Special Matrix Forms, Variables, and the Number of Solutions

Note: The notation **(augmented) matrix** refers to an object which is either a true matrix or an augmented matrix.

As discussed in the previous notes, we will try to solve a given linear system by finding an equivalent system in which the solution is easy to determine. We will do this by transforming the corresponding augmented matrix using elementary row operations into an augmented matrix which has a special from. The process which we use to affect these transformations is referred to as **row reduction**. These notes will describe the various special forms and what sort of information can be determined from them.

Before we define these forms, we need to define the notion of a leading entry. If A is a (augmented) matrix, then the first non-zero entry in a given row of A is called a **leading entry**. If a row does not have a non-zero entry, then we refer to that row as a **zero row**. Therefore every row either contains a leading entry or it is a zero row.

Most of the theorems in these notes will be given without proof as their proofs are beyond our scope. We will give these proofs in an appendix which will appear later.

1. Echelon Form

The first form we will look at is the most basic of the special forms:

Definition 1.1. *We say that a (augmented) matrix is in* **echelon form** *if*

(1.) *The leading entry in row k occurs to the right of the leading entry in row $k-1$.*

(2.) *All zero rows lie below every non-zero row.*

Condition (1.) says that there are only zeros below each leading entry. Basically a (augmented) matrix is in echelon form if its non-zero entries from a triangle. It turns out that the echelon form contains quite a bit of information. Recall that the columns of the coefficient matrix of a linear system correspond to distinct variables, and this correspondence is the same in the augmented matrix of this same system. Under this correspondence we have the following definitions:

Definition 1.2. *Suppose that A is the coefficient or augmented matrix of a linear system and that A is in echelon form then:*

(1.) *If a column of A contains a leading entry of some row, then that column is called a* **pivot column** *and if there is a corresponding variable, then it is called a* **lead variable**.

(2.) *If a column of A does not contain a leading entry of some row, then that column is called a* **non-pivot** *column and, if there is a corresponding variable, then it is called a* **free variable**.

It turns out that every (augmented) matrix can be put into echelon form using elementary row operations. Moreover, although this echelon form will not be unique, the position of the pivot and non-pivot columns will be the same. We record this in the following theorem:

Theorem 1.3. *Suppose that A is a (augmented) matrix, then A can be transformed into an (augmented) matrix U_1 which is in echelon form using elementary row operations. Moreover if U_2 is another (augmented) echelon* *matrix obtained from A using elementary row operations, then column k of U_1 is a pivot column if and only if the column k of U_2 is a pivot column.*

We will refer to the (augmented) matrix U_1 (or U_2) mentioned in the theorem above as an **echelon form** of A.

2. The Number of Solutions of a Linear System

Suppose that A is the augmented matrix of a linear system and U is an echelon form of A. It is possible to tell whether the corresponding system is consistent just from U. We can also tell the number of solutions if it is consistent. We encode this information in the following theorem:

Theorem 2.1. *Suppose that A is the augmented matrix of a linear system and U is an echelon form of A, then*

(1.) *If the last column of U (ie. the augmented part) is a pivot column then the linear system has* **no solution**.

(2.) *If the last column of U is the only non-pivot column in U then the linear system has a* **unique solution**.

(3.) *If the last column of U is a non-pivot column and U has another non-pivot column, then the linear system has* **infinitely many solutions**

We could restate this theorem using variables, for example part (2.) says that is the system is consistent and that all the variables are lead variables, then the system has a unique solution. Also part (3.) says that if the system is consistent and there at least one of the variables is a free variable, then there are infinitely many solutions.

3. Row Echelon Form

In this section we define another special matrix form. It is an intermediate form and, as such, we will only define it and then move to then next section.

Definition 3.1. *We say that a (augmented) matrix is in* **row echelon form** *if*

(1.) *It is in echelon form and*

(2.) *each leading entry is 1.*

The only difference between a matrix in echelon form and row echelon for is the condition on the leading entries. It is easy to see that a (augmented) matrix which is in echelon form can be transformed into row echelon form by simply dividing each non-zero row by is leading entry. This is referred to as **normalizing** the leading entries.

The process of transforming a (augmented) matrix into echelon form (or row echelon form) using elementary row operations is called **Gaussian elimination**. If a system is in in one of these two forms, then it can be solved using **back substitution**.

4. Reduced Row Echelon Form

Our last special form is the most important of the forms. First we will define it and then give some of its properties.

Definition 4.1. *We say that a (augmented) matrix is in* **reduced row echelon form** *if*

(1.) *It is in row echelon form and*

(2.) *each leading entry is the only non-zero entry in its column.*

Condition (2.) says that there are only zeros above and below each leading entry in a reduced row echelon (augmented) matrix. The importance of this form is contained in the theorem below, which we will prove:

Theorem 4.2. *If A is a (augmented) matrix, then A can be transformed into a* **unique** *reduced row echelon (augmented) matrix using elementary row operations.*

Note that this theorem implies the results from the previous sections as any echelon form or row echelon form of A must get reduced to the same reduced row echelon (augmented) matrix by the uniqueness condition.

The process of transforming a (augmented) matrix into reduced row echelon form using elementary row operations is called **Gauss-Jordan elimination**. A system in this form will be the easiest of all equivalent systems to solve.

5. Overdetermined, Underdetermined and Homogeneous Linear Systems

In this last section, three specific types of linear systems are defined and some of their properties are listed.

Definition 5.1. *Suppose that a linear system has m equations and n unknowns.*

(1.) *We say that the system is* **underdetermined** *if $n > m$ and*

(2.) *we say that the system is* **overdetermined** *if $m > n$.*

Part (1.) says that an underdetermined system is one which has more unknowns than equations. Part (2.) states that an overdetermined system has more equations than unknowns. Either type of system can be inconsistent or have infinitely many solutions, but an underdetermined system **never** has a unique solution. This is because an underdetermined system must have at least one free variable. Therefore, an underdetermined system which is consistent mush have an infinite number of solutions. We should point out that an overdetermined system will usually be inconsistent. Trying to find an approximate solution to such a system is the main purpose of the study of least squares.

A system of linear equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

is called **homogeneous**. It is important to note that a homogeneous system is always consistent. It is always satisfied by the solution $x_1 = x_2 = \cdots = x_n = 0$. This solution is called the **trivial solution**.