Error Term on the Hardy-Littlewood *k*-tuple Conjecture

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I. Introduction

This paper will explore the error term on the Hardy-Littlewood *k*-tuple Conjecture, which estimates the number of prime constellations in a given range from 2 to *N* (where *N* is a large positive integer). A prime constellation is defined as the set of prime numbers of the form: (p, p+2m_1, p+2m_2, …, p+2m_k), where $m_1, m_2, \dots, m_k$ are positive integers. The conjecture estimates the number of prime constellations using the formula given below:

$$C(m_1, m_2, \dots, m_k) \int_2^N \frac{1}{Log[x]Log[x+2m_1] \dots Log[x+2m_k]} dx, \quad \text{(Eq.1)}$$

where

$$C(m_1, m_2, \dots, m_k) \sim 2^s \prod_q \frac{1 - \frac{w_q(q)}{q}}{(1-\frac{1}{q})^{s+1}}. \quad \text{(Eq. 2)}$$

In Eq. 2, q is every odd prime between 2 and *N*, $w_q(q)$ is the number of distinct residue classes of (0,$m_1, \dots, m_k$) modulo *q,* and *s* is one less than the size of the prime constellation (i.e. a 2-tuple has $s = 1$). In addition, we can note that $w_q(q) = s + 1$ for all q > m_s. Thus for the 2-tuple case, the formula for $C(m_1, m_2, \dots, m_k)$ becomes:

$$C(m_1) \sim 2 \prod_q \frac{1 - \frac{2}{q}}{(1-\frac{1}{q})^2} \prod_{q|m_1} \frac{q-1}{q-2}. \quad \text{(Eq. 3)}$$

The term $\prod_q \frac{1-\frac{2}{q}}{(1-\frac{1}{q})^2}$, when taken over a sufficient number of of odd primes q , converges to approximately 0.66016. This value is called the Twin Primes Constant and is often referred to as c_2. This gives us the final formula:

$$C(m_1) \sim 2c_2 \prod_{q|m_1} \frac{q-1}{q-2}. \qquad \text{(Eq. 4).}$$

II. Determining the Upper Bound of α

Using Eq. 1 and Eq. 4, we calculated the conjecture's estimate of twin prime constellations for a fixed *N* and a varying m_1. This paper will measure the error of the conjecture's estimate using an error term α that is defined as:

$$\alpha = \frac{Log[|Empirical - Conjecture|]}{Log[N]}, \qquad \text{(Eq. 5)}$$

where "Empirical" is the actual number of prime constellations that exist in a given range 2 to *N* and "Conjecture" is the value of the estimate. We chose to study α because we anticipate the error of the conjecture, $|Empirical - Conjecture|$, to be of the form:

$$|Empirical - Conjecture| \leq AN^{\alpha}. \qquad \text{(Eq. 6)}$$

While this paper does not explain why we anticipate the error to be of this form, we use this prediction due to the nature of pre-existing work done on error estimates of conjectures. The paper will begin its study by looking at the behavior of α for various sets of 2-tuples at a fixed *N* value. The graphs in Appendix A demonstrate the behavior of α for different ranges of *N* and m_1. The graphs show that α is bounded by ½.

III. Determining the Upper Bound of *A*

Using α = ½, we calculated *A* in Eq. 6; however, we are interested in the maximum value of *A* since we are looking for the error bound. Thus, to find where *A* is maximized, we calculated

A for a fixed m_1 at various *N*. The general trend for *A* tended to be a monotonically decreasing function as seen in the table below for $m_1 = 2$ and various *N* values.

For the table below $m_1 = 2$.

N	*A*
2^{20}	3.84914*10^{-5}
2^{21}	1.98577*10^{-5}
2^{22}	5.50649*10^{-6}
2^{23}	8.85031*10^{-6}
2^{24}	1.0991*10^{-6}

Table 1 Behavior of *A*: Values for *A* were calculated at many *N* values, only a few are show here.

The trend seen above holds true for all m_1 we have studied, $m_1 = 1$ to $20, 100,1000$; thus, we can reasonably assume that *A* is inversely proportional to *N*. This result agrees with the general claim of the conjecture itself that states: for a sufficiently large *N*, the conjecture can accurately predict the number of prime constellations in a given range 2 to *N*. From these results, we complied a table that has various $m_1$values and their corresponding *A* values (See Appendix B).

IV. Conclusion

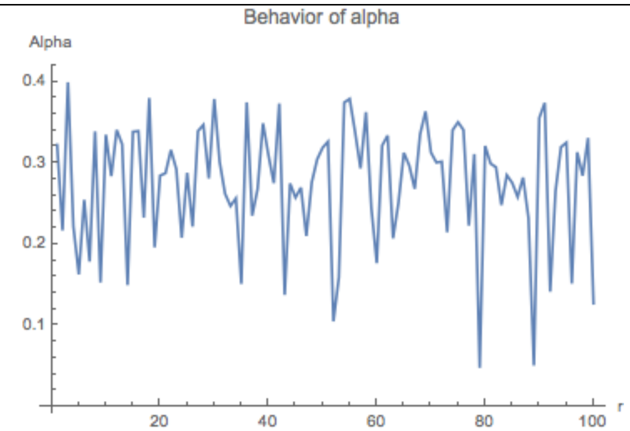
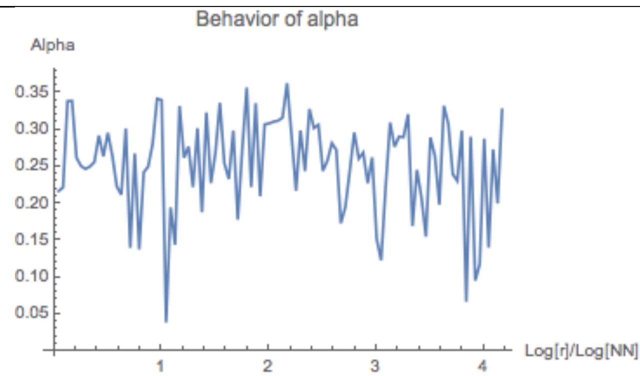
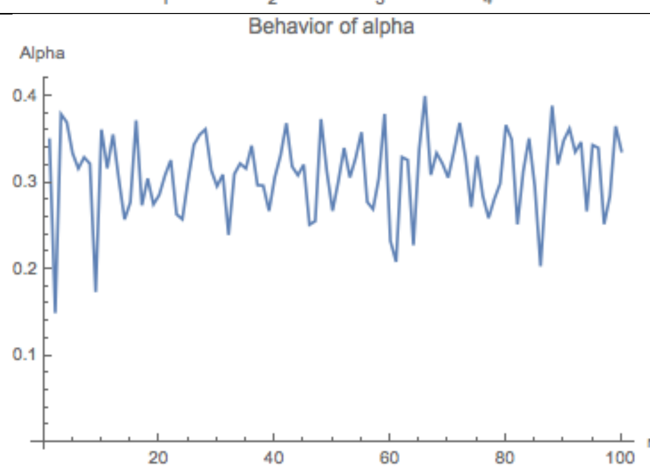
From the results above, we can gather that the error term on the Hardy-Littlewood *k*-tuple Conjecture does in fact have an error approximation that can be modeled by a function of the form AN^{α}, where *A* is a constant dependent on *N*, *N* is the upper bound of the range of interest, and $\alpha = \frac{1}{2}$. While this paper does not explicitly explore other size tuples, previous work with 3-tuples suggests that their error term behavior would be similar to the 2-tuples. Thus, we may reasonably assume that error terms for various tuples can be modeled in a similar form to the one illustrated above.

Note: Calculations for this project were done using Wolfram Alpha's *Mathematica* Software; some sample code that was used to calculate these results is included in Appendix C. In the references below, I have listed two papers. The first paper, Brent (1974), was useful in understanding the conjecture itself. The second paper, Granville (2007), contains research relevant to this project's topic.

References:

1. Brent, Richard P. "The Distribution of Small Gaps Between Successive Primes." *Mathematics of Computation* 28.125 (1974): 315. Web.
2. Granville, Andrew. "Refinements of Goldbach's Conjecture, and the Generalized Riemann Hypothesis." *Functiones Et Approximatio Commentarii Mathematici* 37.1 (2007): 159-73. Web.

Appendix A: Behavior of α Graphs

Value of N	Value of m_1	Graph
2^{24}	1 to 100	
2^{24}	2 to 2^{100}	
2^{28}	1 to 100	

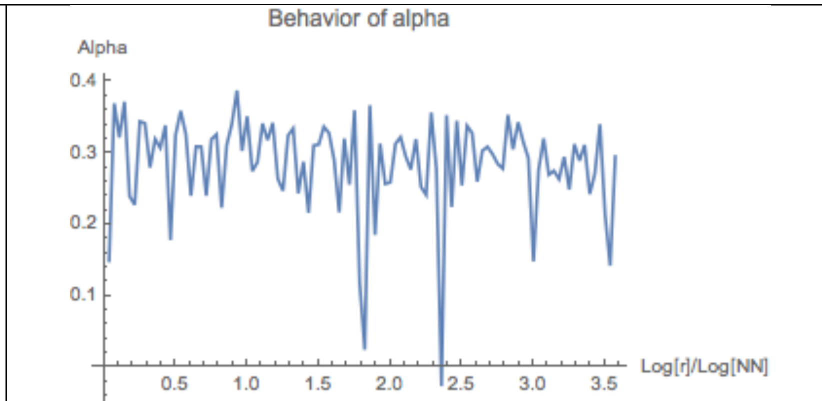
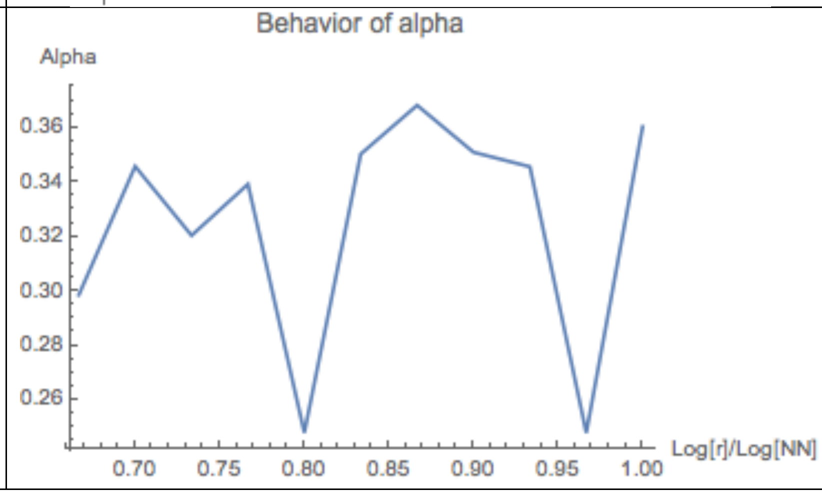
<table>
<tr><td>2^{28}</td><td>2 to 2^{100}</td><td>
</td></tr>
<tr><td>2^{30}</td><td>2^{20} to 2^{30}</td><td>
</td></tr>
</table>

Figure 1: The table above shows how the value of alpha is bounded from above by ½ for various for m_1 and *N*. Note: In the graphs the axes have "r" and "NN" these correspond to m_1 and N (the naming difference is due to cross referencing multiple sources).

Appendix B: Behavior of A

Value of m_1	Value of A
1	$2.21274 * 10^{-5}$
2	$3.84914 * 10^{-5}$
3	$4.60216 * 10^{-5}$
4	$3.86429 * 10^{-6}$
5	$2.69231 * 10^{-5}$
6	$4.84824 * 10^{-5}$
7	$1.37947 * 10^{-5}$
8	$1.56882 * 10^{-5}$
9	$1.78449 * 10^{-5}$
10	$2.3626 * 10^{-6}$
11	$4.06328 * 10^{-6}$
12	$7.22066 * 10^{-5}$
13	$2.52207 * 10^{-5}$
14	$4.79284 * 10^{-5}$
15	$7.09623 * 10^{-6}$
16	$1.65303 * 10^{-5}$
17	$1.78241 * 10^{-5}$
18	$1.85252 * 10^{-5}$
19	$3.2703 * 10^{-5}$
20	$8.7101 * 10^{-6}$
100	$3.54925 * 10^{-5}$
1000	$1.54595 * 10^{-5}$

Table 1: Table of A. The values for A are displayed above. The A values above correspond to an N-value of 2^{20}**.** As mentioned in the paper, A generally tends to be a monotonically decreasing as N increases. We chose $N=2^{20}$ as it is the lowest value of N where are results can be applicable to the uses of the conjecture (reducing N to very small integers is not useful as the conjecture operates on the premise N is sufficiently large).

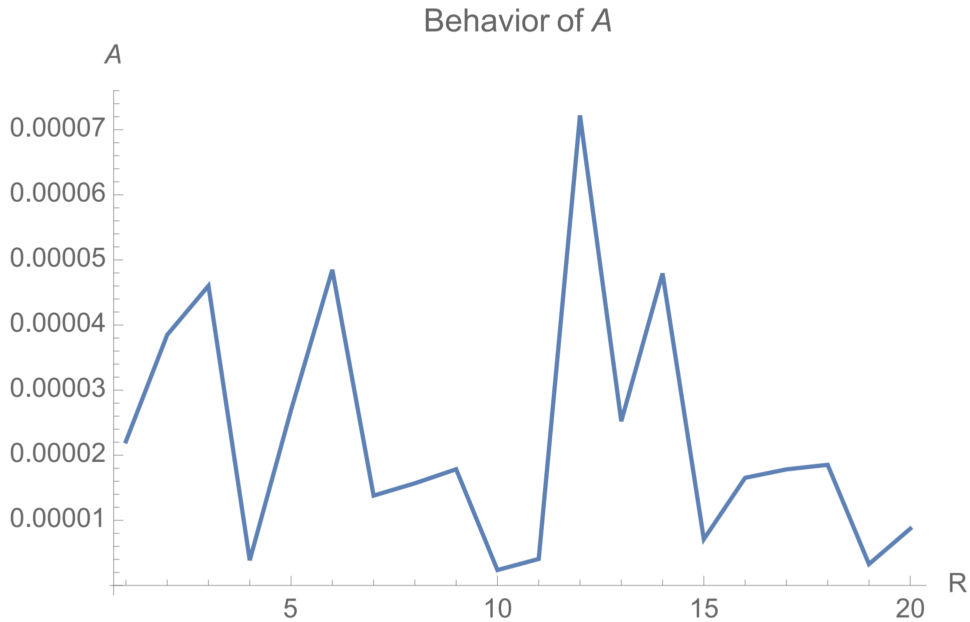


Figure 2: Graph of A. Again, R on the graph above is m_1. A generally tends to be be around 10^{-5} or 10^{-6} order of magnitude. The alpha value for m_1=100 or 1000 is not on the graph (as the values distort the graph); however, this trend hold for those higher values of $m_1$as well.

Appendix C: *Mathematica* Code

```
(*Below is some of the code used to find Alpha*)
c2 = 0.66016;
qlist[r_] := Select[FactorInteger[r], OddQ[#[[1]]] && #[[1]] > 1 &][[All, 1]];
mathprod[lst_] := If[Length[lst] == 0, 1, Times @@ lst]
A[r_] := 2 * c2 * mathprod[(qlist[r] - 1) / (qlist[r] - 2)];

Timing[
 out = {};
 For[j = 1, j ≤ 100, j ++, {
   r = 2^j;
   (* r = 2^j *)
   (* r = Floor[1.5^j] *)
   NN = 2^28;
   (* NN = 2^30 *)
   counter = 0;
   k = 1;
   p = Prime[k];
   While[p ≤ NN, {If[PrimeQ[p + 2 r], counter ++]; k ++; p = Prime[k];}];
   emp = counter;
   conj = A[r] N[Integrate[1 / ((Log[x]) (Log[x + 2 r])), {x, 2, NN}]];
   alpha = Log[Abs[emp - conj]] / Log[NN];
   Print[j, "     ", alpha];
   out = Append[out, {(Log[r] / Log[NN]), alpha}];
  }]]
ListPlot[out, AxesLabel → {"Log[r]/Log[NN]", "Alpha"},
 Joined → True, PlotLabel → Style["Behavior of alpha"], PlotRange → All]
```

```
(*Below is some of the code used to find A*)
```

```
c2 = 0.66016;
qlist[r_] := Select[FactorInteger[r], OddQ[#[[1]]] && #[[1]] > 1 &][[All, 1]];
mathprod[lst_] := If[Length[lst] == 0, 1, Times @@ lst]
A[r_] := 2 * c2 * mathprod[(qlist[r] - 1) / (qlist[r] - 2)];
out242 = {};
For[r = 1, r < 21, r ++, {
   Print["R-Value: " , r];
   Timing[

     For[j = 1, j ≤ 28, j ++, {

        (* r = 2^j *)
        (* r = Floor[1.5^j] *)
        NN = 2^j;
        (* NN = 2^30 *)
        counter = 0;
        k = 1;
        p = Prime[k];
        While[p ≤ NN, {If[PrimeQ[p + 2 r], counter ++];
           k ++;
           p = Prime[k];}];
        emp = counter;
        conj = A[r] N[Integrate[1 / ((Log[x]) (Log[x + 2 r])), {x, 2, NN}]];
        alpha = Log[Abs[emp - conj]] / Log[NN];
        AConstant = ScientificForm[Abs[emp - conj] / (NN) ^1 / 2];
        Print["N Value: ", NN, "       ", "Value of A: ", AConstant];
        out242 = Append[out242, {(r), AConstant}];
      }]];
   ListPlot[out242, AxesLabel → {"r", "A"},
     Joined → True, PlotLabel → Style["Behavior of A"], PlotRange → All]
  }];
```