EXTENSION OF STANLEY'S ALGORITHM TO SPHERICAL VARIETIES
(PRELIMINARY VERSION)

ROY JOSHUA AND SHAUN VAN AULT

ABSTRACT. About 20 years ago, Richard Stanley formulated a remarkable algorithm for computing the intersection cohomology Betti numbers of toric varieties. It was shown in a recent paper by the first author and Michel Brion that one can extend the same algorithm to a large class of spherical varieties. In this paper, the authors discuss the implementation of this algorithm as a program written with the help of LiE and GAP packages.

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1. Introduction: the mathematical framework and some elementary examples

Let $G$ be a complex connected reductive algebraic group and let $B$ be a Borel subgroup. A normal complex algebraic variety $X$, equipped with an action of $G$, is spherical if it contains a dense orbit of $B$. Spherical $G$-varieties enjoy the following properties:

- they contain only finitely many $B$-orbits, and hence only finitely many $G$-orbits,
- each $G$-orbit admits a slice (see ?? below) which is an affine spherical variety under a connected reductive subgroup of $G$, and
- the associated link (see ?? below) is a projective spherical variety, of strictly smaller dimension.

Given a complex connected reductive group $G$ as above, we will restrict to projective $G \times G$-reductive varieties as in [?] and [?, Section 5]. We will recall their definition from [?] presently.

Consider a projective irreducible $G \times G$-variety $X$ equipped with an ample $G \times G$-linearized line bundle $L$. Let $R = \bigoplus_{n=0}^{\infty} \Gamma(X, L^\otimes n)$, this is a graded, finitely generated reduced algebra, where $G \times G$ acts. This defines an affine variety $\tilde{X}$ where $G_m \times G \times G$ acts. Further, the action of $G_m$ is attractive, and the corresponding link is nothing but $X$. We say that the pair $(X, L)$ is a linearized projective $G \times G$-variety.

Put $\tilde{G} = G_m \times G$, this is a connected reductive group with weight lattice $\tilde{\Lambda} = \mathbb{Z} \times \Lambda$. We may regard $\tilde{X}$ as a $\tilde{G} \times \tilde{G}$-variety, where $G_m \times G_m$ acts via its morphism $(t_1, t_2) \mapsto t_1 t_2^{-1}$ to $G_m$. For any $x \in X$ with representative $\tilde{x} \in \tilde{X}$, we obtain readily an exact sequence of isotropy groups:

\[(1.0.1) \quad 1 \to G_m \to (\tilde{G} \times \tilde{G})_{\tilde{x}} \to (G \times G)_x \to 1.\]

Now projective $G \times G$-reductive varieties may be characterized as those linearized projective $G \times G$-varieties $(X, L)$ that satisfy the following conditions:

(i) $X$ is normal.
(ii) There exists $x \in X$, fixed by $\text{diag} T$, such that the orbit $(B^- \times B)x$ is dense in $X$, and that $\text{diag} T$ fixes the fiber of $L$ at $x$.
(iii) The isotropy group $(G \times G)_x$ is connected.

Again, (iii) is equivalent to the assumption of connectivity of $(B^- \times B)x$, or of $(T \times T)x$; and the set of all $x \in X$ satisfying (ii) is a unique $T \times T$-orbit: the orbit of base points, of dimension equal to $\text{rk} X$.

**Examples 1.1.** 1. Take $G = T$ a complex torus. Then the projective $G \times G$-varieties identify with the projective toric varieties associated to both $T$ and all quotient tori of $T$.

2. One can require in addition that the projective reductive varieties contain $G$ as a dense open orbit: we then obtain compactifications of algebraic groups considered in [?], [?] and [?].

Next we recall that projective $G \times G$-reductive varieties are determined combinatorially from certain polytopes just as in the case of toric varieties.

Let $\sigma \subseteq \tilde{\Lambda}_{\mathbb{R}} = \mathbb{R} \times \Lambda_{\mathbb{R}}$ be the cone associated to $\tilde{X}$, and put $\delta = \sigma \cap (1 \times \Lambda_{\mathbb{R}})$. Then $\delta$ is a lattice polytope in $\Lambda_{\mathbb{R}}$, and $\sigma$ is the cone over $\delta$. Since $\sigma$ is $W$-admissible, $\delta$ satisfies the following conditions:

(i) The relative interior $\delta^0$ meets $\Lambda^+_{\mathbb{R}}$.
(ii) The distinct translates $w\delta^0$ ($w \in W$) are disjoint.
A lattice polytope $\delta \subset \Lambda_\mathbb{R}$ satisfying (i) and (ii) is called a \textit{W-admissible polytope}. These classify polarized reductive varieties; we denote by $(X_\delta, L_\delta)$ the linearized reductive variety with polytope $\delta$, then $\dim X_\delta = \text{rk}\delta$. The closure in $X$ of the orbit of base points, equipped with the restriction of $L$, is the linearized toric variety with polytope $\delta$. The $G \times G$-orbit closures in $X_\delta$ are the $X_\phi$, where $\phi \subseteq \delta$ is a $W$-admissible face.

Since $N_W(\sigma) = N_W(\delta)$ and $C_W(\sigma) = C_W(\delta)$, we obtain two subsets $J = J(\delta) \subseteq I = I(\delta) \subseteq \Pi$ satisfying the properties of the previous subsection. Now the description (??) of the isotropy group $(G \times G)_x$ carries over to this projective setting, with $\Lambda_\sigma$ being replaced by the lattice $\Lambda_\delta$ spanned by the \textit{differences} of any two elements of $\Lambda \cap \delta$.

As a consequence, the description of orbits as fibered spaces carries over as well; specifically, the analogues of (??), (??), (??) and (??) hold with $\sigma$ being replaced by $\delta$. Further, projective embeddings of a quotient of $G$ by a connected normal subgroup (resp. of $G$) correspond to $W$-invariant lattice polytopes (resp. with nonempty interior).

1.1. We obtain a combinatorial description of slices and links in reductive varieties.

Consider a $W$-admissible polytope $\delta \subset \Lambda_\mathbb{R}$, and a $W$-admissible face $\varphi \subseteq \delta$. These correspond to a linearized reductive variety $(X_\delta, L_\delta)$ together with a $G \times G$-orbit $\mathcal{O} = \mathcal{O}_\varphi$: the open orbit in $X_\varphi \subseteq X_\delta$. We describe the local structure of $X$ along $\mathcal{O}$, by making explicit the objects introduced in 3.2.

Let $x$ be a base point of $\mathcal{O}$, then $(B^- \times B)x$ is open in $(G \times G)x = \mathcal{O}$. Further, it follows from (??) that the normalizer $P$ of $(B^- \times B)x$ in $G \times G$ equals $P^-_J \times P_J$, where $J = J(\varphi)$. Since $x$ is fixed by $\text{diag}T$, the Levi subgroup $L$ of $P$ equals $L_J \times L_J$. Further, by [AB1] Lemma 2.8, the variety $\Sigma$ is an affine reductive variety for $L_J$; one readily checks that the corresponding $W_J$-admissible cone is generated by the differences $\lambda - \mu$, where $\lambda \in \delta$ and $\mu \in \varphi$.

Now by (??) again, we have $L_x = G_\varphi \times G_\varphi$. Note that $G_\varphi$ is a connected reductive subgroup of $G$, normalized by $T$; further, $T_\varphi$ is a maximal torus of $G_\varphi$, so that the weight lattice of $G_\varphi$ equals $\Lambda_\varphi = \Lambda/\Lambda_\varphi$. The set of simple roots of $G_\varphi$ is $J = J(\varphi)$, with Weyl group $W_J = C_W(\varphi)$; we denote the latter by $W_\varphi$.

By [AB1] Lemma 4.1, the slice $S_x$ is an affine reductive variety for $G_\varphi$. Denote its $W_\varphi$-admissible cone by $\sigma = \sigma_\varphi$; this cone is the image in $\Lambda_\varphi$ of the cone of $\Sigma$. So we may regard $\sigma$ as the normal cone to $\delta$ along its face $\varphi$. Note the equality $\text{rk} S_x = \dim \delta - \dim \varphi$.

To describe the link $\mathbb{P}(S_x)$, note first that the closed convex cone $\sigma$ contains no line. Thus, we may find a linear form $f$ on $\Lambda_{\mathbb{R}}/\Lambda_\varphi \mathbb{R}$ that takes positive values at all non-zero points of $\sigma$. We may assume, in addition, that $f$ takes integer values at all points of $\Lambda/\Lambda_\varphi$, and is invariant under the normalizer of $\sigma$ in $W_\varphi$. Then by [AB1] 3.2, 4.1, $f$ yields a positive $G_\varphi \times G_\varphi$-invariant grading of the algebra of regular functions on $S_x$. In other words, $f$ defines an attractive $G_{m}$-action on $S_x$ that commutes with the action of $G_\varphi \times G_\varphi$. Now $\mathbb{P}(S_x)$ is the reductive variety for $G_\varphi$ associated with the polytope $\sigma \cap (f = n)$, where $n$ is a suitable positive integer. We may regard this polytope as the link of $\delta$ along its face $\varphi$; we have $\text{rk} \mathbb{P}(S_x) = \dim \delta - \dim \varphi - 1$.

1.2. The closure property of the class of group imbeddings. If $X_\delta$ is an embedding of a quotient of $G$ by a connected normal subgroup, then $\delta$ is $W$-invariant, so that $\sigma_\varphi$ is invariant under $W_\varphi$. Thus, $S_x$ is an embedding of a quotient of $G_\varphi$ by a connected normal subgroup. So the class of embeddings of connected reductive groups is stable under taking slices and, likewise, links. Therefor we will mostly restrict to these class of reductive varieties.

We end this section by recalling the key result from [?].
Theorem 1.2. Let $X$ be a projective reductive $G \times G$-variety; let $IP_X(t)$ ($IP_{X,x}(t)$) be the Poincaré polynomial for global intersection cohomology (for the stalks of the intersection cohomology sheaves at $x \in X$, respectively). Then

\[
(1.2.1) \quad IP_X(t) = \sum_x (1-t^2)^{r-r_x} \frac{P_{G/T}(t)}{P_{G_x/T_x}(t)} t^{2d_x} IP_{X,x}(\frac{1}{t})
\]

(sum over representatives of $G$-orbits in $X$), and

\[
(1.2.2) \quad IP_{X,x}(t) = IP_{S_x,x}(t) = \tau_{\leq d_x-1}((1-t^2)IP_{\mathbb{P}(S_x)}(t)).
\]

(sum over representatives of $G$-orbits in $X$), and

\[
(1.2.3) \quad IP_{X,x}(t) = IP_{S_x,x}(t) = \tau_{\leq d_x-1}((1-t^2)IP_{\mathbb{P}(S_x)}(t)).
\]

Here $T$ is a maximal torus of $G$, of dimension $r$; $T_x$ is a maximal torus of $G_x$, of dimension $r_x$; $S_x$ is a slice at $x$ to $Gx$, of dimension $d_x$; $\mathbb{P}(S_x) = (S_x - x)/\mathbb{G}_m$ is the corresponding link for an attractive action of the multiplicative group, and $\tau_{\leq d_x-1}$ denotes the truncation to degrees $\leq d_x - 1$.

(In fact, the Poincaré polynomial $P_{G_x/T_x \times G_x/T_x}(t)$ divides $P_{G/T \times G/T}(t)$, and the quotient has non-negative coefficients. See [?] p. 321.)

Definition 1.3. A face is $W$-admissible if its relative interior intersects with the non-negative Weyl-chamber.

2. The algorithm (in outline)

Step by Step algorithm

Initial Processing

1. Enter the reductive group using the classification scheme in LiE. Generate the Cartan matrix of the algebraic group using LiE: i.e. Cartan(group) from LiE but preferably from Gap as LieCartan(group). (Here Cartan is a built-in function in LiE and LieCartan is the way to call this when invoked from GAP.)

2. Compute the Weyl group of the given group. (There are procedures to do this in LiE.) Also generate and store the Poincaré series for $G/B \times G/B$ (An easy formula for this is discussed in [BJ2].)

3. Read in the ($W$-stable) polytope by giving its vertices in the root lattice

The Recursive or iterative part

For each $W$-admissible face $\phi$ of $\delta$ one calls the following steps recursively or iteratively until the polytope in Step 5 reduces to a point.

4. For each $W$-admissible face $\phi$ of the polytope $\delta$, compute the sets $I(\phi)$ ($J(\phi)$) of simple roots so that reflections in a plane orthogonal to them leaves $\phi$ stable (point-wise fixed, respectively). Compute also the lattice $\Lambda_\phi$ (generated by the differences $\lambda - \mu, \lambda, \mu$ in the face $\phi$) and the quotient $\Lambda/\Lambda_\phi$ by obtaining a set of integral basis vectors for the lattices. Compute also the Weyl sub-group $W_\phi$ generated by the simple roots in $J(\phi)$.

5. Compute $\sigma_\phi = \text{the normal cone to } \phi$ in $\delta$. Obtain the polytope $\sigma_\phi \cap (f = 1)$, for some linear functional that is invariant by the normalizer of the normal cone $\sigma_\phi$ in $W_\phi$, positive and integral on the cone $\sigma_\phi$. 
(6) Compute the Poincaré polynomial \( P_\phi = \sum_{w \in W_\phi} t^{2(l(w))} = P_{G_x/T_x}(t) \) where the point \( x \) is in the orbit corresponding to \( \phi \). \( (G_x) \) is the maximal reductive subgroup of the stabilizer at \( x \) and \( T_x \) is a maximal torus in \( G_x \). Substitute this into the formula (1.2.1). Replace \( W \) by \( W \) and the old \( \Lambda \) by \( \Lambda/\Lambda_\phi \) which is represented by a sub-lattice normal to \( \phi \).

(7) Repeat steps 4 through 7 until the polytope in the last step is a point. When the normal polytope \( \delta_\phi \) is a point, the corresponding reductive variety is the \( G=P \) variety \( G=P_{I(\phi)}(x) \), \( G=P_{I(\phi)}(x) \). The Poincaré polynomial for this is \( (\sum_{w \in W_{I(\phi)}} t^{2(l(w))})^2 \) where \( W_{I(\phi)} \) denotes the subset of representative of minimal length of \( W/W_{I(\phi)} \) with \( W_{I(\phi)} \) denoting the subgroup of \( W \) generated by the simple roots in \( I(\phi) \).

3. Implementation

3.1. Input and Output. For us it seemed more convenient to write a significant part of our code in the GAP package. However, we need several basic Lie-theoretic computations performed by the LiE package. Therefore, we have chosen to write an outer shell that is entirely in GAP; it calls on LiE as necessary to perform various calculations that are more Lie-theoretic and involve the structure of the weight-lattice. Certain features of LiE and GAP have dictated the choice of several aspects of our program: for example, the representation of polytopes is based on how vectors in the Lie-algebra are represented in LiE.

Since the actual simple roots often involve irrational numbers as their Cartesian co-ordinates, representing the actual simple roots in the usual Cartesian co-ordinates would essentially involve numerical approximations. To avoid this and other related issues, vectors are represented in LiE by their co-ordinate vectors with respect to the basis of simple roots. For example, in case the group is \( PGL_3 \) or \( SL_3 \), its Lie-algebra is of type \( A_2 \) and the simple roots are \( \alpha \) and \( \beta \), with \( \alpha \) along the \( x \)-axis and \( \beta \) in the second quadrant making an angle of \( 2\pi/3 \) radians with \( \alpha \). In this case the vector \( \alpha + \beta \) will be represented as \([1, 1]\). The length of the simple roots are normalized in LiE somewhat different from the standard conventions: for example, for type \( A_n \), all simple roots have length \( \sqrt{2} \) and not 1.

Therefore, whenever the given algebraic group is a product of simple groups (or more generally, semi-simple) any admissible (see Definition 1.3) polytope whose vertices lie in the associated weight-lattice is allowed (with the exception that its vertices are all required to be distinct from the origin). In the case the algebraic group is the product of a central torus and simple groups (or more generally, a central torus and a semi-simple group), then the weight lattice is extended by taking its product with a lattice equal to the lattice of characters of the given central torus. In this case any admissible polytope whose vertices line in this extended lattice is allowed, though we again require that none of the vertices coincide with the origin.

Here are some examples.

**Example 3.1. Input:** set default \( A_1A_1 \). (Now the group is \( SL_2 \times SL_2 \).) polytope = \([2, 0], [0, 2], [-2, 0], [0, -2]\)

**Output:** The Poincare polynomial = \( a_0X[0] + a_2X[2] \cdots + a_12X[12] \) where \( a_i \) is the coefficient of \( t^i \).
In the last example, the Weyl-group consists of the reflections about the $x$- and $y$-axes. Therefore the positive Weyl chamber is the first quadrant and the above polytope is symmetric with respect to the action of this Weyl-group. The $W$-admissible face of dimension 1 is the edge joining $[0,2]$ and $[2,0]$.

**Example 3.2. Input:** set default A1T1. (Now the group is $GL_2$ or $SL_2 \times \mathbb{G}_m$.)

polytope $=[[0,1],[2,0],[-2,0]]$

**Output:** The Poincare polynomial $= a_0X[0] + a_2X[2] + \cdots + a_8X[8]$

In the last example, we let the single positive simple root be along the positive $x$-axis. In this case the Weyl-group acts by reflection about the $y$-axis. Therefore the above polytope is $W$-stable. The edges joining $[0,1]$ with $[2,0]$ and $[-2,0]$ with $[2,0]$ are $W$-admissible faces of dimension 1.

**Remark 3.3.** In practice, one first enters the GAP environment, then loads LiE by the commands

Read("liegap.g")
Read("lie.g")

Our program is loaded into this environment by

Read("BJAlgorithm.g")

The command to set a particular group as default is then Set_default("group");

### 3.2. Implementation of the key-steps.

#### 3.2.1. The initial processing and computing the Poincaré polynomials.

As discussed above, one needs to perform certain operations initially: the program, for example computes the Poincaré polynomial $P_{G/B \times G/B}(t)$, the Weyl group of $G$, the fundamental roots etc. Since these are needed for later computations, the program stores these values in some convenient data structure.

The Poincaré polynomial $P_{G/B}(t)$ is computed as follows. There are built-in functions in LiE that compute the following: the longest element of a given group (the LiE-function: $\text{long\_word}$), the set of all Bruhat descendants of a given word in the Weyl-group (the Lie-function: $\text{Bruhat\_desc}$) and the length of a given word in the Weyl-group (the LiE-function: $\text{length}$). Making use of these built-in functions, we have written that computes the Poincaré polynomial $P_{G/B}(t) = \Sigma_{w \in W} t^{l(w)}$ of the flag-variety $G/B$ where $G$ is the given group and $B$ is a Borel subgroup of $G$. In fact, since we also need to compute the Poincaré series, $P_{G_x/B_x}(t)$ for the various stabilizer subgroups $G_x$ appearing in (1.2.3), we have written a routine that takes any subgroup $W'$ of the Weyl-group $W$ of the group $G$ and computes the Poincaré polynomial $P_{W'}(t) = \Sigma_{w' \in W'} t^{l_2(w')}$. 

#### 3.2.2. The iterative part.

Here the main effort is in analyzing the given polytope, constructing admissible normal polytopes, $\delta_\phi$, associated to various $W$-admissible faces $\phi$ and also the constructing the corresponding smaller weight-lattices associated to the groups $G_x$.

First, we have written a routine that takes each face and tests if it is $W$-admissible in the sense of Definition 1.3.

Then for each $W$-admissible face, we have another routine that finds a set of generators for the associated normal cone. In case the $W$-admissible face $\phi$ is a $W$-admissible vertex, $v_0$, this routine simply takes the vectors $v_0v_j$ as $v_j$ varies among the vertices of the given polytope $\delta$ not in $\phi$. If the face $\phi$ is not a vertex, then we have a routine that does the following: Let $v_0$
denote a fixed vertex of the face $\phi$. Let $V$ denote the real vector space associated to the given weight lattice and let $W$ denote the subspace spanned by the vectors $v_0u_i$ as $u_i$ varies among the vertices in $\phi$. Our routine then takes the vectors $v_0v_j$ as $v_j$ varies among the vertices of the polytope $\delta$ not in $\phi$ and produces the orthogonal projections of these vectors orthogonal to the subspace of $W$ of $V$. This makes use of the LiE-function $\text{inprod}$ that computes an appropriate inner product of two given vectors in the vector space $V$.

The next key-step is the construction of a normal polytope and the associated weight-lattice. For each $j$, let $L_j = L - I$ denote the half-line starting at the vertex $v_0$ and contains the vector $v_0v_j$. The normal cone (considered above) does not contain any full line through the vertex $v_0$ so that there exists a hyperplane that intersects all the lines $L_j$ exactly at a point; in fact one can also assume that the normal to this hyperplane makes an acute angle with the above generators of the normal cone. Assume that $\dim(V) = n$ as a vector space and that the vertices of the polytope $\delta$ not in $\phi$ are listed as $v_j$, $j = 1, \ldots, n+k$, with $k$ an integer, which could be negative, 0 or positive. Then the normal vector $N$ to the above hyperplane is a vector in $V$ and therefore may be written as $N = (x_1, \ldots, x_n)$. The above conditions on the angle between $N$ and the vectors $v_0v_j = (a_{j1}, \ldots, a_{jn})$ translates to the inequalities

$$x_1a_{11} + \cdots + x_na_{1n} > 0 \quad (3.2.3)$$

$$\vdots$$

$$x_1a_{n+k1} + \cdots + x_na_{n+k1} > 0$$

The above inequalities may be written as a system of linear equations in $2n + k$-variables by introducing the slack variables $s_1, \ldots, s_{n+k}$:

$$x_1a_{11} + \cdots + x_na_{1n} - s_1 = 0 \quad (3.2.4)$$

$$\vdots$$

$$x_1a_{n+k1} + \cdots + x_na_{n+k1} - s_{n+k} = 0$$

with all the slack variables $s_i > 0$. Standard techniques for solving linear equations in GAP, then finds a solution for the $s_i$ and hence for the $x_i$s. This technique, therefore finds a hyperplane $H$ (through the origin of the vector space $V$) that intersects all the lines along the vectors $v_0v_j$ that generate the normal cone.

For each $W$-admissible face $\phi$, we have also written a routine that finds the subgroup $W_{\phi}$ of the Weyl group $W$ that fixes the face $\phi$ point-wise. Then the subgroup $W_{\phi}$ has an induced action on the vertices of the polytope $\delta$ not in the face $\phi$. By $W_{\phi}$-symmetry each hyperplane $wH$, $w \in W_{\phi}$ also will intersect the lines along the vectors $v_0v_j$ that generate the normal cone. Now one simply takes the sum of these hyperplanes $\sum_{w \in W_{\phi}} wH$ to obtain a hyperplane that is stable with respect to the action of $W_{\phi}$. This hyperplane, $H_0$ intersects the Weyl-chamber in $V$ and the new positive roots are the positive roots of $V$ that lie in $H_0$. 


The normal polytope $\delta_\phi$ has as vertices, the intersections of the lines $L_j$ (along the vectors $v_0 v_j$ that generate the normal cone) with the hyperplane $H_0$.

To continue the iteration, one replaces the original polytope $\delta$ (the original vector space $V$) with the normal polytope $\delta_\phi$ (the vector sub-space $H_0$, respectively). When the normal polytope reduces to a point, we stop the iteration and proceed by examining another $W$-admissible face. Our algorithm proceeds in a depth-first manner searching all the admissible faces and iterated faces starting with a single $W$-admissible face of the original polytope, i.e. We may form a tree with the original polytope as the root, the normal polytopes associated to all its faces as the vertices on the next level, the normal polytopes associated to the faces of the normal polytopes on the first level as the second level etc. Then the iterative algorithm proceeds by exploring this tree in a depth-first manner.

3.3. Explicit examples.

Examples 3.4. We will consider two examples of Type $A_n$. For type $A_n$ the reductive group will be either $PGL_{n+1}$ or $SL_{n+1}$. The semi-simple rank of these groups is $n$, the weight lattice is of dimension $n$ and the Weyl group $W$ identifies with the symmetric group $S_n$. Projective group imbeddings of $G$ are given by $W$-symmetric polytopes in $A_\mathbb{R}$ whose interior meets the positive Weyl chamber $A_\mathbb{R}^+$. Observe that in this case the Poincaré polynomial $P_{G/B}(t) = \Sigma_{i=0}^n t^{2i}$. For the sake of simplicity we will not consider the complication arising from our representation of vectors as co-ordinate vectors with respect to the basis of simple roots (and not as the actual vectors.)

1. We will first work out in detail the case of $GL_2$ considered also in [BJ2].

The weight-lattice for $PGL_2$ is one dimensional and we extend that by an orthogonal component to obtain the following two dimensional lattice representing the weight lattice of $GL_2$. Here the positive simple root is drawn along the positive $x$-axis so that positive Weyl chamber is the the right half-plane. As the polytope $\delta$, we take the triangle with vertices at $a$, $b$ and $c$.

![Diagram](https://example.com/diagram.png)

Of the faces, the 1-dimensional faces $bc$ and $ac$ and the vertices $b$ and $c$ are $W$-admissible. The vertex $b$ is fixed by the action of the Weyl group which is to reflect about the $y$-axis. The following is the corresponding tree diagram as the algorithm explores the various admissible faces iteratively.
2. Here we consider the group $G = PGL_3$ which is of type $A_2$. Therefore the root lattice is two dimensional with the simple roots denoted $\alpha$ and $\beta$. We consider the polytope given by the following hexagon.
In this case the admissible faces are $bc$, $cd$, $b$, $c$ and $d$. Therefore the tree for the iterated calls of our algorithm is as follows. (Here np denotes the normal polytope at a face.)
References


Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA
E-mail address: joshua@math.ohio-state.edu

Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA
E-mail address: ault@math.ohio-state.edu