An Example of Orthogonal Bases and Least Squares Approximations

(1.) Let S be the subspace of $\mathbb{R}^{2\times 2}$ spanned by $\mathbf{x}_1 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$ and $\mathbf{x}_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$,

(a.) Verify that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal basis for S.

To answer this question, we just need to check that $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0$ when $i \neq j$. So:

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 1 - 4 - 6 + 9 = 0$$

$$\langle \mathbf{x}_1, \mathbf{x}_3 \rangle = -3 + 3 = 0$$

$$\langle \mathbf{x}_2, \mathbf{x}_3 \rangle = -3 + 3 = 0$$

Therefore $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal basis for S.

(b.) Find the orthogonal projection $\mathbf{p}$ (or least squares approximation) of $\mathbf{y} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ onto S.

First, $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ where $\mathbf{p}_i$ is the orthogonal projection of $\mathbf{y}$ onto the line spanned by $\mathbf{x}_i$. Also recall that:

$$\mathbf{p}_i = \frac{\langle \mathbf{y}, \mathbf{x}_i \rangle}{\langle \mathbf{x}_i, \mathbf{x}_i \rangle} \mathbf{x}_i$$

So:

$$\mathbf{p}_1 = \frac{\langle \mathbf{y}, \mathbf{x}_1 \rangle}{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} \mathbf{x}_1 = \frac{54}{30} \mathbf{x}_1$$

$$\mathbf{p}_2 = \frac{\langle \mathbf{y}, \mathbf{x}_2 \rangle}{\langle \mathbf{x}_2, \mathbf{x}_2 \rangle} \mathbf{x}_2 = \frac{4}{20} \mathbf{x}_2$$

$$\mathbf{p}_3 = \frac{\langle \mathbf{y}, \mathbf{x}_3 \rangle}{\langle \mathbf{x}_3, \mathbf{x}_3 \rangle} \mathbf{x}_3 = \frac{2}{10} \mathbf{x}_3$$

Therefore

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \frac{54}{30} \mathbf{x}_1 + \frac{4}{20} \mathbf{x}_2 + \frac{2}{10} \mathbf{x}_3 = \begin{pmatrix} 1.4 & 7 \\ 3 & 6.2 \end{pmatrix}$$