Restrictions of $\Omega_m(q)$-modules to alternating groups

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Abstract

We consider the restriction of an irreducible $\mathbf{F}\Omega_m(q)$-module M to a subgroup H where $F^*(H) \cong A_n$ and where $\mathbf{F}$ is algebraically closed with $(\text{char}(\mathbf{F}), q) \neq 1$. Given certain restrictions on the highest weight of M, we show that if $m > n^6$, then $M{\downarrow}_H$ is reducible.

1 Introduction

In the study of the maximal subgroups of classical groups, the following question arises: Given an absolutely irreducible module M for K and a subgroup H, when does $M{\downarrow}_H$ remain absolutely irreducible? In this article $K \cong \Omega_m(q)$ is the commutator subgroup of an m-dimensional orthogonal group over $\mathbf{F}_q$, and $F^*(H) \cong A_n$ is the alternating group of degree n. We treat the case that the field of definition of M has characteristic dividing q.

Let $\mathbf{F}$ be an algebraically closed field containing $\mathbf{F}_q$, the field with q elements, such that $\text{char}(\mathbf{F}) > 3$. Then $K < \overline{K}$ where $\overline{K} \cong \Omega_m(\mathbf{F})$ and we may assume that M is a $\mathbf{F}K$-module. By [6, Theorem 43], every absolutely irreducible $\mathbf{F}K$-module is the restriction of an irreducible $\mathbf{F}\overline{K}$-module of the same weight. So we may assume that $M = M(\lambda)$ is an irreducible $\mathbf{F}\overline{K}$-module with highest weight λ. Let $\ell = \lfloor m/2 \rfloor$ be the Lie rank of $\overline{K}$ and let $\{\lambda_i\}$ be the fundamental dominant weights of $\overline{K}$. The labeling of these weights corresponds to the labeling of the Dynkin diagrams for $\overline{K}$ as given in [3].

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Hypothesis 1.1. *Assume the following are true:*

(1.) *If m is even, then* $\lambda = \left(\sum_{i=1}^{\ell-2} a_i \lambda_i \right) + a_{\ell-1}(\lambda_{\ell-1} + \lambda_\ell)$*;* $a_i \in \mathbf{Z}$*,* $a_i \geq 0$.

(2.) *If m is odd, then* $\lambda = \left(\sum_{i=1}^{\ell-1} a_i \lambda_i \right) + 2a_\ell \lambda_\ell$*;* $a_i \in \mathbf{Z}$*,* $a_i \geq 0$.

(3.) *If* $\mu_i = \sum_{j=i}^{\ell-1} a_j$*, m even or if* $\mu_i = \sum_{j=i}^{\ell} a_j$*, m odd then*

 (a.) $\mu_1 < p = \mathrm{char}(\mathbf{F}_q)$*;*

 (b.) $1 < \sum \mu_i = k < \ell$.

Conditions (1.) and (2.) imply that M is not a faithful module for any proper covering group of $\overline{K}$. We now state our main result:

Theorem 1.2. *Assume that H, K and $M = M(\lambda)$ are as above with $n, m \geq 10$ and $(q, 6) = 1$. Suppose further that λ satisfies hypothesis 1.1. If $m > n^6$, then $M{\downarrow_H}$ is reducible.*

Our strategy is to produce a small subspace in M with a large stabilizer in H and then, using Frobenius reciprocity, produce an upper bound for $\dim(M)$. We produce a lower bound for $\dim(M)$ as an $\mathbf{F}\overline{K}$-module using the length of the Weyl group orbit of a subdominant weight in M. The result then follows by comparing these two bounds.

2 A construction of $\overline{W}(\lambda)$

In this section we construct the Weyl module $\overline{W}(\lambda)$ of $\overline{K}$ with highest weight λ. Then M is a homomorphic image of this module. Our construction proceeds by first constructing the Weyl module $W(\lambda)$ for a complex Lie group G of the same type and rank as $\overline{K}$, then we use Kostant's $\mathbf{Z}$-form to produce $\overline{W}(\lambda)$. For notational convenience we assume that $\{\lambda_i\}$ are the fundamental dominant weights for G as well as for $\overline{K}$, and accordingly, assume that λ is a dominant weight of G.

Let V be a complex, m-dimensional vector space possessing a non-degenerate orthogonal form $\mathbf{f}(\ ,\ )$ and let $\mathcal{B}$ be a basis for V so that

$$\mathcal{B} = \begin{cases} \{e_i, f_i \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is even} \\ \{e_i, f_i, x \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is odd} \end{cases}$$

with $\mathbf{f}(e_i, e_j) = \mathbf{f}(f_i, f_j) = \mathbf{f}(x, e_i) = \mathbf{f}(x, f_i) = 0$, $\mathbf{f}(e_i, f_j) = \delta_{i,j}$ and $\mathbf{f}(x, x) = 2$. We then define $G = \Omega(V)$ and let T be the maximal torus of G with respect to $\mathcal{B}$. Set $V_e = \langle e_i \mid 1 \leq i \leq \ell \rangle$ and $V_f = \langle f_i \mid 1 \leq i \leq \ell \rangle$.

Suppose that λ satisfies hypothesis 1.1 and $d = \max\{i \mid \mu_i \neq 0\}$ so that $\mu = (\mu_1, \dots, \mu_d)$ is a proper partition of k. Let $\mathcal{T}$ be the tableau of shape μ with entries $t_{i,j} = j + \sum_{s<i} \mu_s$. Define the following subgroups of the symmetric group $\mathcal{S}_k$:

$$\mathcal{R}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same row as } t_{i,j} \text{ for all } i, j\}$$

$$\mathcal{C}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same column as } t_{i,j} \text{ for all } i, j\}$$

and elements of $\mathbf{C}\mathcal{S}_k$:

$$r_\mu = \sum_{\sigma \in \mathcal{R}_\mu} \sigma \qquad \text{and} \qquad c_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \operatorname{sgn}(\sigma)\sigma$$

Define $\kappa_{i,j} : V^{\otimes k} \to V^{\otimes(k-2)}$ by $\kappa_{i,j}(v_{l_1} \otimes \cdots \otimes v_{l_k}) = f(v_{l_i}, v_{l_j})(v_{l_1} \otimes \cdots \otimes \widehat{v_{l_i}} \otimes \cdots \otimes \widehat{v_{l_j}} \otimes \cdots \otimes v_{l_k})$ for $1 \leq i < j \leq k$ and set

$$\mathcal{K} = \bigcap_{i,j} \ker(\kappa_{i,j})$$

$\mathcal{S}_k$ acts on $V^{\otimes k}$ by place permutation, specifically:

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(k)}}$$

This action commutes with the diagonal action of G on $V^{\otimes k}$.

Given $v \in V^{\otimes k}$, we define one additional element r_μ^v of the group algebra $\mathbf{C}\mathcal{S}_k$ as follows: Let $\mathcal{R}_\mu^v = \{\sigma \in \mathcal{R}_\mu \mid \sigma(v) = v\}$ and let $\{s_i\}$ be a left transversal for $\mathcal{R}_\mu^v$ in $\mathcal{R}_\mu$. Define $r_\mu^v = \sum_i s_i$ Notice that $r_\mu(v) = |\mathcal{R}_\mu^v| r_\mu^v(v)$

By [2, Theorem 19.22], $W(\lambda) = c_\mu r_\mu \left(V^{\otimes k}\right) \cap \mathcal{K}$ is the Weyl module for G with highest weight λ. Since V is a complex vector space, $c_\mu r_\mu \left(V^{\otimes k}\right) = \langle c_\mu r_\mu^v(v) \mid v \in V^{\otimes k} \rangle$.

Define $V_{\mathbf{Z}} = \mathbf{Z}[\mathcal{B}]$ and let $\overline{V} = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}$. Then $\overline{\mathbf{f}}(\ ,\ ) = \mathbf{f}(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a non-degenerate orthogonal form on $\overline{V}$. Without loss of generality, we may assume that $\overline{K} = \Omega(\overline{V})$. Moreover if $\overline{e}_i = e_i \otimes 1_{\mathbf{F}}$, $\overline{f}_i = f_i \otimes 1_{\mathbf{F}}$ and $\overline{x} = x \otimes 1_{\mathbf{F}}$, then

$$\overline{\mathcal{B}} = \begin{cases} \{\overline{e}_i, \overline{f}_i \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is even} \\ \{\overline{e}_i, \overline{f}_i, \overline{x} \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is odd} \end{cases}$$

is a standard basis for $\overline{V}$ with respect to $\overline{\mathbf{f}}(\ ,\ )$. We identify r_μ and c_μ with the elements $r_\mu \otimes 1_{\mathbf{F}}$ and $c_\mu \otimes 1_{\mathbf{F}}$ of $\mathbf{F}\mathcal{S}_k$.

Suppose that $L \subset \mathrm{End}(V)$ is the adjoint module for G so that L is a complex Lie algebra of type D_ℓ or B_ℓ. Let $\Delta = \{r_1, \ldots, r_\ell\}$ be the set of simple roots corresponding to the torus T and let Φ be the root system generated by Δ. Set $\Delta_0 = \{r_1, \ldots, r_{\ell-1}\}$ and let $\Phi_0 \subset \Phi$ be the subset generated by Δ_0. Using the setup of [1, §11.2], $\{\epsilon_r, h_{r_i} \mid r \in \Phi, 1 \leq i \leq \ell\}$ is a Chevalley basis for L and $\{\epsilon_r, h_{r_i} \mid r \in \Phi_0, 1 \leq i \leq \ell - 1\}$ is a Chevalley basis for $L_0 \subset L$ where L_0 is a Lie algebra of type $A_{\ell-1}$. Let $G_0 < N_G(V_e \oplus V_f)$ such that $G_0 \cong SL_\ell(\mathbf{C})$. Then, by [1, Theorem 11.3.2], $G = \langle\ \exp(\zeta\epsilon_r) \mid r \in \Phi,\ \zeta \in \mathbf{C}\rangle$ and $G_0 = \langle\ \exp(\zeta\epsilon_r) \mid r \in \Phi_0,\ \zeta \in \mathbf{C}\rangle$. Note that neither G nor G_0 is the adjoint group for L or L_0, respectively. We may consider V_e to be the natural module for G_0. Under this identification, V_f is the dual of V_e.

Assume that $\mathcal{U}(L)$ is the universal enveloping algebra of L. From [3, §26], Kostant's $\mathbf{Z}$ -form $\mathcal{U}_{\mathbf{Z}}(L)$ is the $\mathbf{Z}$-span of $\{\ \epsilon_r^m/m! \mid r \in \Phi, m \in \mathbf{Z}^+\}$. Given any vector v of weight λ in $W(\lambda)$, $\mathcal{U}_{\mathbf{Z}}(L)v \bigotimes_{\mathbf{Z}} \mathbf{F} = \overline{W}(\lambda)$ where $\overline{W}(\lambda)$ is the Weyl module for $\overline{K}$ with highest weight λ. By the previous remarks, $\mathcal{U}_{\mathbf{Z}}(L_0) \subset \mathcal{U}_{\mathbf{Z}}(L)$, which implies that $\mathcal{U}_{\mathbf{Z}}(L_0)v \bigotimes_{\mathbf{Z}} \mathbf{F} \subset \overline{W}(\lambda)$.

Define $v_{\mu_i} = \bigotimes_{j=1}^{\mu_i} e_i$ and $v_\mu = \bigotimes_{i=1}^{d} v_{\mu_i}$

Lemma 2.1. *We have*

(1.) $c_\mu(v_\mu)$ *is a vector of weight* λ *in* $W(\lambda)$*;*

(2.) $\mathcal{U}_{\mathbf{Z}}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \ldots, e_\ell]^{\otimes k}$.

Proof: First note that $\mathcal{R}_\mu^{v_\mu} = \mathcal{R}_\mu$ so that $r_\mu^{v_\mu}(v_\mu) = v_\mu$ and that $c_\mu(v_\mu) \neq 0$. This implies that $c_\mu(v_\mu) \in c_\mu r_\mu\left(V^{\otimes k}\right)$. It is clear that $c_\mu(v_\mu) \in \mathcal{K}$ so we have $c_\mu(v_\mu) \in W(\lambda)$. Let $t \in T$ and write $t = \mathrm{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1})$ or $t = \mathrm{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1}, t')$ depending on the parity of m. Then

$$tv = c_\mu(tc_\mu(v_\mu)) = c_\mu\left(\bigotimes_{i=1}^{d} t_i^{\mu_i} v_{\mu_i}\right) = \left(\prod_{i=1}^{d} t_i^{\mu_i}\right) c_\mu(v_\mu)$$

From the definition of μ it follows that $c_\mu(v_\mu)$ is a vector of weight λ and so (1.) follows. With the identification of V_e with the natural module of G_0, we see by [2, Theorem 15.15] that $c_\mu r_\mu\left(V_e^{\otimes k}\right)$ is the Weyl module for G_0 corresponding to the partition μ of k via the Schur functor. The argument above restricted to $t \in T \cap G_0$ shows that $c_\mu(v_\mu)$ is a highest weight vector in $c_\mu r_\mu\left(V_e^{\otimes k}\right)$. In particular $\mathcal{U}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right)$. Using the proof of [4, Theorem 8.3.1], we have

$$\mathcal{U}_{\mathbf{Z}}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \dots, e_\ell]^{\otimes k}$$

which completes our proof. □

Lemma 2.2. *Suppose* $\overline{v} = \overline{v}_{i_1} \otimes \cdots \otimes \overline{v}_{i_k}$ *where* $\{\overline{v}_i\}$ *is a collection of mutually orthogonal, linearly independent singular vectors. Then*

(1.) *If* $\mathrm{sgn}(\sigma_c)\sigma_c\sigma_r(\overline{v}) \neq -\overline{v}$ *for all* $\sigma_c \neq 1 \in \mathcal{C}_\mu, \sigma_r \in \mathcal{R}_\mu$, *then* $c_\mu r_\mu^{\overline{v}}(\overline{v}) \neq 0$*;*

(2.) $c_\mu r_\mu^{\overline{v}}(\overline{v}) \in \overline{W}(\lambda)$.

Proof: Since $\overline{v}$ is a summand of $c_\mu r_\mu^{\overline{v}}(\overline{v})$ and all other summands of $c_\mu r_\mu^{\overline{v}}(\overline{v})$ have the form $\mathrm{sgn}(\sigma_c)\sigma_c\sigma_r(\overline{v})$, part (1.) must hold. There is $g \in \overline{K}$ such that $g(\overline{v}_{i_j}) = \alpha_{i_j}\overline{e}_{i_j}$ such that $\alpha_{i_j} \neq 0$ for all $1 \leq i \leq k$. If $w = e_{i_1} \otimes \cdots \otimes e_{i_k}$, then $r_\mu^{\overline{v}} = r_\mu^w$. As

$$c_\mu r_\mu^w(w) \in c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \dots, e_\ell]^{\otimes k},$$

lemma 2.1 implies that $c_\mu r_\mu^w(w) \in \mathcal{U}_{\mathbf{Z}}(L)v$. Writing $\overline{w} = \alpha_{i_1}\overline{e}_{i_1} \otimes \cdots \otimes \alpha_{i_k}\overline{e}_{i_k}$, we then have

$$c_\mu r_\mu^{\overline{w}}(\overline{w}) \in \mathcal{U}_{\mathbf{Z}}(L)v \bigotimes_{\mathbf{Z}} \mathbf{F} = \overline{W}(\lambda)$$

Finally, as $\overline{W}(\lambda)$ is a $\mathbf{F}\overline{K}$-module, $g^{-1}c_\mu r_\mu^{\overline{w}}(\overline{w}) = c_\mu r_\mu^{\overline{v}}(\overline{v}) \in \overline{W}(\lambda)$. □

Though $W(\lambda)$ is a irreducible module for G, $\overline{W}(\lambda)$ may not be an irreducible module for $\overline{K}$; however, it does possess a unique maximal submodule by [6, Lemma 80] which we denote by $\mathrm{Rad}(\overline{W}(\lambda))$. Moreover, $M \cong \overline{W}(\lambda)/\mathrm{Rad}(\overline{W}(\lambda))$.

We now discuss the orthogonal forms on $V^{\otimes k}$ and $W(\lambda)$. Suppose $v, w \in V^{\otimes k}$ where $v = v_1 \otimes \cdots \otimes v_k$ and $w = w_1 \otimes \cdots \otimes w_k$. We define $\mathbf{f}^k(\ ,\ )$ by

$$\mathbf{f}^k(v, w) = \prod_{i=1}^{k} \mathbf{f}(v_i, w_i)$$

$\mathbf{f}^k(\ ,\ )$ is a non-degenerate, G-invariant orthogonal form on $V^{\otimes k}$. This form is also invariant under the action of $\mathcal{S}_k$. Note that

$$
\begin{aligned}
\mathbf{f}^k[c_\mu(v), c_\mu(w)] &= \sum_{\sigma\in\mathcal{C}_\mu} \operatorname{sgn}(\sigma)\mathbf{f}^k[\sigma(v), c_\mu(w)] \\
&= \sum_{\sigma\in\mathcal{C}_\mu} \operatorname{sgn}(\sigma)\mathbf{f}^k[v, \sigma^{-1}c_\mu(w)] \\
&= \sum_{\sigma\in\mathcal{C}_\mu} \mathbf{f}^k[v, c_\mu(w)] \\
&= |\mathcal{C}_\mu|\mathbf{f}^k[v, c_\mu(w)].
\end{aligned}
$$

We define $\mathbf{f}^k_\mu(\ ,\ )$ on $c_\mu\left(V^{\otimes k}\right)$ by

$$
\mathbf{f}^k_\mu[c_\mu(v), c_\mu(w)] = \mathbf{f}^k[v, c_\mu(w)]
$$

By a similar argument as above, we see that $\mathbf{f}^k[v, c_\mu(w)] = \mathbf{f}^k[w, c_\mu(v)]$, so this form is symmetric. Since $\mathbf{f}^k(\ ,\ )$ is bilinear and G-invariant, $\mathbf{f}^k_\mu(\ ,\ )$ is also bilinear and G-invariant. Therefore $\mathbf{f}^k_\mu(\ ,\ )$ is a G-invariant orthogonal form on $W(\lambda) \subset c_\mu\left(V^{\otimes k}\right)$. As before, $\overline{\mathbf{f}}^k(\ ,\ ) = \mathbf{f}^k(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a $\overline{K}$-invariant orthogonal form on $\overline{V}^{\otimes k}$ and $\overline{\mathbf{f}}^k_\mu(\ ,\ ) = \mathbf{f}^k_\mu(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a $\overline{K}$-invariant orthogonal form on $\overline{W}(\lambda)$. This form is possibly degenerate. We denote the radical of this form as $\overline{W}(\lambda)^\perp$. The following lemma is generally known, although we present a proof:

Lemma 2.3. $\operatorname{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$

Proof: Define $\overline{v}_{-\mu_i} = \bigotimes_{j=1}^{\mu_i} \overline{f}_i$ and $\overline{v}_{-\mu} = \bigotimes_{i=1}^{d} \overline{v}_{-\mu_i}$ and set $\overline{v}_{-\lambda} = c_\mu(v_{-\mu})$. Noting that $r_\mu^{\overline{v}_{-\mu}} = 1$, $\overline{v}_{-\lambda} \neq 0 \in \overline{W}(\lambda)$ by lemma 2.2. A similar argument as in the proof of lemma 2.1 shows that $\overline{v}_{-\lambda}$ is a vector of weight $-\lambda$. Hypothesis 1.1 implies that $d < \ell$. In particular, there is an element ω_0 of the Weyl group of $\overline{K}$ such that $\omega_0(\overline{v}_{-\lambda}) = \overline{v}_\lambda$. This means that $M = M(\lambda)$ must be self-dual. Clearly we have that $\overline{W}(\lambda)^\perp \subset \operatorname{Rad}(\overline{W}(\lambda))$ and that $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ is non-degenerate, so this latter module is also self-dual. Since M is self-dual and is a homomorphic image of $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$, $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ must possess a submodule isomorphic to M. Since $M \cong \overline{W}(\lambda)/\operatorname{Rad}(\overline{W}(\lambda))$ and $\operatorname{Rad}(\overline{W}(\lambda))$ does not possess a constituent which is isomorphic to M, we must have $\operatorname{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$ and our result follows. □

Lemma 2.4. *Let $\{\overline{v}_i, \overline{w}_i \mid 1 \leq i \leq k\}$ be a hyperbolic basis for some $2k$-dimensional subspace of $\overline{V}$. Set $\overline{v} = \overline{v}_1 \otimes \cdots \otimes \overline{v}_k$ and $\overline{w} = \overline{w}_1 \otimes \cdots \otimes \overline{w}_k$. Then*

(1.) $c_\mu r_\mu(\overline{v}) \neq 0,\ c_\mu r_\mu(\overline{w}) \neq 0$;

(2.) $c_\mu r_\mu(\overline{v}),\ c_\mu r_\mu(\overline{w}) \in \overline{W}(\lambda)$;

(3.) $\overline{\mathbf{f}}^k_\mu[c_\mu r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] \neq 0$.

Proof: Parts (1.) and (2.) follow from lemma 2.2 since $r^{\overline{v}}_\mu = r^{\overline{w}}_\mu = r_\mu$ and the $\overline{v}_i$ are distinct, similarly for $\overline{w}_i$. If $\sigma_1, \sigma_2 \in \mathcal{S}_k$, then

$$\overline{\mathbf{f}}^k[\sigma_1(\overline{v}), \sigma_2(\overline{w})] = \prod_{i=1}^k \overline{\mathbf{f}}[\overline{v}_{\sigma_1^{-1}(i)}, \overline{w}_{\sigma_2^{-1}(i)}] = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 \\ 0 & \text{otherwise} \end{cases}$$

Recall that $\mathcal{R}_\mu \cap \mathcal{C}_\mu = 1$. Then we have

$$\begin{aligned} \overline{\mathbf{f}}^k_\mu[c_\mu r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] &= \overline{\mathbf{f}}^k[r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] \\ &= \sum_{\sigma \in \mathcal{R}_\mu} \overline{\mathbf{f}}^k[\sigma(\overline{v}), c_\mu r_\mu(\overline{w})] \\ &= \sum_{\sigma \in \mathcal{R}_\mu} \overline{\mathbf{f}}^k[\sigma(\overline{v}), \sigma(\overline{w})] \\ &= |\mathcal{R}_\mu|. \end{aligned}$$

Part (3.) then follows as $|\mathcal{R}_\mu| = \prod_{i=1}^d \mu_i!$ and $\mu_i < \mathrm{char}(\mathbf{F}_q)$ for all i. □

Lemma 2.5. *M possesses a vector of weight λ_k*

Proof: Let $\{\overline{e}_i, \overline{f}_i \mid 1 \leq i \leq k\}$ be a subset of our standard basis $\overline{\mathcal{B}}$ for $\overline{V}$. By part (2.) of lemma 2.4, $c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k) \in \overline{W}(\lambda)$. An argument similar to that used in lemma 2.1 shows that $c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k)$ is a vector of weight λ_k. Hence λ_k is a subdominant weight of λ. Condition (3.) of hypothesis 1.1 insures that λ is p-restricted. Therefore using the results of [5], M possesses a vector of weight λ_k. □

3 Elementary abelian 3-subgroup E_k

Assume that $k \leq n/3 - 2$ and recall that $F^*(H)$ possesses a subgroup H_0 isomorphic to S_{n-2}. Let

$$E_k \cong \langle\ (123), (456), \ldots, (3k-2,\ 3k-1,\ 3k)\ \rangle < A_n$$

be a subgroup of H_0 generated by commuting 3-cycles in $F^*(H)$ so that E_k is an elementary abelian 3-group of rank k. Then

$$N_k = N_{H_0}(E_k) \cong S_3 \wr S_k \times S_{n-3k-2}$$

$$C_k = C_{H_0}(E_k) E_k \times S_{n-3k-2}$$

and let $H_k < C_k$ so that $H_k \cong S_{n-3k-2}$. Note that $C_{N_k}(H_k) \cong S_3 \wr S_k$ and this subgroup controls fusion in E_k. Let $\sigma \neq 1 \in E_k$ and assume that σ is the product of k_1 disjoint 3-cycles. Then $C_{N_k}(\sigma) \cong \mathbf{Z}_3 \wr S_{k_1} \times S_3 \wr S_{k-k_1} \times S_{n-3k-2}$ which implies $|\sigma^{N_k}| = 2^{k_1}\binom{k}{k_1}$.

Let $\varphi \in E_k^* = \mathrm{Hom}(E_k, \mathbf{F}^*)$. The group N_k acts on this group by $\varphi^g : \sigma \mapsto \varphi(g^{-1}\sigma g)$ for $g \in N_k$, $\sigma \in E_k$. We abuse notation slightly and define φ^{-1} by $\varphi^{-1} : \sigma \mapsto \varphi(\sigma^{-1})$ for all $\sigma \in E_k$. Recall that $\mathrm{In}_{N_k}(\varphi) = \{g \in N_k | \varphi^g = \varphi\}$ is the inertia group of φ in N_k and note that $H_k \in \mathrm{In}_{N_k}(\varphi)$.

If $\varphi \in E_k^*$ is non-trivial, then the previous remarks concerning the action of N_k on E_k imply that $[N_k : \mathrm{In}_{N_k}(\varphi)] = 2^{k_1}\binom{k}{k_1}$ for some k_1, $1 \leq k_1 \leq k$ and that $\varphi^{-1} \in \varphi^{N_k}$. Since $\binom{k}{k_1} \geq k$ unless $k = k_1$, in which case $2^{k_1} \geq 2k$, we have $[N_k : \mathrm{In}_{N_k}(\varphi)] \geq 2k$.

4 Decomposition of $\overline{V}\downarrow_{E_k}$ and C_k-invariant subspace of $\overline{W}(\lambda)$

We continue to assume that $k \leq n/3 - 2$ and we now consider the restriction of $\overline{V}$ to E_k. Since char$(\mathbf{F}) \neq 3$, we have $\overline{V}\downarrow_{E_k} \cong \bigoplus_{\varphi \in E_k^*} \overline{V}_\varphi$ where $\overline{V}_\varphi$ is the homogeneous component of φ. Let $\overline{v}_1 \in \overline{V}_{\varphi_1}$ and $\overline{v}_2 \in \overline{V}_{\varphi_2}$. Then $(g\overline{v}_1, g\overline{v}_2) = \varphi_1(g)\varphi_2(g)(\overline{v}_1, \overline{v}_2)$ for all $g \in E_k$. If $\varphi_1^{-1} \neq \varphi_2$ then $(\overline{v}_1, \overline{v}_2) = 0$ which implies $\overline{V}_{\varphi_1} \perp \overline{V}_{\varphi_2}$ when $\varphi_1^{-1} \neq \varphi_2$. Since $\overline{V}$ is non-degenerate, $\dim(\overline{V}_{\varphi_1}^{\perp}) = \dim(\overline{V}) - \dim(\overline{V}_{\varphi_1})$ and it follows that $\overline{V}_\varphi \oplus \overline{V}_{\varphi^{-1}}$ must be non-degenerate and therefore possesses a hyperbolic basis.

Pick $\varphi \neq 1$ so that $\overline{V}_\varphi \neq 0$. Since $g\overline{V}_\varphi = \overline{V}_{\varphi^g}$ for $g \in N_k$, we may consider $\overline{V}_\varphi$ to be an $\mathbf{F}\mathrm{In}_{N_k}(\varphi)$-module. Let E_{k-1}^* be a maximal subgroup of E_k^* which does not contain φ. Define $\mathcal{O}_+ = \varphi E_{k-1}^* \cap \varphi^{N_k}$ and $\mathcal{O}_- = \varphi^{-1} E_{k-1}^* \cap \varphi^{N_k}$ so that $\mathcal{O}_+ \cup \mathcal{O}_- = \varphi^{N_k}$ and $|\mathcal{O}_+| = |\mathcal{O}_-| \geq k$. Moreover $\varphi_i \in \mathcal{O}_+$ if and only if $\varphi_i^{-1} \in \mathcal{O}_-$. We assume that $\mathcal{O}_+ = \{\varphi_i\}$ and that $\mathcal{O}_- = \{\varphi_i^{-1}\}$. Then $\left(\bigoplus_{\varphi_i \in \mathcal{O}_+} \overline{V}_{\varphi_i}\right) \bigoplus \left(\bigoplus_{\varphi_i^{-1} \in \mathcal{O}_-} \overline{V}_{\varphi_i^{-1}}\right)$ is an $\mathbf{F}N_k$-submodule of $\overline{V}\downarrow_{N_k}$. If $\varphi' \in \varphi^{N_k}$ then, as $C_{N_k}(H_k)$ also controls fusion in E_k^*, there is a $g \in C_{N_k}(H_k)$ such that $g\overline{V}_\varphi = \overline{V}_{\varphi'}$. In particular $\overline{V}_\varphi \cong \overline{V}_{\varphi'}$ as $\mathbf{F}H_k$-modules. Define $D = \dim(\overline{V}_\varphi)$ so that $D = \dim(\overline{V}_{\varphi_i})$ for all i.

Given the above decomposition, we form the following:

$$\overline{V}_+ = \bigotimes_{i=1}^{k} \overline{V}_{\varphi_i} \qquad \text{and} \qquad \overline{V}_- = \bigotimes_{i=1}^{k} \overline{V}_{\varphi_i^{-1}}$$

Recall that $D = \dim(\overline{V}_{\varphi_i})$ and assume that $\{\overline{v}_{i,j}, \overline{w}_{i,j} \mid 1 \leq j_i \leq D\}$ is a hyperbolic basis for $\overline{V}_{\varphi_i} \oplus \overline{V}_{\varphi_i^{-1}}$. Define $\overline{v}^{j_1,\cdots,j_k} = \bigotimes_{i=1}^{k} \overline{v}_{i,j_i}$ and $\overline{w}^{j_1,\cdots,j_k} = \bigotimes_{i=1}^{k} \overline{w}_{i,j_i}$. Then $\{\overline{v}^{j_1,\cdots,j_k}, \overline{w}^{j_1,\cdots,j_k} \mid 1 \leq j_i \leq D\}$ forms a hyperbolic basis for $\overline{V}_+ \oplus \overline{V}_-$. If $\sigma \in \mathcal{S}_k$, then $\sigma(\overline{v}^{j_1,\cdots,j_k}) = \overline{v}^{j_1,\cdots,j_k}$ if and only if $\sigma = 1$ since the V_{φ_i} are distinct. Moreover, $r_\mu^{\overline{v}^{j_1,\cdots,j_k}} = r_\mu$ for all $\overline{v}^{j_1,\cdots,j_k} \in \overline{V}_+$. Similarly for $\overline{w}^{j_1,\cdots,j_k} \in V_-$.

By parts (1.) and (2.) of lemma 2.4, and as $\overline{V}_\pm$ are both totally singular, $c_\mu r_\mu\left(\overline{V}_\pm\right) \subset \overline{W}(\lambda)$. By part (3.) of lemma 2.4, $\overline{\mathbf{f}}_\mu^k[c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1,\cdots,j_k})] \neq 0$. Whenever $(j_1,\ldots,j_k) \neq (j_1',\ldots,j_k')$, we have that $\overline{\mathbf{f}}_\mu^k[c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1',\cdots,j_k'})] = 0$. Therefore $\{c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1,\cdots,j_k}) \mid 1 \leq j_i \leq D\}$ is a hyperbolic basis for

$$c_\mu r_\mu\left(\overline{V}_+\right) \bigoplus c_\mu r_\mu\left(\overline{V}_-\right)$$

Lemma 4.1. *We have*

(1.) $\overline{V}_\pm \cong c_\mu r_\mu(\overline{V}_\pm)$ *as* $\mathbf{F}C_k$*-modules;*

(2.) *If* k *is even, then* C_k *stabilizes a 1-dimensional subspace of* M*;*

(3.) *If* k *is odd, then* C_k *stabilizes a* D*-dimensional subspace of* M*.*

Proof: Given the hyperbolic basis $\{\overline{v}^{j_1,\cdots,j_k}, \overline{w}^{j_1,\cdots,j_k} \mid 1 \leq j_i \leq D\}$ for $\overline{V}_+ \oplus \overline{V}_-$, it is clear that the map $\overline{v}^{j_1,\cdots,j_k} \mapsto c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k})$ is a C_k-invariant bijection. Therefore $\overline{V}_+ \cong c_\mu r_\mu(\overline{V}_+)$ as $\mathbf{F}C_k$-modules. The case for $\overline{V}_-$ follows by a similar argument, proving part (1.). Suppose that k is even and recall that $\overline{V}_{\varphi_i} \cong \overline{V}_{\varphi_j}$ and $\overline{V}_{\varphi_i^{-1}} \cong \overline{V}_{\varphi_j^{-1}}$ as $\mathbf{F}H_k$-modules. As $H_k \cong S_{n-3k-2}$ and all irreducible $\mathbf{F}S_{n-2k-2}$ are self-dual, H_k stabilizes a 1-dimensional subspace of $\overline{V}_{\varphi_i} \otimes \overline{V}_{\varphi_j}$. It follows by induction that H_k stabilizes a 1-dimensional subspace of $\overline{V}_+$. If k is odd, then the same argument leads to a D-dimensional subspace being stabilized by H_k. As E_k acts as scalars on $\overline{V}_\pm$, these spaces are, in fact, stabilized by C_k. Using part (1.), C_k stabilizes a subspace $\overline{W}_0$ of one of these dimensions in $\overline{W}(\lambda)$. Since $c_\mu r_\mu\left(\overline{V}_+\right) \bigoplus c_\mu r_\mu\left(\overline{V}_-\right)$ possesses a hyperbolic basis, $\overline{W}_0 \cap \overline{W}(\lambda)^\perp = 0$. If we let

$$M_0 = \left(\overline{W}_0 + \overline{W}(\lambda)^\perp\right) / \overline{W}(\lambda)^\perp$$

then lemma 2.3 implies that $M_0 \subset \overline{W}(\lambda)/\overline{W}(\lambda)^\perp \cong M$, hence (2.) and (3.). □

5 Proof of Theorem 1.2

We are now in a position to prove theorem 1.2:

Since M possesses a vector $\overline{v}_{\lambda_k}$ of weight λ_k by lemma 2.5, we can produce a lower bound for $\dim(M)$ as follows: Let $\mathrm{Weyl}(\overline{K})$ be the Weyl group of $\overline{K}$ and recall that ℓ is the Lie rank of $\overline{K}$. We compute $C_{\mathrm{Weyl}(\overline{K})}(\lambda_k)$ using [3, §13.1], and compute $|\lambda_k^{\mathrm{Weyl}(\overline{K})}|$, whence

$$\dim(M) \geq |\lambda_k^{\mathrm{Weyl}(\overline{K})}| = 2^k \binom{\ell}{k} \tag{1}$$

Case 1: First suppose that $k \geq n/3 - 1$. We assume that $\dim(\overline{V}) \geq 2n^4$, so $\ell \geq n^4$. Since $\dim(M) \leq \sqrt{|H|} \leq \sqrt{n!}$, Eq. (1) implies that $2^k \binom{\ell}{k} \leq \sqrt{n!}$. Trivially, $2^{n^4/2} > \sqrt{n!}$ for all $n \geq 1$, so that $k < n^4/2 \leq \ell/2$. Using the fact that $\binom{\ell}{k_1} < \binom{\ell}{k_2}$ if $k_1 < k_2 < \ell/2$, $\binom{\ell}{k}$ will be minimal when $k = n/3 - 1$ and $\ell = n^4$. Note also that $\binom{\ell}{k} = \prod_{i=1}^{k} \frac{(\ell-i+1)}{(k-i+1)} \geq \frac{(\ell-k+1)^k}{k^k}$. We have:

$$2^{n/3-1}\binom{n^4}{n/3-1} < \sqrt{n!},$$

$$2^{n/3-1}\frac{(n^4 - n/3 + 2)^{n/3-1}}{(n/3)^{n/3-1}} < (n^{1/2})^{n-1},$$

$$2^{n/3-1}(n^3-1)^{n/3-1} < n^{(n-1)/2},$$

$$n^{n-3} < n^{(n-1)/2},$$

$$n - 3 < (n-1)/2,$$

$$n < 5.$$

This contradicts our assumption that $n \geq 10$, so that $\dim(\overline{V}) \leq 2n^4$ or $k < n/3 - 1$.

Case 2: We assume that $k < n/3 - 1$ and that k is odd. Lemma 4.1 and Frobenius reciprocity imply $\dim(M) \leq D[H : C_k]$. Since $D \geq \frac{\ell}{2k}$ and $[H : C_k] = \frac{n!}{2(3^k)(n-3k-2)!}$, we have $\dim(M) \leq \frac{\ell}{2k}\,\frac{n!}{3^k(n-3k-2)!}$. Combining with (1) we get:

$$2^k\binom{\ell}{k} \leq \frac{\ell}{2k}\,\frac{n!}{2(3^k)(n-3k-2)!},$$

$$2^k\binom{\ell-1}{k-1} < \frac{n^{3k+2}}{3^{k-1}},$$

$$2^k\frac{(\ell-k+1)^{k-1}}{(k-1)^{k-1}} < \frac{n^{3(k-1)}n^5}{3^{k-1}},$$

$$2\frac{\ell-k}{k-1} < \frac{n^3}{3}n^{5/(k-1)}.$$

Observing that $(k-1)n^{5/(k-1)} < n^3$ when $k \geq 3$ and $n \geq 10$, we have

$$2\ell < \frac{n^6+2n}{3} < n^6.$$

Case 3: Finally we assume that $k < n/3 - 1$ and that k is even. Again lemma 4.1 and Frobenius reciprocity imply that $\dim(M) \le [H : C_k] \le \frac{n!}{2(3^k)(n-3k-2)!}$. Combining with (1) we get:

$$2^k \binom{\ell}{k} \le \frac{n!}{3^k(n-3k-2)!},$$
$$2^k \frac{(\ell-k+1)^k}{k^k} < \frac{n^{3k+2}}{3^k} = \frac{n^{3k}}{3^k} n^2,$$
$$2\frac{\ell-k}{k} < \frac{n^3}{3} n^{2/k},$$
$$2\ell < \frac{n^5+3n}{9}.$$

In all cases, $2\ell < n^6$, which implies that $\dim(\overline{V}) \le n^6$. This completes the proof of theorem 1.2. $\square$

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