Math 572
Winter 2010

Solutions to Exam 2 Review

(1a.)

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{pmatrix}\\ &= (-4-\lambda)(5-\lambda)+18\\ &= \lambda^2-\lambda-2\\ &= (\lambda-2)(\lambda+1)\end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. We compute the eigenvectors using the "`null()`" function in Matlab using the "`'r'`" option:

$\texttt{null}(\texttt{A}-2*\texttt{eye}(2),'\texttt{r}') = \begin{bmatrix}-1\\ 1\end{bmatrix}$

This gives $\mathbf{x}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$ as an eigenvector for λ_1.

$\texttt{null}(\texttt{A}+\texttt{eye}(2),'\texttt{r}') = \begin{bmatrix}-2\\ 1\end{bmatrix}$

This gives $\mathbf{x}_2 = \begin{pmatrix}-2\\ 1\end{pmatrix}$ as an eigenvector for λ_2.

(1b.)

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}1-\lambda & 0 & 0\\ 4 & 9-\lambda & 2\\ 2 & 4 & 2-\lambda\end{pmatrix}\\ &= (1-\lambda)[(9-\lambda)(2-\lambda)-8]\\ &= (1-\lambda)(\lambda^2-11\lambda+10)\\ &= (1-\lambda)^2(10-\lambda)\end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 10$. We compute the eigenvectors using the "`null()`" function in Matlab using the "`'r'`" option:

$2*\texttt{null}(\texttt{A}-\texttt{eye}(3),'\texttt{r}') = \begin{bmatrix}-4 & -1\\ 2 & 0\\ 0 & 2\end{bmatrix}$

This gives $\mathbf{x}_1 = \begin{pmatrix}-4\\ 2\\ 0\end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ as eigenvectors for λ_1 and λ_2.

$\texttt{null}(\texttt{A}-10*\texttt{eye}(3),'\texttt{r}') = \begin{bmatrix}0\\ 2\\ 1\end{bmatrix}$

This gives $\mathbf{x}_3 = \begin{pmatrix}0\\ 2\\ 1\end{pmatrix}$ as an eigenvector for λ_3.

(1c.)

$$\det(A-\lambda I) = \det\begin{pmatrix} 5-\lambda & 0 & 0 \\ 3 & 1-\lambda & 1 \\ 7 & -2 & 3-\lambda \end{pmatrix}$$
$$= (5-\lambda)[(1-\lambda)(3-\lambda)+2]$$
$$= (5-\lambda)(\lambda^2 - 4\lambda + 5)$$

Therefore the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 2+i$ and $\lambda_3 = 2-i$. We compute the eigenvectors using the "`null()`" function in Matlab using the "`'r'`" option:

$$22 * \texttt{null}(\texttt{A} - 5 * \texttt{eye}(3), '\texttt{r}') = \begin{bmatrix} 10 \\ 13 \\ 22 \end{bmatrix}$$

This gives $\mathbf{x}_1 = \begin{pmatrix} 10 \\ 13 \\ 22 \end{pmatrix}$ as an eigenvector for λ_1

Since λ_2 and λ_3 are complex conjugates, we only need to calculate the eigenvector for λ_2:

$$2 * \texttt{null}(\texttt{A} - (2+\texttt{i}) * \texttt{eye}(3), '\texttt{r}') = \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix}$$

This gives $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1-i \\ 2 \end{pmatrix}$ as an eigenvector for λ_2 and $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1+i \\ 2 \end{pmatrix}$ as an eigenvector for λ_3.

(2a.)

$$\det(A-\lambda I) = \det\begin{pmatrix} a-\lambda & -b \\ b & a-\lambda \end{pmatrix}$$
$$= (a-\lambda)(a-\lambda) + b^2$$
$$= \lambda^2 - 2a\lambda + (a^2+b^2)$$

Therefore the eigenvalues are:

$$\lambda_1 = \frac{2a + \sqrt{4a^2 - 4(a^2+b^2)}}{2}$$
$$= \frac{2a + 2\sqrt{-b^2}}{2}$$
$$= a + bi$$

and $\lambda_2 = a - bi$.

(2b.) The eigenvalues of A are real if and only if $b = 0$.

(3a.) This system is can be written as:

$$\mathbf{Y}' = A\mathbf{Y}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 5 & -18 \\ 6 & 1 & -1 \end{pmatrix}$$

Using the matlab command "`eig()`", we see that the eigenvalues of A are $\lambda_1 = 2+3i$, $\lambda_2 = 2-3i$ and $\lambda_3 = 2$.

Using the "`null()`" function in Matlab using the "`'r'`" option, we get the corresponding eigenvectors:

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 3+3i \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 3-3i \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_3 = \begin{pmatrix} -3 \\ 33 \\ 5 \end{pmatrix}$$

Therefore we have

$$\mathbf{Y}_1 = \begin{pmatrix} 0 \\ e^{2t}(3\cos 3t - 3\sin 3t) \\ e^{2t}(\cos 3t) \end{pmatrix}, \mathbf{Y}_2 = \begin{pmatrix} 0 \\ e^{2t}(3\cos 3t + 3\sin 3t) \\ e^{2t}(\sin 3t) \end{pmatrix} \text{ and} \mathbf{Y}_3 = \begin{pmatrix} -3e^{2t} \\ 33e^{2t} \\ 5e^{2t} \end{pmatrix}$$

So the general solutions is

$$\mathbf{Y} = c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + c_3\mathbf{Y}_3 = \begin{pmatrix} c_3(-3e^{2t}) \\ c_1e^{2t}(3\cos 3t - 3\sin 3t) + c_2e^{2t}(3\cos 3t + 3\sin 3t) + c_3 33e^{2t} \\ c_1e^{2t}(\cos 3t) + c_2e^{2t}(\sin 3t) + c_3 5e^{2t} \end{pmatrix}$$

(3b.) First we set $y_3 = y_1'$ and $y_4 = y_2'$, this gives us the first order system:

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= -4y_1 + 5y_2 + 5y_3 - y_4 \\ y_4' &= 5y_2 - 4y_3 + 4y_4 \end{aligned}$$

This system is can be written as:

$$\mathbf{Y}' = A\mathbf{Y}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 5 & 5 & -1 \\ 0 & 5 & -4 & 4 \end{pmatrix}$$

Using the matlab command "`[v d]=eig()`", we see that the eigenvalues of A are (rounded to 4 decimals) $\lambda_1 = 5.0000$, $\lambda_2 = 4.8595$, $\lambda_3 = 0.5741$ and $\lambda_4 = -1.4337$.

Using the same command gives us the corresponding eigenvectors:

$$\mathbf{x}_1 = \begin{pmatrix} -0.0000 \\ 0.1961 \\ -0.0000 \\ 0.9806- \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -0.0085 \\ -0.2014 \\ -0.0414 \\ -0.9786 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} -0.8236 \\ -0.2715 \\ -0.4729 \\ -0.1559 \end{pmatrix} \text{ and } \mathbf{x}_4 = \begin{pmatrix} 0.2503 \\ 0.5144 \\ -0.3588 \\ -0.7375 \end{pmatrix}$$

Therefore the general solution is given by

$$\mathbf{Y} = c_1e^{\lambda_1}\mathbf{x}_1 + c_2e^{\lambda_2}\mathbf{x}_2 + c_3e^{\lambda_3}\mathbf{x}_3 + c_4e^{\lambda_3}\mathbf{x}_4$$

(4a.)

$$< \mathbf{x}, \mathbf{y} >= \mathbf{y}^H\mathbf{x} = -4 - 11i$$

$$< \mathbf{y}, \mathbf{x} >= \overline{< \mathbf{x}, \mathbf{y} >} = -4 + 11i$$

$$||\mathbf{x}|| = \sqrt{x^h x} = \sqrt{39}$$

$$||\mathbf{y}|| = \sqrt{y^H y} = \sqrt{94}$$

(4b.)

$$< \mathbf{x}, \mathbf{y} >= \mathbf{y}^H \mathbf{x} = 39 - 25i$$

$$< \mathbf{y}, \mathbf{x} >= \overline{< \mathbf{x}, \mathbf{y} >} = 39 + 25i$$

$$||\mathbf{x}|| = \sqrt{x^h x} = \sqrt{58}$$

$$||\mathbf{y}|| = \sqrt{y^H y} = \sqrt{94}$$

(5a.) Using the matlab command "`[v d]=eig()`", we see that the eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 9$. However, the dimension of $N(A - 9I)$ is one. This means that A is NOT diagonalizable.

(5b.) Using the matlab command "`[v d]=eig()`", we see that the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. However, the dimension of $N(A - I)$ is one. This means that A is NOT diagonalizable.

(5c.) Using the matlab command "`[v d]=eig()`", we see that the eigenvalues of A (rounded to 4 decimals) are $\lambda_1 = -3.4900$, $\lambda_2 = 1.7450 + 3.1836i$, $\lambda_3 = 1.7450 - 3.1836i$ and $\lambda_4 = 4.0000$. Since the eigenvalues are distinct, we know that A is diagonalizable. If we let U be the matrix `v` produced by the above command, then

$$U = \begin{pmatrix} 0.7121 & 0.6988 & 0.6988 & -0.5164 \\ 0.2581 & 0.2061 + 0.4335i & 0.2061 - 0.4335i & -0.2582 \\ -0.5894 & -0.0204 - 0.2687i & -0.0204 + 0.2687i & 0.2582 \\ -0.2807 & -0.4474 - 0.0923i & -0.4474 + 0.0923i & 0.7746 \end{pmatrix}$$

and

$$D = U^{-1}AU = \begin{pmatrix} -3.4900 & 0 & 0 & 0 \\ 0 & 1.7450 + 3.1836i & 0 & 0 \\ 0 & 0 & 1.7450 - 3.1836i & 0 \\ 0 & 0 & 0 & 4.0000 \end{pmatrix}$$

Finally, as $E^A = Ue^D U^{-1}$,

$$E^A = U \begin{pmatrix} 0.0305 & 0 & 0 & 0 \\ 0 & -5.7210 - 0.2404i & 0 & 0 \\ 0 & 0 & -5.7210 + 0.2404i & 0 \\ 0 & 0 & 0 & 54.5982 \end{pmatrix} U^{-1}$$

$$= \begin{pmatrix} -36.8683 & -31.4741 & -28.9211 & -61.8286 \\ -15.7882 & -20.7243 & -13.3081 & -31.1969 \\ 14.6468 & 16.5848 & 10.7463 & 29.9101 \\ 49.6694 & 42.9061 & 35.9406 & 90.0330 \end{pmatrix}$$

(6a.) True - $A^H = A^T$ if A is real.

(6b.) True - by theorem.

(6c.) True - a matrix with n distinct eigenvalues has at least n linearly independent eigenvectors.

(6d.) False - $U^H = U^{-1}$ so this is only true if $U = U^{-1}$.