Review 1 Solutions

(1a.) \[ A = \begin{pmatrix}
3 & -2 & 4 & 3 & -1 \\
2 & 0 & 4 & 1 & 2 \\
-2 & 2 & -2 & -1 & 2 \\
-2 & -2 & -6 & 2 & -6
\end{pmatrix} \]
\[ \text{rref}(A) = \begin{pmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

(1b.) \[ x_1 + 2x_2 + x_5 = 3 \]
\[ x_2 + x_3 + x_5 = -1 \]

(1c.) It is consistent. From (1b.) we have the corresponding linear system:
\[
\begin{align*}
x_1 + 2x_2 &= 3 \\
x_2 + x_3 &= -1 \\
x_4 &= 2
\end{align*}
\]
x_3 and x_5 are free variables. This gives the general solution
\[ x = \begin{pmatrix}
x_3 \\
2 \\
x_5
\end{pmatrix} \]

(2.) A reduces to
\[ \begin{pmatrix}
1 & -1 & 3 \\
0 & 5 & -11 \\
0 & 0 & \alpha + 17
\end{pmatrix} \]

(a.) There is no solution if \( \alpha = -17 \) and \( \beta \neq -7 \).

(b.) There is a unique solution if \( \alpha \neq -17 \).

(c.) There are infinitely many solutions if \( \alpha = -17 \) and \( \beta = -7 \).

(3a.) \[ E = D_2(-3) = \begin{pmatrix}
1 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & 1
\end{pmatrix} \]

(3b.) \[ E = E_{32}(2) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 2 & 1
\end{pmatrix} \]

(4a.) Note that the choices for \( E_i \) are not unique. Once such choice is:
\[ E_1 = P_{23} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \]
\[ E_2 = E_{21}(-1) = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \]
\[ E_3 = D_3(1/7) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/7
\end{pmatrix} \]

(4b.) As \( E_3 E_2 E_1 A = B \),
\[ \det(E_3) \det(E_2) \det(E_1) \det(A) = \det(B) \]
Thus \( (1/7)(-1) \det(A) = \det(B) = -4 \) which gives \( \det(A) = 28 \).

(5.) Row reduce A with the following steps (not unique):
\[ A \rightarrow_{E_{21}(-2)} \rightarrow_{E_{31}(2)} \rightarrow_{E_{32}(5)} = B \]
Therefore

\[
A = E_{21}(-2)^{-1}E_{31}(2)^{-1}E_{32}(5)^{-1}B = E_{21}(2)E_{31}(-2)E_{32}(-5)B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \end{pmatrix}
\]

(6a.) \[ A^{-1} = \begin{pmatrix} -0.0213 & -0.1064 & 0.3830 \\ 0.2340 & 0.1702 & -0.2128 \\ 0.0638 & 0.3191 & -0.1489 \end{pmatrix} \text{ and } \det(A) = 47 \]

(6b.) \[ A^{-1} = \begin{pmatrix} 0.3421 & -0.6579 & 0.0263 & -0.1579 \\ 0.0526 & 0.0526 & 0.1579 & 0.0526 \\ 0.0263 & 0.0263 & 0.0789 & 0.5263 \\ 0.2368 & 0.2368 & -0.2895 & -0.2632 \end{pmatrix} \text{ and } \det(A) = 38 \]

(7a.) False - generally not true.

(7b.) False - generally not true.

(7c.) False - a unique solution means there are no free variables, hence there is at most one solution to \( Ax = b_2 \).

(7d.) False - can be inconsistent.

(7e.) True - elementary matrices are always invertible.

(7f.) True - the reduced row echelon form of \( A \) is \( I_n \).

(7g.) False - \( \det(\alpha A) = \alpha^n \det(A) \)

(8a.)
\[
AA^{-1} = I \\
\Rightarrow \det(AA^{-1}) = \det(I) = 1 \\
\Rightarrow \det(A)\det(A^{-1}) = 1 \\
\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}
\]

(8b.) Let \( b \) be given, then \( A(A^{-1}b) = I b \) so that \( A^{-1}b \) is a solution to \( Ax = b \). Thus a solution exists. Now we need to show that it is unique: Suppose that \( Ay = b \), then \( A^{-1}Ay = A^{-1}b \Rightarrow y = A^{-1}b \). Thus \( A^{-1}b \) is the unique solution to \( Ax = b \).

(8c.) Since \( A \) is invertible, it can be row reduced to the identity matrix \( I \). In terms of elementary matrices, this means that there are elementary matrices \( E_1, \ldots, E_k \) such that \( E_k \cdots E_1 A = I \). Therefore \( A = (E_k \cdots E_1)^{-1} = E_k^{-1} \cdots E_1^{-1} \). Note that we could adjust our notation to match the question as written.

(8d.) \( A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0 \)