Infinite Series and Geometric Distributions

1. Geometric Series

Suppose that $|x| < 1$, then the geometric series in x is absolutely convergent:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Here is how we find this value: Let

$$S_0 = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

then

$$xS_0 = x\sum_{k=0}^{\infty} x^k = x + x^2 + x^3 + \cdots$$

so

$$S_0 - xS_0 = 1$$

$$S_0(1-x) = 1$$

$$\mathbf{\sum_{k=0}^{\infty} x^k = S_0 = \frac{1}{1-x}}$$

In fact we can use this method to find the tail sums of this series:

$$\sum_{k=m}^{\infty} x^k - x\sum_{k=m}^{\infty} x^k = x^m$$

so

$$\mathbf{\sum_{k=m}^{\infty} x^k = \frac{x^m}{1-x}}$$

Now consider another sum which converges absolutely for $|x| < 1$:

$$S_1 = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

then

$$xS_1 = x\sum_{k=0}^{\infty}(k+1)x^k = x + 2x^2 + 3x^3 + \cdots$$

so

$$S_1 - xS_1 = S_0 = \frac{1}{1-x}$$

$$S_1(1-x) = \frac{1}{1-x}$$

$$\mathbf{\sum_{k=0}^{\infty}(k+1)x^k = S_1 = \frac{1}{(1-x)^2}}$$

Finally, one last sum:

$$S_2 = \sum_{k=0}^{\infty}(k+1)^2x^k = 1 + 4x + 9x^2 + \cdots$$

then

$$xS_2 = x\sum_{k=0}^{\infty}(k+1)^2x^k = x + 4x^2 + 9x^3 + \cdots$$

so

$$S_2 - xS_2 = 1 + 3x + 5x^2 + 7x^3 = (2 + 4x + 6x^2 + \cdots) - (1 + x + x^2 + \cdots) = 2S_1 - S_0$$

$$S_1(1-x) = \frac{2}{(1-x)^2} - \frac{1}{1-x} = \frac{1+x}{(1-x)^2}$$

$$\sum_{\mathbf{k=0}}^{\infty} \mathbf{(k+1)^2 x^k = S_2 = \frac{1+x}{(1-x)^3}}$$

2. Geometric Distributions

Suppose that we conduct a sequence of Bernoulli (p)-trials, that is each trial has a success probability of $0 < p < 1$ and a failure probability of $1-p$. The geometric distribution is given by:

$$P(X = n) = \text{ the probability that the first success occurs on trial } n$$

$$\mathbf{P(X = n) = (1-p)^{n-1}p} \qquad \text{where } n \in \{1, 2, \dots\}$$

Note that

$$\sum_{n=1}^{\infty} P(X=n) = \sum_{n=1}^{\infty}(1-p)^{n-1}p = \sum_{k=0}^{\infty}(1-p)^k p = p\sum_{k=0}^{\infty}(1-p)^k$$

As this last sum is a geometric series, and $|1-p| < 1$,

$$\sum_{j=n}^{\infty} P(X=n) = p\frac{1}{1-(1-p)} = p\frac{1}{p} = 1$$

The cumulative distribution function is given by:

$$P(X \le n) = 1 - P(X > n) = 1 - \sum_{k=n+1}^{\infty}(1-p)^{k-1}p = 1 - \sum_{k=n}^{\infty}(1-p)^k p = 1 - p\frac{(1-p)^n}{p}$$

so

$$\mathbf{P(X \le n) = 1 - (1-p)^n}$$

If X is a geometrically distributed random variable with parameter p, then

$$E(X) = \sum_{n=1}^{\infty} n(1-p)^{n-1}p = \sum_{k=0}^{\infty}(k+1)(1-p)^k p = p\sum_{k=0}^{\infty}(k+1)(1-p)^k$$

Using the notes above:

$$E(X) = p\sum_{k=0}^{\infty}(k+1)(1-p)^k = p\frac{1}{[1-(1-p)]^2} = p\frac{1}{p^2}$$

so

$$\mathbf{E(X) = \frac{1}{p}}$$

Also

$$E(X^2) = \sum_{n=1}^{\infty} n^2(1-p)^{j-1}p = \sum_{k=0}^{\infty}(k+1)^2(1-p)^k p = p\sum_{k=0}^{\infty}(k+1)^k(1-p)^k$$

Using the notes above:

$$E(X^2) = p\sum_{k=0}^{\infty}(k+1)^2(1-p)^k = p\frac{1+(1-p)}{[1-(1-p)]^3} = p\frac{2-p}{p^3}$$

so

$$\mathbf{E(X^2) = \frac{2-p}{p^2}}$$

Thus

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

and

$$\mathbf{SD(X)} = \frac{\sqrt{\mathbf{1-p}}}{\mathbf{p}}$$