Math 530

Infinite Series and Geometric Distributions

1. Geometric Series

Suppose that $|x| < 1$, then the geometric series in $x$ is absolutely convergent:

$$
\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}
$$

Here is how we find this value: Let

$$
S_0 = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots
$$

then

$$
xS_0 = x \sum_{k=0}^{\infty} x^k = x + x^2 + x^3 + \cdots
$$

so

$$
S_0 - xS_0 = 1
$$

$$
S_0(1-x) = 1
$$

$$
\sum_{k=0}^{\infty} x^k = S_0 = \frac{1}{1-x}
$$

In fact we can use this method to find the tail sums of this series:

$$
\sum_{k=m}^{\infty} x^k - x \sum_{k=m}^{\infty} x^k = x^m
$$

so

$$
\sum_{k=m}^{\infty} x^k = \frac{x^m}{1-x}
$$

Now consider another sum which converges absolutely for $|x| < 1$:

$$
S_1 = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots
$$

then

$$
xS_1 = x \sum_{k=0}^{\infty} (k+1)x^k = x + 2x^2 + 3x^3 + \cdots
$$

so

$$
S_1 - xS_1 = S_0 = \frac{1}{1-x}
$$

$$
S_1(1-x) = \frac{1}{1-x}
$$

$$
\sum_{k=0}^{\infty} (k+1)x^k = S_1 = \frac{1}{(1-x)^2}
$$

Finally, one last sum:

$$
S_2 = \sum_{k=0}^{\infty} (k+1)^2x^k = 1 + 4x + 9x^2 + \cdots
$$

then

$$
xS_2 = x \sum_{k=0}^{\infty} (k+1)^2x^k = x + 4x^2 + 9x^3 + \cdots
$$
\[ S_2 - xS_2 = 1 + 3x + 5x^2 + 7x^3 = (2 + 4x + 6x^2 + \cdots) - (1 + x + x^2 + \cdots) = 2S_1 - S_0 \]

\[ S_1(1 - x) = \frac{2}{(1 - x)^2} - \frac{1}{1 - x} = \frac{1 + x}{(1 - x)^2} \]

\[ \sum_{k=0}^{\infty} (k + 1)^2x^k = S_2 = \frac{1 + x}{(1 - x)^3} \]

2. **GEOMETRIC DISTRIBUTIONS**

Suppose that we conduct a sequence of Bernoulli \((p)\)-trials, that is each trial has a success probability of \(0 < p < 1\) and a failure probability of \(1 - p\). The geometric distribution is given by:

\[ P(X = n) = \text{the probability that the first success occurs on trial } n \]

\[ P(X = n) = (1 - p)^{n-1}p \quad \text{where } n \in \{1, 2, \ldots\} \]

Note that

\[ \sum_{n=1}^{\infty} P(X = n) = \sum_{n=1}^{\infty} (1 - p)^{n-1}p = \sum_{k=0}^{\infty} (1 - p)^k p = p \sum_{k=0}^{\infty} (1 - p)^k \]

As this last sum is a geometric series, and \(|1 - p| < 1\),

\[ \sum_{n=1}^{\infty} P(X = n) = p \frac{1}{1 - (1 - p)} = p \frac{1}{p} = 1 \]

The cumulative distribution function is given by:

\[ P(X \leq n) = 1 - P(X > n) = 1 - \sum_{k=n+1}^{\infty} (1 - p)^{k-1}p = 1 - \sum_{k=n}^{\infty} (1 - p)^k p = 1 - p \frac{(1 - p)^n}{p} \]

so

\[ P(X \leq n) = 1 - (1 - p)^n \]

If \(X\) is a geometrically distributed random variable with parameter \(p\), then

\[ E(X) = \sum_{n=1}^{\infty} n(1 - p)^{n-1}p = \sum_{k=0}^{\infty} (k + 1)(1 - p)^k p = p \sum_{k=0}^{\infty} (k + 1)(1 - p)^k \]

Using the notes above:

\[ E(X) = p \sum_{k=0}^{\infty} (k + 1)(1 - p)^k = p \frac{1}{[1 - (1 - p)]^2} = p \frac{1}{p^2} \]

so

\[ E(X) = \frac{1}{p} \]

Also

\[ E(X^2) = \sum_{n=1}^{\infty} n^2(1 - p)^{j-1}p = \sum_{k=0}^{\infty} (k + 1)^2(1 - p)^k p = p \sum_{k=0}^{\infty} (k + 1)^2(1 - p)^k \]

Using the notes above:

\[ E(X^2) = p \sum_{k=0}^{\infty} (k + 1)^2(1 - p)^k = p \frac{1 + (1 - p)}{[1 - (1 - p)]^3} = p \frac{2 - p}{p^3} \]

so

\[ E(X^2) = \frac{2 - p}{p^2} \]
Thus

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2 - p}{p^2} - \frac{1}{p^2} = \frac{1 - p}{p^2} \]

and

\[ \text{SD}(X) = \frac{\sqrt{1 - p}}{p} \]