Elementary Matrices and Determinants

1. Preliminary Results

Theorem 1.1. *Suppose that A and B are $n \times n$ matrices and that A or B is singular, then AB is singular.*

Proof: First assume that B is singular. Then there is a non-trivial vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$, which gives $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This means that AB must be singular as there is a non-trivial solution to the homogeneous equation. Without loss of generality, we may assume that A is singular and B is non-singular. Similar to the above, there is a non-trivial vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. Since B is non-singular, $B^{-1}\mathbf{x}$ is non-trivial as well. Thus $AB(B^{-1}\mathbf{x}) = A(BB^{-1})\mathbf{x} = A\mathbf{x} = \mathbf{0}$. This means that AB must be singular in this case as well as there is a non-trivial solution to the homogeneous equation. □

Theorem 1.2. *Suppose that A is an $n \times n$ matrix. Then A is non-singular if and only if A is the product of elementary matrices.*

Proof: If A is non-singular, then A can be row reduced to the identity matrix. This means there is a series of elementary matrices, $\mathcal{E}_1, \ldots \mathcal{E}_k$ such that $\mathcal{E}_1 \cdots \mathcal{E}_k A = I$. So:

$$A = (\mathcal{E}_1 \cdots \mathcal{E}_k)^{-1}$$

$$A = (\mathcal{E}_k)^{-1} \cdots (\mathcal{E}_1)^{-1}$$

which is a product of elementary matrices. Conversely, if A is the product of elementary matrices, then A must be invertible as the product of invertible matrices is also invertible. □

Theorem 1.3. *Suppose that $A = (a_{ij})$ is an $n \times n$ matrix which is either upper triangular, lower triangular or diagonal. Then $\det(A) = a_{11}a_{22} \cdots a_{nn}$.*

Proof: This can be proved using induction on n. We will not give this argument. □

2. Effect of Elementary Matrices on Determinants

Theorem 2.1. *Suppose that A is an $n \times n$ matrix.*

(1) *If $E = P_{ij}$ is an elementary matrix of permutation type, then* $\det(EA) = -\det(A) = \det(E)\det(A)$.
(2) *If $E = D_i(t)$ is an elementary matrix of diagonal type, then* $\det(EA) = t\det(A) = \det(E)\det(A)$.
(3) *If $E = E_{ij}(t)$ is an elementary matrix of unipotent type, then* $\det(EA) = \det(A) = \det(E)\det(A)$.

Proof: The proofs of (1.) and (3.) are done using induction, so we will only give a proof for (2.). Here just compute $\det(A)$ and $\det(D_i(t)A)$ by expanding along the ith row. The minors are identical and hence the cofactors are as well. The only difference is that where a_{ij} occurs in A, there is a ta_{ij} in $D_i(t)A$. Thus $\det(D_i(t)A) = t\det(A)$. Also note that $\det(D_i(t)) = t$ by theorem 1.3, so $\det(D_i(t)A) = t\det(A) = \det(D_i(t))\det(A)$. □

Theorem 2.2. *Suppose that A is an $n \times n$ matrix and $\mathcal{E}_1, \ldots \mathcal{E}_k$ are elementary matrices such that $\mathcal{E}_1 \cdots \mathcal{E}_k A = U$ where U is an echelon matrix. Furthermore suppose that $\mathcal{E}_i$ is not of diagonal type and that the number of permutation type matrices in $\mathcal{E}_1, \ldots \mathcal{E}_k$ is exactly r. Then*

$$\det(A) = (-1)^r \det(U) = (-1)^r (u_{11} u_{22} \cdots u_{nn})$$

where $U = (u_{ij})$.

Note: It is always possible to put A into echelon form without using elementary matrices of diagonal type.

Proof: This is a direct consequence of theorems 1.3 and 2.1. □

3. Main Results

Theorem 3.1. *Suppose that A is a singular $n \times n$ matrix. Then* $\det(A) = 0$.

Proof: Row reduce A to an echelon matrix as in theorem 2.1. Since A is singular and $n \times n$, one of the columns of U must be a non-pivot column. In particular, $u_{ii} = 0$ for some i. This means $\det(U) = 0$ using theorem 1.3 and, therefore $\det(A) = 0$ from theorem 2.1. □

Theorem 3.2. *Suppose that A and B are $n \times n$ matrices. Then* $\det(AB) = \det(A)\det(B)$.

Proof: Suppose that A is singular, then AB is also singular from theorem 1.1. Using theorem 3.1, $\det(A) = 0$ and $\det(AB) = 0$. This implies $0 = \det(AB) = \det(A)\det(B) = 0$. Now we may assume that A is non-singular. By theorem 1.2, $A = \mathcal{E}_1 \cdots \mathcal{E}_k$ so $\det(AB) = \det(\mathcal{E}_1 \cdots \mathcal{E}_k B)$. Repeated applications of theorem 2.1 then give $\det(AB) = \det(\mathcal{E}_1) \cdots \det(\mathcal{E}_k)\det(B) = \det(A)\det(B)$. □