Modular Forms

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Define

* $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\}$
* $\mathfrak{h} = \{z \in \mathbb{C} : Im(z) > 0\}$, $\mathfrak{h}$ is called the complex upper half plane

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ Then for any $z \in \mathbb{C}$ we define the action of $SL_2(\mathbb{Z})$ on $\mathfrak{h}$ by a linear functional transformation:

$$\gamma(z) = \frac{az + b}{cz + d} \in \mathfrak{h}$$

Let R be an open subset of $\mathbb{C}$. We define $f : R \to \mathbb{C}$ to be *holomorphic* at a point $z \in R$ if f is complex differentiable at z i.e. for $z \in R$ the limit:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, and it is the same if we take h along any path.
We said that f is holomorphic on R if f is holomorphic at all points $z \in R$.
$f(z)$ is holomorphic at ∞ if $f(z)$, is bounded, or $f(1/z)$ is holomorphic at 0.

A modular form of weight k on $SL_2(\mathbb{Z})$ is a function $f : \mathfrak{h} \to \mathbb{C}$ such that

i If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then $f(\gamma(z)) = (cz+d)^k f(z)$ $\forall z \in \mathfrak{h}$

ii $f(z)$ is holomorphic on $\mathfrak{h}$ and at ∞

Let f a modular form, take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ then $\forall z \in \mathfrak{h}$ we have $f(z+1) = f(\gamma(z)) = (cz+d)^k f(z) = f(z)$ then $f(z)$ is periodic, also f is holomorphic at ∞, thus f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

Let $q(z) = e^{2\pi i z}$, then $f(z) = \sum_{n=0}^{\infty} a_n q^n$ and this is called the *q-expansion* of f about ∞.

Let $k \in \mathbb{Z}, k > 2$. Define the Eisenstein series of weight k to be a series

$$G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(nz+m)^k}$$

Theorem

For $k > 2$, the function $G_k(z)$ is modular of weight k on $SL_2(\mathbb{Z})$

Proof: We must prove that $G_k(z+1) = G_k(z)$, and $G_k(\frac{-1}{z}) = z^k G_k(z)$ $\square$

The beginning of the q-expansion of the first few $G_k(z)$ are given by:

$$G_4(z) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \cdots$$

$$G_6(z) = -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots$$

In general,

$$G_k = \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $\zeta(s)$ is the Riemann zeta function.

Consider the set M_k of all modular forms of weight k. We have that:

* $f(z) = 0 \in M_k$
* If $g \in M_k$, then $\alpha g \in M_k$ $\forall \alpha \in \mathbb{C}$
* If $g, h \in M_k$, then $g + h \in M_k$

Thus M_k is a vector space over $\mathbb{C}$. Then $G_k \in M_k$ $\forall k \in \mathbb{Z}$, $k > 2$. We can calculate the dimension of each vector space M_k

k	0	2	4	6	8	10	12	$\cdots$
dimM_k	1	0	1	1	1	1	2	$\cdots$

Note that if $f \in M_k$, $g \in M_h$, then $fg \in M_{k+h}$.

Example: $G_4^2 \in M_8$, and by previous observation $G_8 \in M_8$. Since M_8 is 1-dimensional, $G_8 = \alpha G_4^2$ for some $\alpha \in \mathbb{C}$. Comparing the q-expansions, we get the non-obvious identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$$

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