Review 2

Math 571
Autumn 2009

1. Let \( A = \begin{pmatrix} 3 & 6 & 3 & 3 & 6 & 1 \\ 5 & 10 & 4 & 2 & -3 & 6 \\ 7 & 14 & 2 & 0 & -11 & 7 \\ 1 & 2 & 0 & 0 & -1 & 2 \end{pmatrix} \).

(a.) Find a basis for the row space of \( A \).
(b.) Find a basis for the column space of \( A \).
(c.) Find a basis for the nullspace of \( A \). DO NOT USE MATLAB.
(d.) The rank of \( A \). DO NOT USE MATLAB.

2. Suppose that \( x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \), \( x_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \), \( x_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \) and \( x_4 = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} \). Find a basis for \( \text{Span}(x_1, x_2, x_3, x_4) \).

Note: there is more than one answer.

3. Determine if \( 1, e^x \) and \( \cos x \) are linearly independent in \( C[0, 1] \).

4. Find a basis for the subspace \( S \) of \( V \) where:
   (a.) \( V = \mathbb{R}^4 \) and \( S = \{(a - b + c, a + c, a + 2b - c, b - 3c)^T \mid a, b, c \text{ are real numbers} \} \).
   (b.) \( V = C[0, 1] \) and \( S = \text{Span}(1, \sin 2x, \sin x \cos x) \).
   (c.) \( V = \mathbb{P}_4 \) and \( S \) is the set of all polynomials \( p(x) \) in \( V \) with \( p(0) = 0 \) and \( p(1) = 0 \).

5. Let \( A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) and let \( S = \{ B \in \mathbb{R}^{2 \times 2} \mid AB = BA \} \).

(a.) Show that \( S \) is a subspace of \( \mathbb{R}^{2 \times 2} \).
(b.) Find a basis for \( S \).

6. Let \( x_1 = 1, x_2 = 2, x_3 = -1 \) and \( x_4 = -2 \). Then
\[
\langle p(x), q(x) \rangle = p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3) + p(x_4)q(x_4)
\]
defines an inner product on \( \mathbb{P}_4 \). If \( p(x) = x^3 - 6x^2 + 2 \) and \( q(x) = x^3 + x - 1 \), find the following with respect to THIS inner product:
(a.) \( \langle p(x), q(x) \rangle \), \( ||p(x)|| \), \( ||q(x)|| \),
(b.) the orthogonal projection of \( p(x) \) onto the line spanned by \( q(x) \) and
(c.) the angle between \( p(x) \) and \( q(x) \).

7. True or False?
   (a.) A linearly independent set can not contain \( 0 \).
   (b.) If \( x \) and \( y \) are vectors in the inner product space \( V \), then \( ||x + y||^2 = ||x||^2 + ||y||^2 \).
   (c.) If \( A \) is an \( m \times n \) matrix, then \( \dim(\text{Col}(A)) + \dim(\text{N}(A)) = m \).
   (d.) If \( A \) is an \( m \times n \) matrix, then \( Ax = b \) is consistent if and only if \( b \) is in the column space of \( A \).
   (e.) If \( A \) is a singular \( n \times n \) matrix, then the columns of \( A \) form a basis for \( \mathbb{R}^n \).
   (f.) A spanning set can never be linearly independent.

8. Prove the following:
   (a.) Let \( S \) be the subset of \( \mathbb{R}^{n \times n} \) consisting of matrices \( A \) such that \( A^T = A \). Show that \( S \) is a subspace of \( \mathbb{R}^{n \times n} \).
   (The matrices in \( S \) are called symmetric matrices.)
   (b.) If \( \{x_1, \ldots, x_n\} \) are linearly independent vectors in a vector space \( V \), then \( \{x_2, \ldots, x_n\} \) do not span \( V \).
   (c.) If \( \{x_1, \ldots, x_n\} \) are linearly independent vectors in a vector space \( V \) which do not span \( V \), then there is a vector \( x_0 \) in \( V \) such that \( \{x_0, x_1, \ldots, x_n\} \) is also linearly independent.
   (d.) If \( p \) is the orthogonal projection of \( x \) onto the line spanned by \( y \) then \( p \) and \( x - p \) are orthogonal.