Solutions to HW 5

6.3

(1b.) Use ~~Mathematica~~ eig(a) $\Rightarrow \lambda_1 = 2 \quad \lambda_2 = 1$

null(A - λI,'r') $\Rightarrow \bar{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\Rightarrow \boxed{X = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}} \text{ and } X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

(1f.) $\lambda_1 = 2, \ \lambda_2 = 0, \ \lambda_3 = 0$

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \bar{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{X = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}} \text{ and } X^{-1}AX = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4a.) first, diagonalize $A \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 0$ eigenvalues

$$\bar{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow X = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = X^{-1}AX \Rightarrow \text{let } \boxed{B' = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{then } (XB'X^{-1})^2 = XB'^2X^{-1} = XDX^{-1} = A \Rightarrow \boxed{B = XB'X^{-1} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}}$$

(4b.) Use same procedure as $A \Rightarrow \lambda_1 = 9, \ \lambda_2 = 4, \ \lambda_3 = 1$

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \bar{x}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = X^{-1}AX \Rightarrow \text{let } B' = \begin{pmatrix} \sqrt{9} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{1} \end{pmatrix}$$

$$\text{So } B' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \boxed{B = XB'X^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$\boxed{6.3}$

(12.) If A is nilpotent then $A^k = \bar{0}$ for some $k \Rightarrow$ $\lambda_i^k = 0$ for every eigenvalue of $A \Rightarrow \lambda = 0$. So every eigenvalue of A is 0. Now, if $A \neq 0 \Rightarrow N(A) \neq \mathbb{R}^n$ $\Rightarrow \dim(N(A)) < n$. This means A has n eigenvalues (all 0) but less than n lin. indep. eigenvectors.

(27a.) Eigenvalues $\Rightarrow \lambda_1 = 0$, $\lambda_2 = 1$

$$\bar{x}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow X = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix} \Rightarrow D = X^{-1}AX = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^D = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2.7183 \end{pmatrix} \Rightarrow e^A = Xe^DX^{-1} = \boxed{\begin{pmatrix} -5.4366 & -2.7183 \\ 16.3097 & 8.1548 \end{pmatrix}}$$

(27b.) Eigenvalues $\Rightarrow \lambda_1 = 1$, $\lambda_2 = -1$

$$\bar{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow X = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow D = X^{-1}AX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow e^D = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} = \begin{pmatrix} 2.7183 & 0 \\ 0 & 0.3679 \end{pmatrix} \Rightarrow e^A = Xe^DX^{-1} = \boxed{\begin{pmatrix} 5.0687 & 4.7008 \\ -2.3504 & -1.9825 \end{pmatrix}}$$

(27c.) Eigenvalues $\Rightarrow \lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \bar{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \bar{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow D = X^{-1}AX = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow e^D = \begin{pmatrix} e^1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.7183 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2.7183 & 1.7183 & 1.718 \\ -1.7183 & -0.7183 & -1.718 \\ 1.7183 & 1.7183 & 2.718 \end{pmatrix}}$$

$\boxed{6.4}$

(1a) $\|\bar{z}\| = \sqrt{\bar{z}^H \bar{z}} = \sqrt{36} = 6$

$\|\bar{w}\| = \sqrt{\bar{w}^H \bar{w}} = \sqrt{9} = 3$

$\langle \bar{z}, \bar{w} \rangle = \bar{w}^H \bar{z} = -4 + 4i$

$\langle \bar{w}, \bar{z} \rangle = \overline{\langle \bar{z}, \bar{w} \rangle} = -4 - 4i$

(1b.) $\|\bar{z}\| = \sqrt{16} = 4$

$\|\bar{w}\| = \sqrt{49} = 7$

$\langle \bar{z}, \bar{w} \rangle = -4 + 10i$

$\langle \bar{w}, \bar{z} \rangle = -4 - 10i$