Computational Exploration of The Chebyshev Bias

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Abstract

The Chebyshev Bias states that there are "many more" primes congruent to 3 modulo 4 than those congruent to 1. If the Generalized Riemann Hypothesis (GRH) is true, then the natural density of the bias should not exist [2][3]. We aim to find computational evidence for this prediction.

Introduction

Prime numbers (also commonly known as primes), are integers greater than one with no positive divisors than 1 and themselves. The Prime Number Theorem (PNT) estimates the number of primes below a given number x and is denoted by $\pi(x)$. $li(x) = \int_2^x dt/log(t)$ and $x/log(x)$ are two such estimates of $\pi(x)$.

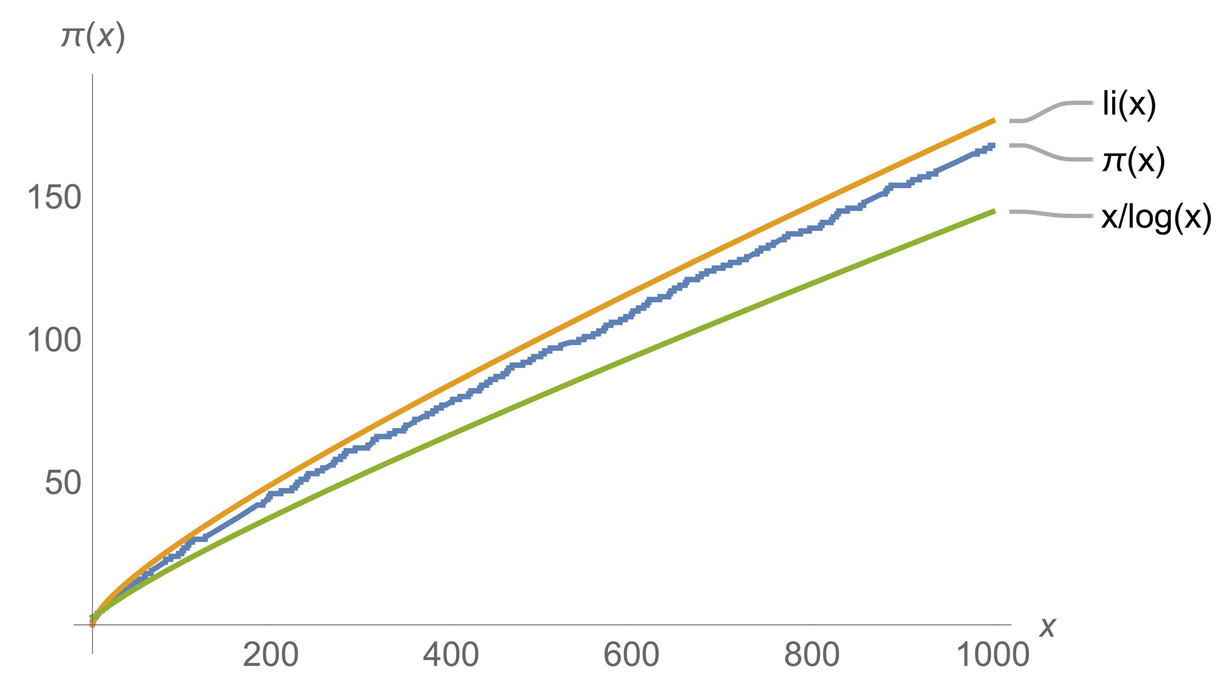

Figure 1: Prime Counting Function and its Estimates

In 1853, Russian mathematician Pafnuty Chebyshev observed an unexpected bias in the seemingly random distribution of primes. For primes less than a given number n, he noticed that there were "many more" primes with remainder 3 when divide by 4, denoted by $\pi(n;4,3)$, than those with remainder 1 when divide by 4, denoted by $\pi(n;4,1)$.

n	$\pi(n)$	$\pi(n;4,3)$	$\pi(n;4,1)$	Bias
15485863	1.0×10^6	500201	499798	403
23879519	1.5×10^6	750237	749762	475
32452843	2.0×10^6	1000240	999759	481
41161739	2.5×10^6	1250096	1249903	193
49979687	3.0×10^6	1500119	1499880	239

Table 1: The Chebyshev Bias for n $\leq 3\times10^6$

Main Objectives

1. Quantify the amount of the Chebyshev bias in a given interval.
2. Extend the range of previous computations by Daniel Shanks.
3. Investigate Shanks' conjecture through computation.

Methodology

In 1959, Shanks calculated the Chebyshev bias for $n \leq 3\times10^6$ by defining $\Delta(n) = \pi(n;4,3) - \pi(n;4,1)$ and conjectured that the mean value of $\tau(n) = \Delta(n)\cdot\sqrt{n}/\pi(n)$, would approach a positive limit [4].

$$\lim_{N\to\infty}\frac{1}{N}\sum_{2}^{N}\tau(n) = 1$$

We investigate this conjecture computationally, guided by Richard Brent's work in the analogous case of the $\pi(x)$ and $li(x)$ race [1]. We define minimum (θ_{min}) and maximum (θ_{max}) biases to examine the amount of the bias between tabulated values. We also shift $\tau(n)$ by subtracting 1, in accordance with Shanks' conjecture and to aid in visualization.

$$\theta_{min}(a,b) = \underset{n\in P\cap(a,b]}{min}\ \tau(n) - 1$$

$$\theta_{max}(a,b) = \underset{n\in P\cap(a,b]}{max}\ \tau(n) - 1$$

Results

No.	a	b	θ_{min}	θ_{max}
1	570079	598582	-0.556	0.221
2	451781	474370	-0.248	0.263
3	4556251	4784063	-0.712	-0.067
4	1714419	1800139	-0.527	0.064
5	42016613	44117443	-0.549	0.137
6	21500440	22575462	-0.099	0.533
7	622283210	653397370	-0.044	0.623
8	172392354	181011971	0.199	0.855
9	2325178799	2441437738	-0.463	0.087
10	7066089014	7419393464	-0.706	0.074

Table 2: Some dyadic intervals (a,b] where b/a $\sim$ 1.05

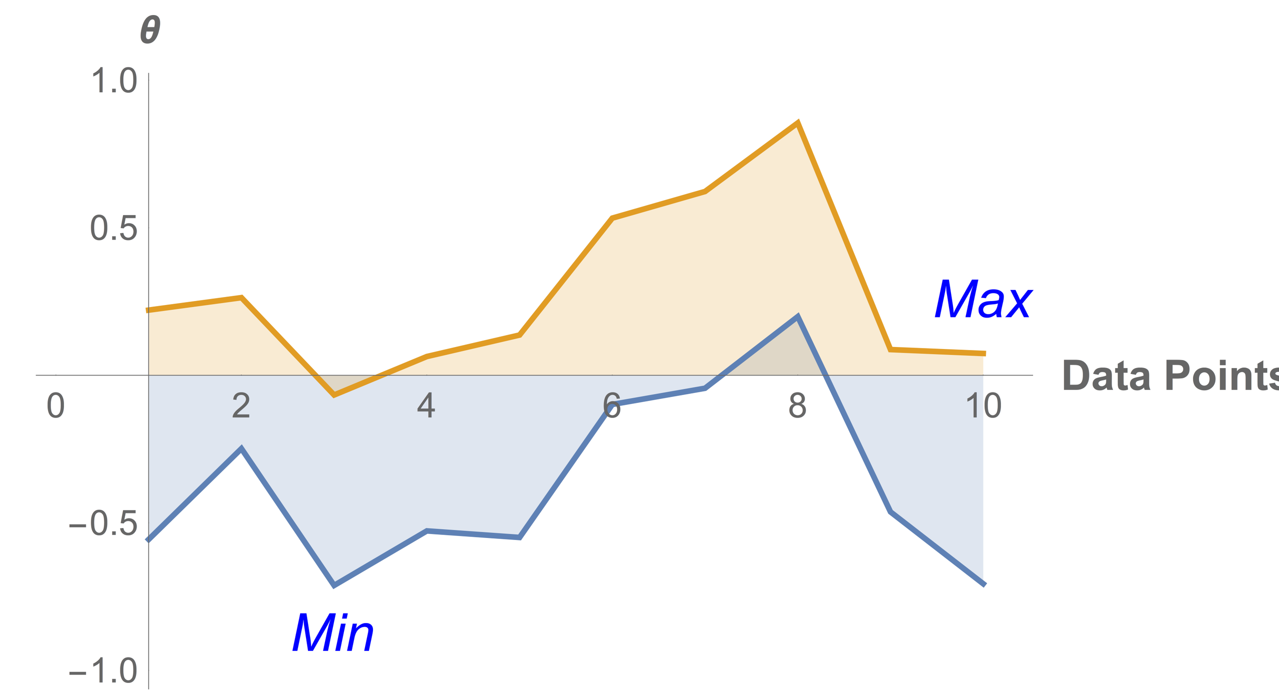

Figure 2: Minimum Bias (θ_{min}) and Maximum (θ_{max}) for n $\in [10^5, 8\times10^9]$

Conclusions

Negative tendencies (where Min is large and negative, while Max is small) manifest over initial intervals. Simultaneously, positive tendencies (where Max is large and positive, while Min is small) manifest over later intervals. These tendencies show instability in the convergence of the limit, suggesting that the natural density and mean do not exist. Our computations fall in accordance with the GRH prediction.

References

[1] Richard P. Brent. Irregularities in the distribution of primes and twin primes. *Mathematics of Computation*, 29(129):43–56, 1975.

[2] Jerzy Kaczorowski. On the shanks-rnyi race problem. *Acta Arithmetica*, 74(1):3146, 1996.

[3] Michael Rubinstein and Peter Sarnak. Chebyshev's bias. *Experiment. Math.*, 3(3):173–197, 1994.

[4] Daniel Shanks. Quadratic residues and the distribution of primes. *Mathematics of Computation*, 13(68):272272, 1959.

Acknowledgements

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