Row Echelon Form and Number of Solutions

1. Row Echelon Form

In these notes we will define one of the most important forms of a matrix. It is one of the "easier" forms of a system to solve, in particular, only back-substitution is needed to complete the solution of the corresponding linear system. Perhaps more importantly, this form allows us to determine the number of solutions that the corresponding linear system has. Recall that a linear system can either be inconsistent (have no solution) or be consistent. Furthermore, if the system is consistent it can either have a unique solutions or infinitely many solutions. There are no other possibilities.

First we need some definitions:

Definition 1.1. *Let A be a coefficient or $[A|\mathbf{b}]$ an augmented matrix of some linear system.*

(1.) *The* **leading entry** *in a row of A or $[A|\mathbf{b}]$ is the first non-zero entry (from left to right) in that row of A or $[A|\mathbf{b}]$.*

(2.) *A* **zero row** *in A or $[A|\mathbf{b}]$ is a row in A or $[A|\mathbf{b}]$ which consists of only zeros.*

Note that a row contains no leading entry if and only if it is a zero row.

Definition 1.2. *Let A be a coefficient or $[A|\mathbf{b}]$ is an augmented matrix of some linear system. We say that A is in* **row echelon form** *if*

(1.) *the leading entry in row k occurs to the left of the leading entry in row $(k+1)$ in A or $[A|\mathbf{b}]$ and*

(2.) *any zero row occurs below every non-zero row in A or $[A|\mathbf{b}]$.*

Equivalently, we could replace condition (1.) with condition (1.)': the leading entry in row k occurs to the right of the leading entry in row $(k-1)$ in A or $[A|\mathbf{b}]$.

2. Number of Solutions

Suppose that $[A|\mathbf{b}]$ is the **augmented matrix** of a linear system that is in **row echelon form**. Note that it is very important for the following result that this is the augmented matrix, not just the coefficient matrix and that this augmented matrix is in row echelon form. Then the number of solutions to the corresponding linear system will be:

(1.) No Solution if the last (augmented) column of $[A|\mathbf{b}]$ (ie. the column which contains $\mathbf{b}$ contains a leading entry from some row of $[A|\mathbf{b}]$.

(2.) Exactly One Solution if the last (augmented) column of $[A|\mathbf{b}]$ (ie. the column which contains $\mathbf{b}$) does not contain a leading entry from any row of $[A|\mathbf{b}]$ and all of the columns of the coefficient part (ie. the columns of A) do contain a leading entry from some row of $[A|\mathbf{b}]$.

(3.) Infinitely Many Solutions if the last (augmented) column of $[A|\mathbf{b}]$ (ie. the column which contains $\mathbf{b}$ does not contain a leading entry from some any of $[A|\mathbf{b}]$ and at least one of the columns of the coefficient part (ie. the columns of A) also does not contain a leading entry from some row of $[A|\mathbf{b}]$.

Given an coefficient matrix A or augmented matrix $[A|\mathbf{b}]$ in row echelon form, we can make the following definitions:

Definition 2.1. *Suppose that x_i is the variable that corresponds to the i-th column in A then we say that*

(1.) *x_i is a* **lead variable** *if the i-th column of A contains a leading entry from some row of A and that*

(2.) *x_i is a* **free variable** *if the i-th column of A does not contain a leading entry from any row of A.*

So our result on solutions says that a consistent system will have exactly one solution if there are no free variables in the corresponding linear system and it will have infinitely many solutions if there is at least one free variable.

If we have just the **coefficient matrix** A in **row echelon form**, then we can't be as sure to the number of solutions to the corresponding linear system. We have the weaker result which is nevertheless useful for many purposes (again note that it is important for this result that the matrix is in row echelon form):

(1.) If the last row of A is a non-zero row, then the system is consistent. Furthermore there will be a unique solution if there are no free variables and infinitely many solutions if there are free variables.

(2.) If the last row of A is a zero row, then the system **might** be inconsistent. In particular, three is always a choice for $\mathbf{b}$ such that $[A|\mathbf{b}]$ will correspond to an inconsistent linear system.