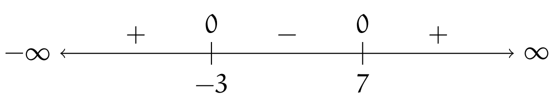
Example 7. Find the domain of

$$f(x) = \frac{\sqrt{x^2 - 4x + 21}}{\ln(x+3)}.$$

$$x^2 - 4x + 21 \geq 0 \qquad \text{and} \qquad x + 3 > 0 \qquad \text{and} \qquad \ln(x+3) \neq 0$$

$$x^2 - 4x - 21 \geq 0$$

$$(x-7)(x+3) <\geq$$



$$(-\infty, -3] \cup [7, \infty)$$

$$x + 3 > 0$$

$$x > -3$$

$$\ln(x+3) \neq 0$$

$$x + 3 \neq 1$$

$$x \neq -4$$

$$x \leq -3 \quad \text{or} \quad 7 \leq x \qquad \text{and} \qquad x > -3 \qquad \text{and} \qquad x \neq -4$$

$$\text{dom}(f) = [7, \infty)$$

This example demonstrates the most common types of problems you can have. The constraint $x^2 - 4x + 21 \geq 0$ comes from the fact that $x^2 - 4x + 21$ is under a square root. We're taking a logarithm of $x + 3$, so we get $x + 3 > 0$. We're dividing by $\ln(x+3)$, so it can't be zero. We solve each of these inequalities individually, then combine them into the overall solution.

Here's how we can deal with piecewise-defined functions.

Example 8. Find the domain of

$$g(x) = \begin{cases} \sqrt{x^2 - 1} & \text{if } x < 0 \\ \frac{1}{x+1} & \text{if } x \geq 0 \end{cases}$$

$$\left(x < 0 \quad \text{and} \quad x^2 - 1 \geq 0\right) \qquad \text{or} \qquad \left(x \geq 0 \quad \text{and} \quad x + 1 \neq 0\right)$$
$$\left(x < 0 \quad \text{and} \quad (x \leq -1 \quad \text{or} \quad x \geq 1)\right) \qquad \text{or} \qquad \left(x \geq 0 \quad \text{and} \quad x \neq -1\right)$$
$$x \leq -1 \qquad \text{or} \qquad x \geq 0$$
$$\text{dom}(g) = (-\infty, 1] \cup [0, \infty)$$

For each branch, we can add the constraint on the branch to the list of constraints. Notice that $x \geq 0$ and $x \neq 1$ are redundant. This reflects that fact that the one bad point in $\frac{1}{x+1}$ occurs at $x = -1$, but we only use that branch for $x \geq 0$.