Converses, Contrapositives and Proof by the Contrapositive

The converse of the implication $P \Rightarrow Q$ is the reverse implication $Q \Rightarrow P$. It is very important to realize that these two implications are not logically equivalent.

**Example 1:** From calculus, if $f(x)$ is continuous on $[a, b]$ then the Riemann integral $\int_a^b f(x) \, dx$ exists. But the converse of this statement: if the Riemann integral $\int_a^b f(x) \, dx$ exists then $f(x)$ is continuous on $[a, b]$ is not true. There are functions which are not continuous on $[a, b]$ which are integrable over $[a, b]$. In particular, if $f(x)$ is a function with a finite number of discontinuities on $[a, b]$ it will be integrable over $[a, b]$.

Sometimes replacing an implication by its contrapositive leads to an easier implication to prove. We can also form the contrapositive of a biconditional: if $P ⇔ Q$ then $¬Q ⇔ ¬P$. These two biconditionals are also logically equivalent.

**Example 2:** Another example from calculus: if $f(x)$ is differentiable at $a$ then $f(x)$ is continuous at $a$.

(a.) The converse of this statement is: if $f(x)$ is continuous at $a$ then it is differentiable at $a$. This statement is false, the classic example being $f(x) = |x|$ at $a = 0$.

(b.) The contrapositive of this statement is: if $f(x)$ is not continuous at $x = a$ then $f(x)$ is not differentiable at $a$. This statement is true as it is the contrapositive of a true statement.

**Example 3:** Show that if $x \neq 5$ then $x^2 - 10x + 25 \neq 0$ is always true.

**Proof:** Let $P$ be the statement $x \neq 5$ and $Q$ be the statement $x^2 - 10x + 25 \neq 0$, then we wish to show that $P \Rightarrow Q$ is always true. We will do this by showing that the contrapositive is always true. Namely, $¬Q \Rightarrow ¬P$.

The contrapositive is: if $x^2 - 10x + 25 = 0$ then $x = 5$. Using the chain of biconditionals:

$$x^2 - 10x + 25 = 0 ⇔ (x - 5)^2 = 0 ⇔ x - 5 = 0 ⇔ x = 5$$

we see that $¬Q \Rightarrow ¬P$ is always true. We actually have proved a stronger statement, that $¬Q ⇔ ¬P$.

As we see in this example, sometimes it is easier to prove a stronger statement than what is being asked.