Progress Report

Title: Explicit Bound for Hurwitz Zeta Function in Rane's Derivation

Writer: Hao Liu (Major in Mathematics)

Advisor: Prof. Hiary (Ghaith Hiary)

In this report, I will summary the current accomplishment of this research, reflect the experience that I gained and show the goal in future.

Project summary:

This project focuses on the theorem 1 put forward by Rane in his research paper "on the mean square value of Dirichlet L series". In theorem 1, he ended with the result

$$\zeta(\frac{1}{2}+it,\alpha)=\sum_{0\le n\le m}(n+\alpha)^{-\frac{1}{2}-it}+(\frac{2\pi}{t})^{it}e^{i(t+\frac{\pi}{4})}\sum_{1\le n\le q}e^{-2\pi in\alpha}n^{-\frac{1}{2}+it}$$
$$+(\frac{2\pi}{t})^{\frac{1}{4}}e^{if(\alpha,t)}\Psi(2y-2q+\beta-\alpha)+O(t^{-\frac{3}{4}})$$

Where the $O(t^{-\frac{3}{4}})$ is the term that I try to find an explicit bound for this error term. To be specific, the project consists of ten steps. In each step, there is an error term in form of "big O" that I need to find an explicit upper bound. In the past three months, I worked out seven of them, results as follows (all the definitions of variables are strictly followed by Rane):

1. $e^{-\pi t}\sum_{n=1}^{q}n^{-\frac{1}{2}}<c_1t^{-\frac{3}{4}}$, Where c_1 =1, and $t\ge 1$
2. $2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{q}(2i)^{-1}e^{i((\frac{\pi s}{2})+2\pi n\alpha)}<c_2(e^{-\pi t}\sum_{n=1}^{q}n^{-\frac{1}{2}})$, Where c_2 =1, and $t\ge 1$
3. $\int_{G_1}\frac{w^{s-1}e^{-(m+\alpha)w}}{e^w-1}<c_3\eta^{\frac{1}{2}}e^{-t(\frac{\pi}{2}+A)}$, Where c_3 =2 π , $A>0$ and $t>4\pi^2$
4. $\int_{G_3}\frac{w^{s-1}e^{-(m+\alpha)w}}{e^w-1}<c_4\eta^{\frac{1}{2}}e^{-t(\frac{\pi}{2}+A)}$, Where c_4 =6, $A>0$ and $t>4\pi^2$
5. $G_2(c\eta,\eta(1+c))(-c\eta,\eta(1-c))$, Where c_5 =1, $A>0$, $t>4\pi^2$ and $C<\frac{3\pi}{4}$

6. $a_n < c_6 t^{-\frac{1}{6}}$, Where $c_6 = 1$ and $t > 4\pi^2$

7. $r_N(z) < c_7 |z|^N (\frac{5e}{2N\sqrt{t}})^{\frac{1}{3}N}$, Where $c_7 = 50$, $N \le \frac{27}{50}t$, $|z| \le \frac{20}{21}(\frac{2N\sqrt{t}}{5})^{\frac{1}{3}}$ and $t > 4\pi^2$

The remaining of the project should focus on the integral $\int_{G_2} \frac{w^{s-1}e^{-(m+\alpha)w}}{e^w - 1}$, where G_2 is the line from $(c\eta, \eta(1+c))$ to $(-c\eta, \eta(1-c))$, and the difficulty is that there is a pole on the integrant for this integral.

Experience:

This is the first time that I really conduct a research. I appreciate Prof. Hiary (Ghaith Hiary) for his semester long teaching before the research, to make sure that I am equipped with prerequisite knowledge, and his guidance on my research project. This research chosen by Prof. Hiary is suitable for me, an undergraduate have explored analytic number theory and have knowledge on complex analysis. Though it seems scary at first that I have to really solve a problem that untouched by other, it is interesting to find that there is extra source that inspires you. (In this research, I follow the process of Titchmarsh in his book "The Theory of Riemann Zeta Function", with modification on the restriction of the Hurwitz Zeta Function). The good part of the research that it is explicitly decomposed into ten steps, and the final result comes from the combination of them. It is exciting if you can solve any one of them. Yet, the difficulty lies on the flexibility of methods, where the method you depend on may be totally different with tiny change on the condition. (The difficulty on calculating integral on G_2 climbs rapidly, even though the integrant is the same).

Having explored these difficulties and rigid thoughts, I am honored to have such research experience and have Prof. Hiary to guide me on this.

Goal:

Since this research consists of ten steps and I have finished only seven of them, it still remains unfinished. Knowing that the application deadline for JUROS is on February 1th, I will try my best to solve the remaining part, and submit it to the board. Also, having an idea of how research is conducted, I have more intuition on how math looks like. In order to have more experience on math research, and see how professors cooperates, I will try to apply for URO in this summer to touch more aspects of mathematics.

Reference

V. V. Rane, On the mean square value of Dirichlet L-series. Journal of the London Mathematical Society, 2(2): 203-215, 1980

Titchmarsh E C, Heath-Brown D R. The theory of the Riemann Zeta Function. Oxford University Press, 1986.