Math 572

Winter 2011

Solutions to Homework 1

(5.) For all parts, let $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$ and α be a scalar.

(5a.) YES:

$$L(\mathbf{x}+\mathbf{y}) = (x_2+y_2, x_3+y_3)^T = (x_2, x_3)^T + (y_2, y_3)^T \text{ and } L(\mathbf{x}) + L(\mathbf{y}) = (x_2, x_3)^T + (y_2, y_3)^T$$

so $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$.

$$L(\alpha\mathbf{x}) = (\alpha x_2, \alpha x_3)^T = \alpha(x_2, x_3)^T \text{ and } \alpha L(\mathbf{x}) = \alpha(x_2, x_3)^T$$

so $\alpha L(\mathbf{x}) = \alpha L(\mathbf{x})$. Thus L is a linear transformation.

(5b.) YES:

$$L(\mathbf{x}+\mathbf{y}) = (0,0)^T \text{ and } L(\mathbf{x}) + L(\mathbf{y}) = (0,0)^T + (0,0)^T = (0,0)^T$$

so $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$.

$$L(\alpha\mathbf{x}) = (\alpha 0, \alpha 0)^T = \alpha(0,0)^T = (0,0)^T \text{ and } \alpha L(\mathbf{x}) = \alpha(0,0)^T = (0,0)^T$$

so $\alpha L(\mathbf{x}) = \alpha L(\mathbf{x})$. Thus L is a linear transformation.

(5c.) NO: Consider $2L(1,1,1) = 2(2,1) = (4,2)$, but $L(2,2,2) = (3,2)$. Since these are not equal, L is not a linear transformation.

(5d.) YES:

$$L(\mathbf{x}+\mathbf{y}) = (x_3+y_3, x_1+y_1+x_2+y_2)^T = (x_3, x_1+x_2)^T+(y_3, y_1+y_2)^T \text{ and } L(\mathbf{x})+L(\mathbf{y}) = (x, x_1+x_2)^T+(y, y_1+y_2)^T$$

so $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$.

$$L(\alpha\mathbf{x}) = (\alpha x_3, \alpha x_1 + \alpha x_2)^T = \alpha(x_3, x_1+x_2)^T \text{ and } \alpha L(\mathbf{x}) = \alpha(x_3, x_1+x_2)^T$$

so $\alpha L(\mathbf{x}) = \alpha L(\mathbf{x})$. Thus L is a linear transformation.

(9a.) YES:

$$L(p(x)+q(x)) = x(p(x)+q(x)) = xp(x)+xq(x) \text{ and } L(p(x)) + L(q(x)) = xp(x) + xq(x)$$

so $L(p(x)+q(x)) = L(p(x)) + L(q(x))$.

$$L(\alpha p(x)) = x(\alpha p(x)) = \alpha(xp(x)) \text{ and } \alpha L(p(x)) = \alpha(xp(x)$$

so $\alpha L(p(x)) = \alpha L(p(x))$. Thus L is a linear transformation.

(9b.) NO: Consider $2L(x) = 2(x^2+x) = 2x^2+2x$ and $L(2x) = x^2+2x$. Since these are not equal, L is not a linear transformation.

(9c.) YES:

$$L(p(x)+q(x)) = p(x)+q(x)+x(p(x)+q(x))+x^2(p(x)+q(x))' = p(x)+xp(x)+x^2p'(x)+q(x)+xq(x)+x^2q'(x)$$

and

$$L(p(x)) + L(q(x)) = p(x) + xp(x) + x^2p'(x) + q(x) + xq(x) + x^2q'(x)$$

so $L(p(x)+q(x)) = L(p(x)) + L(q(x))$.

$L(\alpha p(x)) = \alpha p(x) + x\alpha p(x) + x^2(\alpha p(x))' = \alpha(p(x) + xp(x) + x^2 p'(x))$ and $\alpha L(p(x)) = \alpha(p(x) + xp(x) + x^2 p'(x))$

so $\alpha L(p(x)) = \alpha L(p(x))$. Thus L is a linear transformation.

(11a.) YES:

$$L(f+g) = (f+g)(0) = f(0) + g(0) \text{ and } L(f) + L(g) = f(0) + g(0)$$

so $L(f+g) = L(f) + L(g)$.

$$L(\alpha f) = (\alpha f)(0)) = \alpha f(0) \text{ and } \alpha L(f) = \alpha f(0)$$

so $\alpha L(f) = \alpha L(f)$. Thus L is a linear transformation.

(11b.) NO: Consider $-2L(1) = -2|1| = -2$ and $L(-2) = |-2| = 2$. Since these are not equal, L is not a linear transformation.

(11c.) YES:

$L(f+g) = [(f+g)(0) + (f+g)(1)]/2 = [f(0) + f(1)]/2 + [g(0) + g(1)]/2$ and $L(f) + L(g) = [f(0) + f(1)]/2 + [g(0) + g(1)]/2$

so $L(f+g) = L(f) + L(g)$.

$$L(\alpha f) = [(\alpha f)(0)) + \alpha(f(1))]/2 = \alpha[f(0) + f(1)]/2 \text{ and } \alpha L(f) = \alpha\alpha[f(0) + f(1)]/2$$

so $\alpha L(f) = \alpha L(f)$. Thus L is a linear transformation.

(11b.) NO: Consider $L(x+1) = \left[\int_0^1 (x+1)^2 dx\right]^{1/2} = \left[\int_0^1 x^2 + 2x + 1 dx\right]^{1/2} = \sqrt{7/3}$ and $L(x) + L(1) = \left[\int_0^1 x^2 dx\right]^{1/2} + \left[\int_0^1 1 dx\right]^{1/2} = \sqrt{1/3} + 1$. Since these are not equal, L is not a linear transformation.

(13.) Let $\mathbf{v}$ be a vector in V, then $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$. So

$$L_1(\mathbf{v}) = L_1(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = \alpha_1 L_1(\mathbf{v}_1) + \cdots + \alpha_n L_1(\mathbf{v}_n)$$

Similarly,

$$L_2(\mathbf{v}) = L_2(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = \alpha_1 L_2(\mathbf{v}_1) + \cdots + \alpha_n L_2(\mathbf{v}_n)$$

As $L_1(\mathbf{v}_i) = L_2(\mathbf{v}_i)$ for all i, we get

$$L_1(\mathbf{v}) = \alpha_1 L_1(\mathbf{v}_1) + \cdots + \alpha_n L_1(\mathbf{v}_n) = \alpha_1 L_2(\mathbf{v}_1) + \cdots + \alpha_n L_2(\mathbf{v}_n) = L_2(\mathbf{v})$$

This means $L_1(\mathbf{v}) = L_2(\mathbf{v})$ for every vector in V. Therefore $L_1 = L_2$

(19.) Recall that $\ker(L) = \{\mathbf{v} \mid L(\mathbf{v}) = 0\}$ and the image of L is the subspace $L(V) = \text{Span}(L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n))$ where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Note that $\{1, x, x^2\}$ is a basis for $\mathbb{P}_3$.
(19a.) If $p(x)$ in $\ker(L)$, then $xp'(x) = 0$ which can only happen if $p'(x) = 0$. This means that $p(x)$ is a constant, so

$$\ker(L) = \text{Span}(1) = \mathbb{P}_1$$

$$L(\mathbb{P}_3) = \text{Span}(L(1), L(x), L(x^2)) = \text{Span}(0, x, 2x^2) = \text{Span}(x, x^2) = \{ax + bx^2 \mid a, b \text{ are in } \mathbb{R}\}$$

(19b.) If $p(x)$ in $\ker(L)$, then $p(x) = p'(x)$. Since there is no polynomial which satisfies this equation aside from $\mathbf{0}$,

$$\ker(L) = \text{Span}(0) = \{\mathbf{0}\}$$

$$L(\mathbb{P}_3) = \text{Span}(L(1), L(x), L(x^2)) = \text{Span}(1, x-1, x^2-2x) = \text{Span}(1, x, x^2) = \mathbb{P}_3$$

(19c.) If $p(x)$ in $\ker(L)$, then $p(0)x + p(1) = 0$. In this case we must have $p(0) = 0$ and $p(1) = 0$. Thus x and $x-1$ are factors of $p(x)$. So $p(x)$ is a scalar multiple of $(x)(x-1)$:

$$\ker(L) = \text{Span}((x)(x-1)) = \{a(x)(x-1) \mid a \text{ is in } \mathbb{R}\}$$

$$L(\mathbb{P}_3) = \text{Span}(L(1), L(x), L(x^2)) = \text{Span}(x+1, 1, 1) = \text{Span}(1, x) = \mathbb{P}_2$$