Suppose that $v$ is a vector in $\mathbb{R}^n$ of length $n$. Prove the following:

(a.) A basis for the row space of $A$.
(b.) A basis for the column space of $A$.
(c.) A basis for the nullspace of $A$.
(d.) The rank and nullity of $A$.

(b.) Suppose that the initial distributions of states is given by $v_1, v_2, v_3, v_4$. Find a basis for the subspace of $v_1, v_2, v_3, v_4$.

Note: there is more than one answer.

(c.) Find the transition matrix of this process.
(d.) Determine if $1, e^x$ and $\cos x$ are linearly independent in $C[0, 1]$.
(e.) Determine if $1, e^x$ and $\cos x$ are linearly independent in $C[0, 1]$.
(f.) Find a basis for the subspace $S$ of $V$ where:

(a.) $V = \mathbb{R}^{2 \times 2}$ and $S$ is the set of $2 \times 2$ matrices $A$ with $\det(A) = 0$.
(b.) $V = \mathbb{R}^{2 \times 2}$ and $S$ is the set of $2 \times 2$ upper triangular matrices.
(c.) $V = \mathbb{R}^2$ and $S = \{(x_1, x_2)^T \mid |x_1| = |x_2|\}$.
(d.) $V = \mathbb{P}_3$ and $S$ is the set of all polynomials $p(x)$ in $V$ such that $p(1) = 0$.
(e.) $V = C[-1, 1]$ and $S$ is the set of odd functions in $V$.

(1.) Let $A = \begin{pmatrix} 1 & 2 & -1 & 2 & 2 & 4 & 4 \\ 3 & 6 & -5 & -8 & 7 & 6 & -7 \\ -2 & -4 & -3 & 1 & 3 & 1 & -12 \\ -2 & 4 & 1 & 1 & 0 & 4 & 13 \\ 2 & 4 & -2 & 4 & 4 & 8 & 8 \end{pmatrix}$ and find:

(a.) A basis for the row space of $A$.
(b.) A basis for the column space of $A$.
(c.) A basis for the nullspace of $A$.
(d.) The rank and nullity of $A$.

(2.) Suppose that $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$. Find a basis for $\text{Span}(v_1, v_2, v_3, v_4)$.

(3.) Determine if $S$ is a subspace of $V$ where:

(a.) $V = \mathbb{R}^{2 \times 2}$ and $S$ is the set of $2 \times 2$ matrices $A$ with $\det(A) = 0$.
(b.) $V = \mathbb{R}^{2 \times 2}$ and $S$ is the set of $2 \times 2$ upper triangular matrices.
(c.) $V = \mathbb{R}^2$ and $S = \{(x_1, x_2)^T \mid |x_1| = |x_2|\}$.
(d.) $V = \mathbb{P}_3$ and $S$ is the set of all polynomials $p(x)$ in $V$ such that $p(1) = 0$.
(e.) $V = C[-1, 1]$ and $S$ is the set of odd functions in $V$.

(4.) Determine if $1, e^x$ and $\cos x$ are linearly independent in $C[0, 1]$.

(5.) Find a basis for the subspace $S$ of $V$ where:

(a.) $V = \mathbb{R}^4$ and $S = \{(a - b + c, a + c, a + 2b - c, b - 3c)^T \mid a, b, c, d \text{ are real numbers} \}$.
(b.) $V = C[0, 1]$ and $S = \text{Span}(1, \sin 2x, \sin x \cos x)$.
(c.) $V = \mathbb{P}_4$ and $S$ is the set of all polynomials $p(x)$ in $V$ with $p(0) = 0$ and $p(1) = 0$.

(6.) True or False?

(a.) A linearly independent set can contain $0$.
(b.) A subspace of a vector space must contain $0$.
(c.) If $A$ is an $m \times n$ matrix, then $\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = m$.
(d.) If $A$ is an $m \times n$ matrix, then $Ax = b$ is consistent if and only if $b$ is in the column space of $A$.
(e.) If $A$ is a singular $n \times n$ matrix, then the columns of $A$ form a basis for $\mathbb{R}^n$.
(f.) A spanning set can never be linearly independent.

(7.) Suppose that a molecule has three excited states which are denoted by $A$, $B$ and $C$. Each second, the probability that it transitions from one state to the other is as follows: From $A$ to $B$: 0.2, from $A$ to $C$: 0.3, from $B$ to $A$: 0.4, from $B$ to $C$: 0.2, from $C$ to $A$: 0.5 and from $C$ to $B$: 0.2. Note that these transitions are a Markov process.

(a.) Find the transition matrix of this process.
(b.) Determine that the initial distributions of states is 100 in state $A$, 75 in state $B$ and 25 in state $C$. Find the resulting distributions after 10 seconds.
(c.) Find the steady state probability vector for this process.

(8.) Prove the following:

(a.) Let $S$ be the subset of $\mathbb{R}^{n \times n}$ consisting of matrices $A$ such that $A^T = -A$. Show that $S$ is a subspace of $\mathbb{R}^{n \times n}$. (The matrices in $S$ are called skew-symmetric matrices.)
(b.) If $\{v_1, \ldots, v_n\}$ are linearly independent vectors in a vector space $V$, then $\{v_2, \ldots, v_n\}$ does not span $V$.
(c.) If $\{v_1, \ldots, v_n\}$ are linearly independent vectors in a vector space $V$ which do not span $V$, then there is a vector $v_{n+1}$ in $V$ such that $\{v_1, \ldots, v_n, v_{n+1}\}$ is also linearly independent.