Elementary Matrices and Determinants

1. Preliminary Results

Theorem 1.1. Suppose that $A$ and $B$ are $n \times n$ matrices and that $A$ or $B$ is singular, then $AB$ is singular.

Proof: First assume that $B$ is singular. Then there is a non-trivial vector $x$ such that $Bx = 0$, which gives $ABx = A0 = 0$. This means that $AB$ must be singular as there is a non-trivial solution to the homogeneous equation. Without loss of generality, we may assume that $A$ is singular and $B$ is non-singular. Similar to the above, there is a non-trivial vector $x$ such that $Ax = 0$. Since $B$ is non-singular, $B^{-1}x$ is non-trivial as well. Thus $AB(B^{-1}x) = A(BB^{-1})x = Ax = 0$. This means that $AB$ must be singular in this case as well as there is a non-trivial solution to the homogeneous equation. □

Theorem 1.2. Suppose that $A$ is an $n \times n$ matrix. Then $A$ is non-singular if and only if $A$ is the product of elementary matrices.

Proof: If $A$ is non-singular, then $A$ can be row reduced to the identity matrix. This means there is a series of elementary matrices, $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A = I$. So:

$$A = (E_1 \cdots E_k)^{-1}$$

$$A = (E_k)^{-1} \cdots (E_1)^{-1}$$

which is a product of elementary matrices. Conversely, if $A$ is the product of elementary matrices, then $A$ must be invertible as the product of invertible matrices is also invertible. □

Theorem 1.3. Suppose that $A = (a_{ij})$ is an $n \times n$ matrix which is either upper triangular, lower triangular or diagonal. Then $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

Proof: This can be proved using induction on $n$. We will not give this argument. □

2. Effect of Elementary Matrices on Determinants

Theorem 2.1. Suppose that $A$ is an $n \times n$ matrix.

1. If $E = P_{ij}$ is an elementary matrix of permutation type, then $\det(EA) = -\det(A) = \det(E)\det(A)$.
2. If $E = D_i(t)$ is an elementary matrix of diagonal type, then $\det(EA) = t\det(A) = \det(E)\det(A)$.
3. If $E = E_{ij}(t)$ is an elementary matrix of unipotent type, then $\det(EA) = \det(A) = \det(E)\det(A)$.

Proof: The proofs of (1.) and (3.) are done using induction, so we will only give a proof for (2.). Here just compute $\det(A)$ and $\det(D_i(t)A)$ by expanding along the $i$th row. The minors are identical and hence the cofactors are as well. The only difference is that where $a_{ij}$ occurs in $A$, there is a $ta_{ij}$ in $D_i(t)A$. Thus $\det(D_i(t)A) = t\det(A)$. Also note that $\det(D_i(t)) = t$ by theorem 1.3, so $\det(D_i(t)A) = t\det(A) = \det(D_i(t))\det(A)$. □
Theorem 2.2. Suppose that $A$ is an $n \times n$ matrix and $E_1, \ldots, E_k$ are elementary matrices such that $E_1 \cdots E_k A = U$ where $U$ is an echelon matrix. Furthermore suppose that $E_i$ is not of diagonal type and that the number of permutation type matrices in $E_1, \ldots, E_k$ is exactly $r$. Then
\[ \det(A) = (-1)^r \det(U) = (-1)^r (u_{11} u_{22} \cdots u_{nn}) \]
where $U = (u_{ij})$.

Note: It is always possible to put $A$ into echelon form without using elementary matrices of diagonal type.

Proof: This is a direct consequence of theorems 1.3 and 2.1. □

3. Main Results

Theorem 3.1. Suppose that $A$ is a singular $n \times n$ matrix. Then $\det(A) = 0$.

Proof: Row reduce $A$ to an echelon matrix as in theorem 2.1. Since $A$ is singular and $n \times n$, one of the columns of $U$ must be a non-pivot column. In particular, $u_{ii} = 0$ for some $i$. This means $\det(U) = 0$ using theorem 1.3 and, therefore $\det(A) = 0$ from theorem 2.1. □

Theorem 3.2. Suppose that $A$ and $B$ are $n \times n$ matrices. Then $\det(AB) = \det(A) \det(B)$.

Proof: Suppose that $A$ is singular, then $AB$ is also singular from theorem 1.1. Using theorem 3.1, $\det(A) = 0$ and $\det(AB) = 0$. This implies $0 = \det(AB) = \det(A) \det(B) = 0$. Now we may assume that $A$ is non-singular. By theorem 1.2, $A = E_1 \cdots E_k$ so $\det(AB) = \det(E_1 \cdots E_k B)$. Repeated applications of theorem 2.1 then give $\det(AB) = \det(E_1) \cdots \det(E_k) \det(B) = \det(A) \det(B)$. □