

IRREDUCIBLE MODULES FOR CLASSICAL AND ALTERNATING GROUPS

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1. INTRODUCTION

In this article we consider the following question: Given groups $H < G$ and an absolutely irreducible $\mathbb{F}G$ -module M , when does $M \downarrow_H$ remain irreducible? This question arises naturally in the study of the maximal subgroups of the classical groups. We assume that $H \cong A_n$ or $2.A_n$, $n \geq 15$ and that G is a simply connected classical algebraic group over \mathbb{F} , where \mathbb{F} is an algebraically closed field of characteristic $p \geq 0$. In particular $G = SL(V)$, $Sp(V)$ or $Spin(V)$ where V is a vector space over \mathbb{F} . For convenience, we assume that $\dim(V)$ is even whenever $G = Spin(V)$ and $p = 2$. In this context we prove the following theorem:

Theorem 1.1. *Suppose that $M = M(\lambda)$ is a tensor indecomposable $\mathbb{F}G$ -module such that $M \downarrow_H$ is absolutely irreducible and $\dim(M) > \dim(V)$.*

1. *If $G = SL(V)$ then either*
 - (a) $\dim(V) \leq n^3$ or
 - (b) λ or $\rho(\lambda) = 3\lambda_1, 2\lambda_1 + \lambda_2, 2\lambda_1 + \lambda_\ell, \lambda_2 + \lambda_\ell$ or λ_3 and $\dim(V) \leq n^4$.
2. *If $G = Sp(V)$ or $Spin(V)$ then either*
 - (a) $\dim(V) \leq 2n^4$ or
 - (b) $\lambda = 3\lambda_1, 2\lambda_1 + \lambda_2$ or λ_3 and $\dim(V) \leq 2n^6$.

Furthermore, if $n \geq 74$ and $p \neq 2$, then $G \neq Sp(V)$.

This result strengthens the main result of [3]. Throughout we will assume that λ is the highest weight of M as an $\mathbb{F}G$ -module and that M satisfies the hypothesis of Theorem 1.1. As M is tensor indecomposable, we may replace M with a Galois conjugate and assume that λ is a p -restricted dominant weight when $p > 0$. Then we proceed as follows: Using a result of Premet, suitably extended, we find a vector of weight μ in M which has a small stabilizer W_μ in the Weyl group W of

G . Then $[W : W_\mu] \leq \dim(M)$. Conversely, we produce a large subgroup \tilde{C} of H such that $M \downarrow_{\tilde{C}}$ contains a small submodule M_0 . Using Frobenius reciprocity, we get $\dim(M) \leq \dim(M_0)[H : \tilde{C}]$. The result follows from comparing these bounds.

Our result is analogous to that achieved in [5] for H a classical group. In similar notation, they find an upper bound for $\dim(M)$ in terms of $\dim(V)$. This is in contrast to ours where we find an upper bound for $\dim(V)$ in terms of n . This reflects the fact that there is an infinite family of examples where $\dim(V) < n$ and $M \downarrow_H$ is irreducible. In particular if V is the minimal $n - 1$ or $n - 2$ dimensional module for H over \mathbb{F} , $G = Spin(V)$ or $SL(V)$ and $M = \wedge^k V$, $k < \frac{n}{2}$ then $M \downarrow_H$ is absolutely irreducible.

2. NOTATION AND PRELIMINARY RESULTS

We use the following notation throughout this article:

$\ell = \lfloor \dim(V)/2 \rfloor = \text{Lie rank of } G$ and $e = \lfloor \log_n(\ell) \rfloor$.

$\bar{e} = e$ or $\frac{3e}{2}$ if $e \leq 3$ or $e > 3$, respectively.

T is a fixed maximal torus of G .

If $G = Spin(V)$ or $Sp(V)$, we write $t = (t_1, \dots, t_\ell)$ for an arbitrary $t \in T$.

If $G = SL(V)$, we write $t = (t_1, \dots, t_{\ell+1})$ for $t \in T$, considering T as a subgroup of $GL(V)$.

$\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ are the simple roots of G with respect to T , ordered as in [2].

Φ is the root system generated by Π and Φ^+ is the set of positive roots.

$\{\lambda_1, \dots, \lambda_\ell\}$ are the fundamental dominant weights of G ordered according to the ordering of Π . For convenience we define $\lambda_0 = \lambda_{\ell+1} = 0$.

If $G = SL(V)$ then ρ is the permutation of Π and $\{\lambda_1, \dots, \lambda_\ell\}$ induced by the non-trivial symmetry of the Dynkin diagram of G .

M_μ is the μ -weight space of M .

Λ is the weight lattice of G and Λ^+ is the set of dominant weights in Λ .

$\lambda = \sum_{i=1}^{\ell} a_i \lambda_i$ is the highest weight of M .

$k_s = \sum_{i < \ell/2} i a_i$, $k_d = \sum_{i \geq \ell/2} (\ell - i + 1) a_i$ and $k_d^* = \ell + 1 - k_d$.

W is the Weyl group of G and W_μ is the stabilizer of $\mu \in \Lambda$ in W .

ϵ_i is the element of $\text{Hom}(T, F^*)$ such that $\epsilon_i(t) = t_i$.

It will be convenient to know α_i and λ_i in terms of ϵ_i :

$G = SL(V) :$

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad \lambda_i = \sum_{j=1}^i \epsilon_j$$

$G = Sp(V) :$

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & i < \ell \\ 2\epsilon_\ell & i = \ell \end{cases} \qquad \lambda_i = \sum_{j=1}^i \epsilon_j$$

$G = Spin(V)$, $\dim(V)$ odd :

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & i < \ell \\ \epsilon_\ell & i = \ell \end{cases} \qquad \lambda_i = \begin{cases} \sum_{j=1}^i \epsilon_j & i < \ell \\ \frac{1}{2} \sum_{j=1}^\ell \epsilon_j & i = \ell \end{cases}$$

$G = Spin(V)$, $\dim(V)$ even :

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & i < \ell \\ \epsilon_{\ell-1} + \epsilon_\ell & i = \ell \end{cases} \qquad \lambda_i = \begin{cases} \sum_{j=1}^i \epsilon_j & i < \ell - 1 \\ \frac{1}{2} \sum_{j=1}^\ell \epsilon_j & i = \ell - 1 \\ \frac{1}{2} \left(\sum_{j=1}^{\ell-1} \epsilon_j - \epsilon_\ell \right) & i = \ell \end{cases}$$

Note that by considering $T < GL(V)$, we have a simpler characterization of λ_i in terms of the coordinate functions ϵ_i when $G = SL(V)$.

For $\mu \in \Lambda^+$, write $\mu = \sum_{i=1}^\ell b_i \lambda_i$ and define $m_s(\mu) = \max\{0, i \mid i < \frac{\ell}{2} \text{ and } b_i \neq 0\}$, $m_d(\mu) = \max\{0, \ell - i + 1 \mid i \geq \frac{\ell}{2} \text{ and } b_i \neq 0\}$ and $m_d^*(\mu) = \ell + 1 - m_d(\mu)$.

If U is a \mathbb{F} -vector space, we write $U^{\otimes i} = \otimes_{j=1}^i U$ and define $U^{\otimes 0} = \mathbb{F}$.

$\{v_i\}$ is a basis for V such that $tv_i = t_i v_i$ if $i \leq \ell$, $tv_i = t_{\ell-i+1}^{-1} v_i$ if $\ell < i \leq 2\ell$ and $tv_{2\ell+1} = v_{2\ell+1}$ if $G = Spin(V)$ with $\dim(V)$ odd. Moreover, we assume that $\langle v_i \rangle$ is a maximal totally singular subspace of V .

If $G = SL(V)$, then $\{v_i^*\}$ is a basis for V^* such that $tv_i^* = t_i^{-1} v_i^*$. Note that $v_{\ell+1}^*$ is a vector of weight λ_ℓ .

Our initial result concerns binomial coefficients:

Lemma 2.1. *Suppose that i_1, i_2 and j are integers.*

1. *If $1 \leq i_1 \leq j$ then $\binom{j}{i_1} \geq \left(\frac{j-i_1+1}{i_1}\right)^{i_1}$.*
2. *If $e \geq 3$, $j \geq n^e$ and $\frac{n}{e} \leq i_1 \leq \frac{j}{2}$ then $\binom{j}{i_1} > \dim(M)$.*
3. *If $0 \leq i_1 \leq i_2 \leq j$ then $\frac{j!}{i_1! i_2! (j-i_1-i_2)!} \geq \binom{j}{i_1+i_2}$.*

Proof: Part (1.) follows by writing $\binom{j}{i_1} = \prod_{m=1}^{i_1} \frac{j-m+1}{i_1-m+1} \geq \prod_{m=1}^{i_1} \frac{j-i_1+1}{i_1} = \left(\frac{j-i_1+1}{i_1}\right)^{i_1}$. For (2.), $\frac{n}{e} \leq i_1 \leq \frac{j}{2}$ and $j \geq n^e$ implies that $\binom{j}{i_1} \geq \binom{j}{n/e} \geq \binom{n^e}{n/e}$. Applying (1.) to the last expression gives $\binom{j}{i_1} \geq \left(\frac{n^e - n/e + 1}{n/e}\right)^{n/e}$. As $e \geq 3$, we have

$n^e - \frac{n}{e} + 1 > \frac{n^e}{e}$ and $\frac{e-1}{e} \geq 1/2$. This gives $\binom{j}{i_1} > \left(\frac{n^e/\bar{e}}{n/\bar{e}}\right)^{n/\bar{e}} \geq n^{n/2}$. Now, M is an irreducible $\mathbb{F}H$ -module, so $\dim(M)$ is bounded above by $\chi(1)$ for some $\chi \in \text{Irr}(H)$. Using the orthogonality of such characters, $\chi(1) < \sqrt{|H|} \leq \sqrt{n!} < n^{n/2}$, hence $\dim(M) < n^{n/2}$. This proves (2.). Part (3.) follows by expanding $\binom{j}{i_1+i_2} = \frac{j!}{(i_1+i_2)!(j-i_1-i_2)!}$ and comparing denominators. In particular $(i_1+i_2)! \geq i_1!i_2!$, hence the result. \square

Lemma 2.2. *Suppose that $e \geq 3$ and $\mu \in \Lambda^+$. If M possesses a vector of weight μ , then $m_s(\mu) + m_d(\mu) < \frac{n}{e}$. Furthermore $m_d(\mu) = 0$ if $G = Sp(V)$ or $Spin(V)$.*

Proof: Write $\mu = \sum_{i=1}^{\ell} b_i \lambda_i$. If $b_i > 0$ then $W_\mu \leq W_{\lambda_i}$ which implies that $\dim(M) \geq [W : W_\mu] \geq [W : W_{\lambda_i}]$. Let $G = Sp(V)$ or $Spin(V)$ so that $[W : W_{\lambda_i}] = 2^i \binom{\ell}{i}, 2^{\ell-3}(\ell^2 - \ell)$ or $2^{\ell-1}$, the latter two holding if and only if $G = Spin(V)$ with $\dim(V)$ even and $i = \ell - 2$ or $i > \ell - 2$, respectively. If $0 \neq m_d(\mu) \leq \ell - 3$ then $b_{m_d^*(\mu)} > 0$ and $\dim(M) \geq 2^{m_d^*(\mu)} \binom{\ell}{m_d^*(\mu)} \geq 2^{m_d^*(\mu)} \geq 2^{\ell/2} \geq 2^{n^3/2} > \dim(M)$, which is a contradiction. Similar calculations handle the case $m_d(\mu) \geq \ell - 2$. Therefore $m_d(\mu) = 0$ and $m_s \neq 0$, so that $\dim(M) \geq 2^{m_s(\mu)} \binom{\ell}{m_s(\mu)} \geq \binom{\ell}{m_s(\mu)}$. Since $m_s(\mu) \leq \frac{\ell}{2}$, Part (2.) of Lemma 2.1 implies that $\binom{\ell}{m_s(\mu)} > \dim(M)$ if $m_s(\mu) \geq \frac{n}{e}$, hence $m_s(\mu) < \frac{n}{e}$.

Now suppose that $G = SL(V)$, so that $[W : W_{\lambda_i}] = \binom{\ell+1}{i}$. Since $m_d^*(\mu) = \ell + 1 - m_d(\mu)$, we have $\binom{\ell+1}{m_d^*(\mu)} = \binom{\ell+1}{m_d(\mu)}$ and $m_d(\mu) \leq \frac{\ell}{2}$. Arguing as before, $\binom{\ell+1}{m_d(\mu)} > \dim(M)$ if $m_d(\mu) \geq \frac{n}{e}$. Thus $m_d(\mu) < \frac{n}{e}$. Similarly $m_s(\mu) < \frac{n}{e}$. Thus, as $e \geq 3$, $m_s(\mu) + m_d(\mu) < \frac{\ell}{2}$. Now $W_\mu < W_{\lambda_{m_s(\mu)}} \cap W_{\lambda_{m_d^*(\mu)}}$ so that $[W : W_\mu] \geq [W : W_{\lambda_{m_s(\mu)}} \cap W_{\lambda_{m_d^*(\mu)}}] = \frac{(\ell+1)!}{m_s(\mu)!m_d(\mu)!(\ell+1-m_s(\mu)-m_d(\mu))!}$. Using Part (3.) of Lemma 2.1, we get $[W : W_\mu] \leq \binom{\ell+1}{m_s(\mu)+m_d(\mu)}$. Again, Lemma 2.1 forces $m_s(\mu) + m_d(\mu) \leq \frac{n}{e}$. \square

Note that this Lemma implies that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$.

The last lemma in this section is a technical one:

Lemma 2.3. *Suppose that $k_t \leq \frac{n}{e}$ and that $r = 5, 7, 11$ or 13 .*

1. *If $e \geq 3$ then $r \lceil \frac{k_t}{r-1} \rceil < n - 2$*
2. *If $e \geq 4$ then $r \lceil \frac{2k_t}{r-1} \rceil < n - 2$*

Proof: We will give a proof for the case $e \geq 3$ and $r = 7$, all other cases following from an identical argument. As $\bar{e} \geq e$, we may assume that $k_t \leq \frac{n}{3}$. By way of

contradiction, assume that $r \lceil \frac{k_i}{r-1} \rceil \geq n-2$. As $r=7$, $7 \lceil \frac{k_i}{r-1} \rceil < 7(\frac{k_i}{6} + 1)$. Then $n-2 < \frac{7k_i}{6} + 7 \leq \frac{7n}{18} + 7$. Thus $n < \frac{144}{11} < 14$, which is a contradiction. \square

3. WEIGHT MODULES

Let $V_{\mathbb{Z}}$ be the \mathbb{Z} -lattice generated by $\{v_i\}$ and let $\bar{V} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Moreover assume that \bar{G} is the complex version of G so that $\bar{G} = SL(\bar{V}), Sp(\bar{V})$ or $Spin(\bar{V})$. Let \mathcal{L} be the adjoint module for \bar{G} , which is a complex Lie algebra and let \mathcal{U} be the universal enveloping algebra of \mathcal{L} . We will assume that $\{\lambda_i\}$ are the fundamental dominant weights for \mathcal{U} , that Π is the corresponding set of simple roots and that Φ is the root system of \mathcal{U} .

Let $\{x_{\alpha}, \alpha \in \Phi; h_j, 1 \leq j \leq \ell\}$ be a Chevalley basis for \mathcal{L} . The Kostant \mathbb{Z} -form $\mathcal{U}_{\mathbb{Z}}$ is generated by $\{x_{\alpha}^m/m! \mid \alpha \in \Phi, m \in \mathbb{Z}^+\}$. Moreover the hyperalgebra \mathcal{H} corresponding to G is defined to be $\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$.

Define $v_{\lambda_i} = v_1 \wedge \cdots \wedge v_i$, which is naturally an element of $\wedge^i V_{\mathbb{Z}}$ as well as of $\wedge^i V$. For convenience we identify these two elements. Set $\bar{v}_{\lambda_i} = v_{\lambda_i} \otimes_{\mathbb{Z}} 1_{\mathbb{C}}$. Then \bar{v}_{λ_i} is a vector of weight λ_i unless $G = Spin(V)$ and $i \geq \ell - 1$.

Lemma 3.1. *$\wedge^i V$ and $\wedge^{\dim(V)-i} V^*$ contain a highest weight vector of weight λ_i unless $G = Spin(V)$ and $i \geq \ell - 1$.*

Proof: By [1, §§15.3, 17.3 and 19.5], $(\wedge^i \bar{V}) \cap \mathcal{K}$ is the $\mathbb{F}\mathcal{U}$ -module with highest weight λ_i , where \mathcal{K} is the intersection of kernels of contraction maps. Moreover, as $\bar{v}_{\lambda_i} \in \mathcal{K}$, it is a highest weight vector. If $\mathcal{U}_{\mathbb{Z}}$ is the Kostant \mathbb{Z} -form of \mathcal{U} , it follows that $\mathcal{U}_{\mathbb{Z}} v_{\lambda_i} \otimes_{\mathbb{Z}} \mathbb{F}$ is the Weyl module for G of highest weight λ_i . Since $v_{\lambda_i} \in \wedge^i V$, $\wedge^i V$ contains a highest weight vector of weight λ_i , and we have the first claim. The second follows from the $\mathbb{F}SL(V)$ -isomorphism $\wedge^i V \cong \wedge^{\dim(V)-i} V^*$. \square

As a direct result of this lemma we have:

Proposition 3.2. *M occurs as a section of*

1. $V^{\otimes k_s} \otimes V^{*\otimes k_d}$ if $G = SL(V)$ or
2. $V^{\otimes k_s}$ if $G = Sp(V)$ or $Spin(V)$.

Proof: First note that if v_{μ_1} and v_{μ_2} are highest weight vectors for G of weights μ_1 and μ_2 , respectively, then $v_{\mu_1} \otimes v_{\mu_2}$ is a highest weight vector of weight $\mu_1 + \mu_2$. In particular, if \mathcal{H}^+ are the elements of \mathcal{H} corresponding to positive root elements and

$x \in \mathcal{H}^+$, then $x(v_{\mu_1} \otimes v_{\mu_2}) = (xv_{\mu_1}) \otimes v_{\mu_2} + v_{\mu_1} \otimes (xv_{\mu_2}) = 0$. From the universal property of Weyl modules it is enough to show that the modules above contain a highest weight vector of weight λ . Let $I = \{i | a_i \neq 0\}$ and write $M_i = \wedge^i V$ and $M_i^! = \wedge^i V^*$. Consider the module $\otimes_I M_i^{a_i}$. From the above, the highest weight of this module is $\sum a_i \lambda_i = \lambda$. The claim follows by recalling the definition of k_s and k_d , and replacing $M_{\ell-i+1}$ with $M_i^!$ when $G = SL(V)$ and $i \geq \ell/2$. \square

The next lemma extends the results of Premet [6] and Suprunenko [7].

Lemma 3.3. *Let $\mu \neq 0$ be a subdominant weight of λ . If $G = Sp(V)$ and $p = 2$, assume further that $\mu = \sum b_i \lambda_i$ with $b_\ell = 0$ and that α_ℓ is not in the support of $\lambda - \mu$. Then M possesses a vector of weight μ .*

Proof: By Premet's result [6], we need to only consider the case where $p = 2$ and $G = Sp(V)$. Define $\Pi_0 = \{\alpha_1, \dots, \alpha_{\ell-1}\}$ and let Φ_0 be the root system generated by Π_0 . Suppose \mathcal{U}_0 is the subalgebra of \mathcal{U} corresponding to Π_0 , then $(\mathcal{U}_0)_{\mathbb{Z}}$ is generated by $\{x_{\alpha}^m / m! \mid \alpha \in \Phi_0, m \in \mathbb{Z}^+\}$. Note that \mathcal{U}_0 is the universal enveloping algebra of a complex Lie algebra of type $A_{\ell-1}$. Write $\Phi^+ = \{\beta_1, \dots, \beta_j\}$ such that $\beta_i \in \Phi_0^+$ if $i \leq |\Phi_0^+|$.

Now, M is a homomorphic image of $\mathcal{U}_{\mathbb{Z}} v_{\lambda} \otimes_{\mathbb{Z}} \mathbb{F}$ where v_{λ} is a highest weight vector of weight λ . Moreover, M_{μ} is the image of the space $\langle \left(\prod x_{-\beta_i}^{m_i} / m_i! \right) v_{\lambda} \mid \lambda - \mu = \sum m_i \beta_i \rangle$. From our restrictions on λ and μ , it is clear that $m_i = 0$ if $\beta_i \notin \Phi_0$. In particular M_{μ} is a weight space contained in the image of $(\mathcal{U}_0)_{\mathbb{Z}} v_{\lambda} \otimes_{\mathbb{Z}} \mathbb{F}$. By [7], this space is non-trivial, hence M contains a vector of weight μ . \square

Lemma 3.4. *M possesses a vector of weight $\lambda_{k_s} + \lambda_{k_d^*}$.*

Proof: Recall that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$ and that $\lambda_0 = \lambda_{\ell+1} = 0$. By the previous lemma, it is enough to show that $\lambda_{k_s} + \lambda_{k_d^*}$ is a subdominant weight of λ and that $\lambda - \lambda_{k_s} - \lambda_{k_d^*}$ is supported on $\{\alpha_1, \dots, \alpha_{\ell-1}\}$ when $G = Sp(V)$ and $p = 2$. Using the descriptions from § 2 write $\lambda = b_i \epsilon_i$ and note that $b_1 \geq b_2 \geq \dots \geq b_m$ where $m = \max\{i | b_i \neq 0\}$. We will consider the case $G = Sp(V)$ or $Spin(V)$ first. Here $\sum b_i = k_s$ so that $k_s - m \geq 0$. We will induct on $k_s - m$. If $k_s - m = 0$ then $m = k_s$ and $\lambda = \lambda_{k_s}$, so the claim follows in this case. Assume $k_s - m > 0$. This implies that $b_i > 1$ for some i , so pick m_0 be maximal subject to $b_{m_0} > 1$. By Lemma 2.2, $m < k_s < \frac{n}{\epsilon} < \ell - 1$ so that $\alpha = \epsilon_{m_0} - \epsilon_{m_0+1}$ is a positive

root. Moreover, using the descriptions from §2, $\lambda - \alpha = \sum b_i \epsilon_i - \epsilon_{m_0} + \epsilon_{m+1} = \mu$ is a dominant weight. As $m+1 < \ell$, $\lambda - \mu$ is supported on $\{\alpha_1, \dots, \alpha_{\ell-1}\}$. By induction, λ_{k_s} is a subdominant weight of μ , hence of λ . Moreover $\lambda - \lambda_{k_s}$ contains no α_ℓ term. Thus the result holds in this case.

Now suppose that $G = SL(V)$. Here we write $\lambda = \mu_1 + \mu_2$ where $\mu_1 = \sum_{i < \ell/2} b_i \lambda_i$ and $\mu_2 = \sum_{i \geq \ell/2} b_i \lambda_i$. The previous argument shows that λ_{k_s} is a subdominant weight of μ_1 and that λ_{k_d} is a subdominant weight of $\rho(\mu_2)$. Thus $\rho(\lambda_{k_d}) = \lambda_{k_d^*}$ is a subdominant weight of μ_2 and $\lambda_{k_s} + \lambda_{k_d^*}$ is a subdominant weight of $\mu_1 + \mu_2 = \lambda$. \square

Note that Lemma 2.2 implies that $k_s + k_d \leq \frac{n}{e}$.

4. THE SUBGROUP \tilde{C} AND SUBMODULE M_0

Assume that $r \neq p$ is an odd prime and that $k > 0$ is an integer such that $kr < n-2$. Let H_1 be a subgroup of $H/Z(H)$ such that $H_1 \cong S_{n-2}$ and let $E_{r,k}$ be a subgroup of H_1 generated by k disjoint r -cycles. Then $E_{r,k}$ is an elementary abelian r -group of rank k . Let $C = C_{H_1}(E_{r,k})$ and $N = N_{H_1}(E_{r,k})$. Then $C = E_{r,k} C_1$ where $C_1 \cong S_{n-kr-2}$ and $N = N_1 C_1$ where $N_1 \cong \mathbb{Z}_r : \mathbb{Z}_{r-1} \wr S_k$ and $[N_1, C_1] = 1$. The action of N on $E_{r,k}$ is easy to see, in particular if $1 \neq \sigma \in E_{r,k}$ then $\sigma^N = \sigma^{N_1}$ and $\sigma^i \in \sigma^N$ for $(i, r) = 1$. Moreover, if σ is the product of m disjoint r -cycles, then $C_N(\sigma) \cong \mathbb{Z}_r \wr S_m \times \mathbb{Z}_r : \mathbb{Z}_{r-1} \wr S_{k-m} \times C_1$. Thus $|\sigma^N| = (r-1)^m \binom{k}{m}$. As $\binom{k}{m} \geq k$ for $0 < m < k$ and $(r-1)^{k-1} \geq k$ for all r, k as above, we have that $|\sigma^N| = (r-1)^m \binom{k}{m} \geq (r-1)k$.

Let $\tilde{E}_{r,k}, \tilde{C}, \tilde{C}_1, \tilde{N}$ and \tilde{N}_1 be the full preimage in H of $E_{r,k}, C, C_1, N$ and N_1 , respectively. Since $r > 2$, we may identify $E_{r,k} = O_r(\tilde{E}_{r,k})$ and note that the action of \tilde{N} on $E_{r,k}$ is the same as the action of N on $E_{r,k}$, so the above observations still hold. Define $E_{r,k}^* = \text{Hom}(E_{r,k}, \mathbb{F}^*)$. Then the \tilde{N} action on this subgroup is just the dual of its action on $E_{r,k}$. If $1 \neq \varphi \in E_{r,k}^*$, we abuse notation and write φ^i for the element $\varphi^i : \sigma \mapsto \varphi(\sigma^i)$ of $E_{r,k}^*$.

Now, $V \downarrow_{E_{r,k}} = \bigoplus_{\varphi \in E_{r,k}^*} V_\varphi$ where V_φ is the homogeneous component of φ on V .

Moreover, each V_φ is an $\mathbb{F}\tilde{C}$ -module.

Lemma 4.1. *Assume that $V_{\varphi_1}, V_{\varphi_2} \neq 0$. Then*

1. *If $\varphi_1 \in \varphi_2^{\tilde{N}}$ then $V_{\varphi_1} \cong V_{\varphi_2}$ as $\mathbb{F}\tilde{C}_1$ -modules.*

2. If $\varphi_1 \neq 1$ then V_{φ_1} is a self-dual $\mathbb{F}\tilde{C}_1$ -module.
3. $V_{\varphi_1} \perp V_{\varphi_2}$ if $\varphi_1 \neq \varphi_2^{-1}$.
4. If $\varphi_i \neq 1$ then V_{φ_i} is totally singular.

Proof: From the earlier comments, $\varphi_2^{\tilde{N}} = \varphi_2^{\tilde{N}_1}$ and $[\tilde{C}_1, \tilde{N}_1] = 1$. This implies that $V_{\varphi_1} \cong V_{\varphi_2}$ as $\mathbb{F}\tilde{C}_1$ -modules. Let $\sigma \in \tilde{N}_1$ be an element which interchanges φ_1 and φ_1^{-1} . As $(V_{\varphi_1})^* \cong V_{\varphi_1^{-1}}$ as an $\mathbb{F}E_{r,k}$ -module, $V_{\varphi_1} \oplus (V_{\varphi_1})^*$ is a module for $(E_{r,k}\langle\sigma\rangle)\tilde{C}_1$. As σ centralizes \tilde{C}_1 , $V_{\varphi_1} \cong V_{\varphi_1}^*$ as $\mathbb{F}\tilde{C}_1$ -modules, proving (2.). We may assume that $G = Sp(V)$ or $Spin(V)$ for the remaining parts. Let \mathbf{f} be the non-degenerate, G -invariant, bilinear form on V and let Q be the associated quadratic form when $G = Spin(V)$. Suppose $u_i \in V_{\varphi_i}$ so that $gu_i = \varphi_i(g)u_i$ for all $g \in E_{r,k}$. Then $\mathbf{f}(u_1, u_2) = \mathbf{f}(gu_1, gu_2) = \varphi_1(g)\varphi_2(g)\mathbf{f}(u_1, u_2)$ for all $g \in E_{r,k}$. If $\mathbf{f}(u_1, u_2) \neq 0$ then $\varphi_1 = \varphi_2^{-1}$, hence (3.) is true. (4.) follows from (3.) by taking $\varphi_1 = \varphi_2$, unless $G = Spin(V)$ and $p = 2$. In this case, however, $Q(u_1) = Q(gu_1) = \varphi_1^2(g)Q(u_1)$. As $r > 2$, $\varphi_1^2 = 1$ if and only if $\varphi_1 = 1$. Thus $Q(u_1) = 0$, proving (4.). \square

From this point forward, we will assume that $r = 5, 7$ or 11 and $(r, p) = 1$. Also, we assume that $e \geq 3$ if $G = SL(V)$ and that $e \geq 4$ if $G = Sp(V)$ or $Spin(V)$. Let $k_t = k_s + k_d$. By the remark following Lemma 3.4, $k_t < \frac{n}{e}$. By Lemma 2.3, if $G = SL(V)$, then $r \lceil \frac{k_t}{r-1} \rceil < n - 2$. Similarly, if $G = Sp(V)$ or $Spin(V)$, then $r \lceil \frac{2k_t}{r-1} \rceil < n - 2$. Therefore we can take $k = \lceil \frac{k_t}{r-1} \rceil$ if $G = SL(V)$ and $k = \lceil \frac{2k_t}{r-1} \rceil$ if $G = Sp(V)$ or $Spin(V)$.

Let $1 \neq \varphi \in E_{r,k}^*$ such that $V_\varphi \neq 0$. Set $\varphi_1 = \varphi$ and write $\varphi^{\tilde{N}} = \{\varphi_1, \dots, \varphi_m\}$ such that $\varphi_i = \varphi_{m-i+1}^{-1}$, so that $\bigoplus_{i=1}^{m/2} V_{\varphi_i}$ is a totally singular subspace. Recall from the initial remarks in this section that $m \geq (r-1)k$. Hence $m \geq k_t$ when $G = SL(V)$ and $m/2 \geq k_t$ when $G = Sp(V)$ or $Spin(V)$.

By replacing T with a suitable conjugate, we may assume that each V_{φ_i} with $i \leq m/2$ has a basis which is a subset of $\{v_j\}$ where $j \leq \dim(V)/2$. Moreover, when $G = SL(V)$, we may assume that $E_{r,k} < T$. Write $V^* \downarrow_{E_{r,k}} = \bigoplus_{\psi \in E_{r,k}^*} V'_\psi$. Because $t \in T$ acts as t^{-1} on V^* and $E_{r,k} < T$, we have that $V'_\psi \cong V_{\psi^{-1}}$ as $\mathbb{F}E_{r,k}$ -modules. In particular, if $\{v_{i_1}, \dots, v_{i_j}\}$ is a basis for V_ψ then $\{v_{i_1}^*, \dots, v_{i_j}^*\}$ is a basis for V'_ψ . Moreover, $V'_\psi \cong V_\psi$ as $\mathbb{F}\tilde{C}_1$ -modules as V_ψ is a self-dual $\mathbb{F}\tilde{C}_1$ -module.

Define $U_i = V_{\varphi_i}$ if $i \leq k_s$ and $U_i = V'_{\varphi_i}$ if $k_s < i \leq k_s + k_d$.

Lemma 4.2. $M \downarrow_{\tilde{\mathbb{C}}}$ possesses a submodule isomorphic to $U = \bigotimes_{i=1}^{k_i} U_i$.

Proof: From Proposition 3.2, M occurs as a section of $V^{\otimes k_s} \otimes V^{*\otimes k_d}$. Thus it contains an $\mathbb{F}G$ -submodule \overline{M} such that M is a homomorphic image of \overline{M} . Let ω be the corresponding homomorphism of \overline{M} . Recall that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$.

Notice that $\otimes_{i \leq k_s} U_i \subset V^{\otimes k_s}$ and $\otimes_{i > k_s} U_i \subset V^{*\otimes k_d}$, so $U \subset V^{\otimes k_s} \otimes V^{*\otimes k_d}$. Each basic tensor in $V^{\otimes k_s} \oplus V^{*\otimes k_d}$ is a weight vector for T . Moreover, U has a basis consisting of basic tensors. Since the tensor factors in U are distinct, every basic tensor in U corresponds to a distinct weight. Pick $u_i \in U_i$ such that $u_i \in \{v_j\}$ or $u_i \in \{v_j^*\}$ and let $u = \otimes_{i=1}^{k_i} u_i$. It is clear that u_i is a weight vector. Because of our definition of U_i , in particular because the v_i and v_j^* that occur all have distinct indices, there is an element $w \in W$ such that $u_i = v_i$ if $i \leq k_s$ and $u_i = v_{i+1-i}^*$ if $k_s < i \leq k_s + k_d$. This means that u^w is a vector of weight $\lambda_{k_s} + \lambda_{k_d}^*$. Thus every element of U is conjugate, under W , to u^w .

Now recall from [3, §2] that the group $\mathcal{S} = \mathcal{S}_{k_s} \times \mathcal{S}_{k_d}$ acts on $V^{\otimes k_s} \otimes V^{*\otimes k_d}$ by permuting the tensor factors. A straightforward calculation, which we omit, shows that $\langle \mathbb{F}\mathcal{S}u^w \rangle$ is the entire $\lambda_{k_s} + \lambda_{k_d}^*$ weight space of $V^{\otimes k_s} \otimes V^{*\otimes k_d}$. Using Lemma 3.4, we see that $\omega(\langle \mathbb{F}\mathcal{S}u^w \rangle \cap \overline{M})$ must be a non-trivial subspace of M . Pick $\sigma \in \mathbb{F}\mathcal{S}$ such that $\sigma u^w \in \overline{M}$ and $\omega(\sigma u^w) \neq 0$. We claim that $\sigma U \subset \overline{M}$ and that $\omega|_{\sigma U}$ is injective.

First, \overline{M} is a $\mathbb{F}G$ -module containing σu^w . Thus if w' is any element of W , then $(\sigma u^w)^{w'} = \sigma u^{ww'}$. This is because the action of G commutes with the action of \mathcal{S} . Since $u \in U$ was arbitrary and all basic vectors of U are W -conjugate to u , $\sigma U \subset \overline{M}$. Now, suppose that $u' \neq u$ is any basic tensor in U . Then u' and u are vectors of distinct weights, and so are σu and $\sigma u'$. In particular, σU has a basis consisting of vectors of distinct weights. Moreover, their images under ω remain distinct, and span a subspace of M of dimension $\dim(U)$. Thus $\omega|_{\sigma U}$ is injective. Note that this implies that σU is isomorphic to U as an $\mathbb{F}\tilde{\mathbb{C}}$ -module. As σU is a $\mathbb{F}\tilde{\mathbb{C}}$ -module and ω is a homomorphism, we have that $\omega(\sigma U)$ is a $\mathbb{F}\tilde{\mathbb{C}}$ -submodule of M which is isomorphic to σU , hence to U . \square

Lemma 4.3. *Let $\delta = 0$ if k_t is even and $\delta = 1$ if k_t is odd. Then M possesses a $\mathbb{F}\tilde{C}$ -submodule M_0 such that $\dim(M_0) \leq \ell^\delta$.*

Proof: Note that $k_t \geq 2$ so that $\dim(U_i) \leq \ell$. From Lemma 4.1, V_{φ_i} is self-dual as an $\mathbb{F}\tilde{C}_1$ -module. Similarly, V'_{φ_i} is also self-dual. Moreover, $V_{\varphi_i} \cong V'_{\varphi_i}$ as $\mathbb{F}\tilde{C}_1$ -modules. We induct on $|k_t|$. If $k_t = 1$ then the result is immediate. If $|k_t| = 2$ then $U = U_1 \otimes U_2$ where U_1 and U_2 are equivalent, self-dual $\mathbb{F}\tilde{C}_1$ -modules. Hence \tilde{C}_1 stabilizes a 1-dimensional subspace \overline{M}_0 of U . If $k_t > 2$ then write $U = U_1 \otimes U_2 \otimes U'$. As U' has $k_t - 2$ tensor factors, by induction $U' \downarrow_{\tilde{C}_1}$ has a subspace \overline{M}_1 of dimension at most ℓ^δ . As $E_{r,k}$ acts as scalars on U , \tilde{C} stabilizes $\overline{M}_0 \otimes \overline{M}_1$. Take M_0 to be the image of $\overline{M}_0 \otimes \overline{M}_1$ in M under the homomorphism from Lemma 4.2. \square

5. PROOF OF THEOREM 1.1

In this section we begin the proof of Theorem 1.1. Using Lemma 4.3 and Frobenius reciprocity, we have $\dim(M) \leq \ell^\delta [H : \tilde{C}]$. In particular $\dim(M) \leq \ell^\delta \frac{n!}{r^k(n-rk-2)!}$. And by Lemma 3.4, $[W : W_{\lambda_{k_s} + \lambda_{k_d}}] \leq \dim(M)$. By combining these inequalities, we have:

$$(1) \quad [W : W_{\lambda_{k_s} + \lambda_{k_d}}] \leq \ell^\delta \frac{n!}{r^k(n-rk-2)!}$$

Throughout this section we will assume that $r = 7$ if $p = 5$ and $r = 5$ otherwise.

Case 1. $G = SL(V)$ and $k_t \geq 13$. Using Part (3.) of Lemma 2.1, $[W : W_{\lambda_{k_s} + \lambda_{k_d}}] \geq \binom{\ell+1}{k_s+k_d} = \binom{\ell+1}{k_t}$. Using Part (1.) of Lemma 2.1, $\binom{\ell+1}{k_t} \geq \left(\frac{\ell-k_t+2}{k_t}\right)^{k_t}$. As in the previous case, $\ell - k_t + 2 < \frac{\ell}{2}$. Then using Inequality 1, we get:

$$(2) \quad \left(\frac{\ell}{2k_t}\right)^{k_t} < \ell^\delta \frac{n!}{r^k(n-rk-2)!}$$

So $\ell^{k_t-\delta} < \frac{n!}{r^k(n-rk-2)!} (2k_t)^{k_t} < n^{rk+2} (2k_t)^{k_t}$. As before $\ell \geq n^3$ so that $1 < n^{rk+2-3k_t+3\delta} (2k_t)^{k_t}$. When $r = 5$, $rk \leq 5\frac{k_t+3}{4}$ and $rk+2-3k_t+3\delta \leq \frac{35-7k_t}{4}$. When $r = 7$, $rk \leq 7\frac{k_t+5}{6}$ and $rk+2-3k_t+3\delta \leq \frac{65-11k_t}{6}$. In both cases $rk+2-3k_t+3\delta < -k_t$ when $k_t \geq 14$. As $k_t < \frac{n}{\ell}$, $n^{rk+2-3k_t+3\delta} k_t^{k_t} < 1$ if $k_t \geq 13$, hence the result in this case. \square

Case 2. $G = SL(V)$ and $k_t \leq 12$. For $k_t \geq 6$, we explicitly compute k and δ for $r = 5, 7$ and 11 . We then evaluate Inequality 2 for these values. In all cases, we have a contradiction for at least two primes. For $k_t = 3$ or $k_t = 4$, we assume $\ell \geq n^4$. As $8 < n$, we may use $r = 3$ as well as $r = 5$. Again using Inequality 2

with these primes, we get a contradiction. For $k_t = 5$, $11 < n$ so we may use $r = 3$ as well as $r = 7$. Here we use Inequality 1, in particular $\binom{\ell+1}{5} > \frac{\ell^\delta}{9} \frac{n!}{(n-11)!}$. Finally, the results of [4] eliminate the possibility $k_t = 2$. This completes Part (1.) of Theorem 1.1. \square

Case 3. $G = Sp(V)$ or $Spin(V)$ and $k_t \geq 16$. In this case, $k_t = k_s$ so that $[W : W_{\lambda_{k_s} + \lambda_{k_s^*}}] = 2^{k_t} \binom{\ell}{k_t}$. By Part (1.) of Lemma 2.1, $2^{k_t} \binom{\ell}{k_t} \geq 2^{k_t} \left(\frac{\ell - k_t + 1}{k_t}\right)^{k_t}$. Since $\ell \geq n^4$ and $k_t < \frac{n}{e}$, $\ell - k_t + 1 > \frac{\ell}{2}$. Thus $2^{k_t} \binom{\ell}{k_t} > \left(\frac{\ell}{k_t}\right)^{k_t}$. Using Inequality 1, we get:

$$(3) \quad \left(\frac{\ell}{k_t}\right)^{k_t} < \ell^\delta \frac{n!}{r^k (n - rk - 2)!}$$

Then $\ell^{k_t - \delta} < \frac{n!}{r^k (n - rk - 2)} k_t^{k_t} < n^{rk+2} k_t^{k_t}$. As $\ell \geq n^4$, we have $n^{4k_t - 4\delta} < n^{rk+2} k_t^{k_t}$ and $1 < n^{rk+2-4k_t+4\delta} k_t^{k_t}$. When $r = 5$, $rk \leq 5 \frac{k_t+1}{2}$ and $rk + 2 - 4k_t + 4\delta \leq \frac{13-3k_t}{2}$. When $r = 7$, $rk \leq 7 \frac{k_t+2}{3}$ and $rk + 2 - 4k_t + 4\delta \leq \frac{32-5k_t}{3}$. In both cases $rk + 2 - 4k_t + 4\delta < -k_t$ when $k_t \geq 16$. As $k_t < \frac{n}{e}$, $n^{rk+2-4k_t+4\delta} k_t^{k_t} < 1$ if $k_t \geq 16$, hence the result in this case. \square

Case 4. $G = Sp(V)$ or $Spin(V)$ and $1 < k_t \leq 15$. When $k_t = 4, 6$ or $k_t \geq 8$, we explicitly compute k and δ for $r = 5, 7, 11$ and 13 . We then evaluate Inequality 3 for these values. In all cases we end up with a contradiction for at least two distinct values of r . We omit these calculations as they are straightforward. The remaining possibilities for k_t , namely $k_t = 2, 3, 5$ and 7 require subtler arguments. When $k_t = 2$, it is clear that $8 < n$. Here we may take $r = 3$ or $r = 5$. These choices for r lead to a contradiction as before. When $k_t = 5$, we take $r = 5$ and this gives a contradiction, except when $p = 5$. For this case we take $r = 11$ and work with Inequality 1. In particular, we have $2^5 \binom{\ell}{5} < \ell \frac{n!}{11(n-13)!}$. As $\ell \geq n^4$, this inequality leads to a contradiction. When $k_t = 7$, $n \geq 42$. Again $19, 21 < n$ so we may take $r = 17$ or $r = 19$ in this case. Here $k = 1$ and we have a contradiction. Finally if $k_t = 3$ then $n \geq 18$. If $\ell > n^6$ then we get a contradiction using $r = 3$ and $r = 7$. This completes Part (2.) of Theorem 1.1. \square

Finally suppose that $G = Sp(V)$, $p > 2$ and that V is a $\mathbb{F}A_n$ -module. As no irreducible $\mathbb{F}A_n$ -module has an invariant symplectic form, V must be reducible. Let V_0 be an irreducible submodule. Then, as before, there can be no invariant symplectic form on V_0 , hence it is totally singular. Thus H is contained in a

parabolic subgroup of G , hence of $GL(M)$. As this is not the case, $V \downarrow_H$ must have one constituent which is a faithful $\mathbb{F}_2 A_n$ -module. By [8], $\dim(V)$ is divisible by $2^{(n-s-1)/2}$ where s is the number of 1's in the binary expansion of n . In particular, $\dim(V) > \frac{2^{(n-3)/2}}{\sqrt{n}}$. By Parts (1.) and (2.) of Theorem 1.1, we have that $\dim(V) \leq 2n^6$. Comparing these bounds we see that $n < 74$. This completes the proof of Theorem 1.1. \square

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