1.4 Conditional Probability and Independence

1. An Example

Suppose that a 6-sided die is rolled, what is the probability that the result is a 1? From our earlier results, if $A$ is the event that a 1 is rolled, then $P(A) = \frac{1}{6}$. Suppose that $B$ is the event that an odd number is rolled. Here we have $P(B) = \frac{1}{2}$. Now suppose that we knew that an odd number was rolled (ie. event $B$ has happened). What then is the probability that a 1 was rolled (ie. event $A$ has happened)? In this case the probability is $\frac{1}{3}$ as there are three odd numbers on the die. This is an example of conditional probability. We say that the probability of $A$ given $B$ is $\frac{1}{3}$ and we write $P(A|B)$ for this probability.

2. Definitions

Suppose that $\Omega$ is a probability distribution space with distribution $P$ and suppose that $A$ and $B$ are both events. Then the **conditional probability of $A$ given $B$** is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is the probability that $A$ has occurred given the knowledge that $B$ has occurred. By rearranging we also have:

$$P(A \cap B) = P(A|B)P(B)$$

**Example:** A box contains 3 white balls and 2 black balls. Two balls are drawn without replacement. What is the probability that the second ball is black given that the first is black? Let $A$ be the event that the first ball is black and let $B$ be the event that the second ball is black. We wish to find $P(B|A)$. First $P(A) = \frac{2}{5}$ given that there are 2 black balls out of a total of 5. Next $A \cap B$ represents that both balls are black. There are $5 \times 4 = 20$ possible ordered pairs of balls drawn. Of these, two correspond to $A \cap B$. Thus $P(A \cap B) = \frac{2}{20}$. So:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{20}}{\frac{2}{5}} = \frac{1}{4}$$

3. Tree Diagrams

Suppose that we wish to consider a sequence of events and determine an ultimate probability of some event happening. We can do this using a tree diagram along with conditional probabilities.

**Example:** Suppose that we have two boxes. The first box has 2 gold coins and a silver coin and the second box has 3 gold coins and two silver coins. If a box is chosen at random and a coin drawn, what is the probability that it is a gold coin?
The probability is then given by summing all the paths that end at a gold coin:

\[ \frac{1}{6} + \frac{1}{6} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{19}{30} \]

4. Average Conditional Probability

Suppose that we want to know the probability that event \( A \) happens, but \( A \) generally occurs after some possible events, say \( B_1, B_2 \) or \( B_n \). Then we can use the average conditional probability to find \( A \).

**Theorem 4.1.** Suppose that \( B_1, B_2, \ldots, B_n \) is a partition of \( \Omega \). Then for any event \( A \),

\[
P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)
\]

**Proof:** Note that \( A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \) and the sets \( A \cap B_i \) are pairwise-disjoint. Thus

\[
P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)
\]

The result follows by replacing \( P(A \cap B_i) \) with \( P(A|B_i)P(B_i) \). \( \square \)

**Example:** Two cards are drawn without replacement from a deck of 52 playing cards. What is the probability that the second card is a face card? Let \( A \) be the event that the second card is a face card, let \( B_1 \) be the event that the first card is a face card and let \( B_2 \) be the event that the first card is not a face card. Note that \( \Omega = B_1 \cup B_2 \) and these events are disjoint. Now \( P(B_1) = \frac{12}{52} \) and \( P(B_2) = \frac{40}{52} \), moreover \( P(A|B_1) = \frac{11}{51} \) and \( P(A|B_2) = \frac{12}{51} \). So:

\[
P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = \frac{11}{51} \cdot \frac{12}{52} + \frac{12}{51} \cdot \frac{40}{52} = \frac{612}{2652} = \frac{3}{13}
\]

5. Independence

We say two events, \( A \) and \( B \) are independent if the occurrence of one does not affect the occurrence of the other. That is, if we know that \( A \) has occurred, then \( B \) still occurs with its usual probability. Similarly, if \( B \)
has occurred, then $A$ occurs with its usual probability. In particular:

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

This leads to a useful formula which is also our definition of independence:

$$P(A \cap B) = P(A)P(B)$$

if and only if $A$ and $B$ are independent events. Make sure that you realize that the idea of independent events is distinct from the idea of exclusive or disjoint events. In the latter case, their intersection is the empty set.

**Example:** Suppose a box contains three white balls and three black balls. The white balls are labeled with 1,2 and 3 respective, the same for the black balls. Suppose that a ball is drawn and it is noted that it is white, what is the probability that it is labeled with a 1? Are these events independent? Let $A$ be the event that a white ball is drawn and let $B$ be the event that a ball labeled 1 is drawn. Our problem is to find $P(B|A)$. As $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{6}$,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3} = P(B)$$

Note that $P(A \cap B) = \frac{1}{6} = P(A)P(B)$, so these events are independent.