# BRAUER GROUPS OF SMOOTH SPLIT TORIC SCHEMES AND THEIR FANS

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ABSTRACT. In this paper we consider the Brauer groups of toric schemes, with the open dense torus assumed to be split. We first discuss the basic theory when the base scheme is an arbitrary field. We also consider toric schemes for a split torus over discrete valuation rings in the sense of Mumford *et al.*, and Dedekind domains. We show that fans of such toric varieties and schemes have considerable bearing on the *m*-torsion part of their Brauer groups, where *m* is a positive integer invertible in the base ring.

We also consider the period-index problem for classes in the Brauer groups of toric varieties for large classes of toric varieties. Our results also find immediate application in determining the Brauer groups of toric stacks as in the work of Dhillon, Iyer and the first author.

### CONTENTS

1.	Introduction	1
2.	Brauer groups of split tori	3
3.	Brauer groups of smooth split toric varieties and their fans	12
4.	Extension of the above results to toric schemes defined over discrete valuation rings and Dedekind	
	domains	18
5.	The period-index problem for toric varieties	27
6.	Examples	31
7.	Appendix: Motivic cohomology over regular Noetherian base schemes	35
Ref	References	

## 1. INTRODUCTION

This paper revisits and greatly expands upon the computation of Brauer groups of toric varieties started in the 1993 paper of DeMeyer and Ford (see [DF]) where they computed the Brauer groups of toric varieties over the complex numbers and also the 1978 paper of Magid (see [Mag]) where he computed the Brauer groups of tori, again over the complex numbers. In section 2, we begin with a computation of the Brauer groups of split tori over any Dedekind domain or a discrete valuation ring that is also assumed to be excellent: the main result here is Theorem 2.1.

The third section discusses the computation of the Brauer groups of smooth split toric varieties over any field, under the assumption that the field contains certain primitive roots of unity. When the field is algebraically closed and of characteristic 0, this computation reduces to the one in [DF]. The main result here is Theorem 3.2. One may observe that Theorem 3.2 provides two explicit descriptions of the *m*-primary

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part of the Brauer group of a split smooth toric variety (when m invertible in the base field), one which is aimed at explicit computations as in the first statement of Theorem 3.2(i) and (ii), and the other a qualitative description in terms of the sublattice spanned by the rays of the fan defining the toric variety as in the second statement of Theorem 3.2(i) and (ii).

The fourth section discusses in detail the computation of the Brauer groups of toric schemes over Dedekind domains and also discrete valuation rings in the sense of [KKMSD]. The main results here are Theorems 4.1 and 4.8: here too we provide two explicit descriptions, one which is aimed at explicit computations, and the other a qualitative description in terms of the sublattice spanned by the rays of the fan defining the toric scheme. These results also find immediate application in determining the Brauer groups of toric stacks as in the work of Dhillon, Iyer and the first author: see [DIJ].

The fifth section discusses the period-index problem for smooth split toric varieties: surprisingly we are able to verify that the expected relation between the period and index holds for the classes in the Brauer groups of all smooth toric varieties over algebraically closed fields.<sup>1</sup> In particular, we show that the counterexamples such as those obtained by Hotchkiss (see [Hotch]) for complex analytic tori do not appear. For smooth split toric varieties over fields that are quasi-algebraically closed, such as the function field of a curve over an algebraically closed field, we also show the expected relationship holds for the classes in the Brauer groups of all smooth toric varieties. For smooth-split toric varieties over a finite field  $\mathbf{F}_q$ , the corresponding results also hold (in a slightly stronger form) but only for *m*-torsion classes in the Brauer group of the corresponding toric varieties, where *m* divides q - 1. We conclude the paper with the computations of the Brauer groups of several toric varieties and schemes making strong use of the main theorems of this paper.

Since we also consider toric schemes over Dedekind domains and discrete valuation rings, we will work in the following slightly more general framework. Let B denote a regular Noetherian excellent scheme of dimension at most 1: B will serve as the *base scheme*. We also consider two basic situations here:

1.1. **Basic hypotheses.** B = Spec R, where R is a Dedekind domain, or a DVR which is assumed to be excellent (for example, the ring of integers  $\mathbb{Z}$  or its localization at a prime p).

Let *m* denote a fixed positive integer invertible in  $\mathcal{O}_{B}$  and let X denote a scheme of finite type over B. Then one begins with the *Kummer sequence* 

(1.1) 
$$1 \to \mu_m(1) \to \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \to 1,$$

which holds on the (small) étale site  $X_{et}$  of X, whenever m is invertible in  $\mathcal{O}_B$ . (See [Gr, section 3] or [Mi, p. 66].) Taking étale cohomology, we obtain the corresponding long-exact sequence:

(1.2) 
$$\rightarrow \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \rightarrow \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(1)) \rightarrow \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \rightarrow \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}}) \rightarrow \cdots,$$

which holds on the étale site when m is invertible in  $\mathcal{O}_{\mathrm{B}}$ .

**Definition 1.1.** The cohomological Brauer group Br(X) is the torsion subgroup of the cohomology group  $H^2_{et}(X, \mathbb{G}_m)$ . In other words,  $Br(X) = H^2_{et}(X, \mathbb{G}_m)_{tors}$ .<sup>2</sup>

 $<sup>^{1}</sup>$ We want to point out that our statements only apply to classes in the Brauer group of the toric varieties and *not* to all classes in the Brauer group of the corresponding function field.

<sup>&</sup>lt;sup>2</sup>If X is a regular integral Noetherian scheme, one may observe that  $H^2_{et}(X, \mathbb{G}_m)_{tors} = H^2_{et}(X, \mathbb{G}_m)$ : see, for example, [CTS, Lemma 3.5.3].

Next assume X is smooth over the base scheme B. Then, by Hilbert's Theorem 90, we obtain the isomorphisms:

(1.3) 
$$\operatorname{Pic}(X) \cong \operatorname{CH}^{1}(X) \cong \operatorname{H}^{1}_{et}(X, \mathbb{G}_{m}) \cong \operatorname{H}^{2,1}_{M}(X, \mathbb{Z}),$$

where  $H_M^{2,1}(X,\mathbb{Z})$  denotes motivic cohomology (in degree 2 and weight 1) whose definition for smooth schemes of finite type over B is worked out in [Geis], and we recall this in the Appendix. Then one also obtains the short-exact sequence:

(1.4) 
$$0 \to \operatorname{Pic}(X)/m \cong \operatorname{NS}(X)/m \to \operatorname{H}^{2}_{\operatorname{et}}(X, \mu_{m}(1)) \to {}_{m}\operatorname{Br}(X) \to 0,$$

where the map  $\operatorname{Pic}(X)/m = \operatorname{H}_{M}^{2,1}(X, \mathbb{Z}/m) \to \operatorname{H}_{\operatorname{et}}^{2}(X, \mu_{m}(1))$  is the cycle map, and therefore,  ${}_{m}\operatorname{Br}(X)$  identifies with the cokernel of the cycle map. Thus it follows that for smooth schemes X over B,  ${}_{m}\operatorname{Br}(X)$  is trivial if and only if the above cycle map is surjective: our approach to the Brauer group adopted in this paper is to consider the above cycle map from motivic cohomology to étale cohomology, and involves a combination of motivic and étale cohomology techniques. Moreover, in view of this, we will always restrict to smooth schemes of finite type over the given base scheme B. However, apart from the restriction to smooth schemes, our approach making use of both motivic and étale cohomology techniques over Dedekind domains offers considerable advantages in various computations: these will become clear in later sections of the paper. The authors have already extended several key results of the present paper to *horospherical* varieties for the action of a connected split reductive group over a field, though these are not included in the present paper. We hope to continue computations of the Brauer groups of other classes of spherical varieties in the sense of [Br].

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## 2. BRAUER GROUPS OF SPLIT TORI

The goal of this section is to prove Theorem 2.1 and to obtain a suitable interpretation of the summands on the right-hand-side in terms of cyclic algebras. Throughout the discussion, m will denote a fixed positive integer invertible in  $\mathcal{O}_{\rm B}$ .

**Theorem 2.1.** Let  $T = \mathbb{G}_m^r$  denote a split torus defined over the base B as in 1.1. Then,

(2.1) 
$${}_{m}\mathrm{Br}(\mathrm{T}) \cong {}_{m}\mathrm{Br}(\mathrm{B}) \oplus \left(\oplus^{r}\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{B},\mu_{m}(0))\right) \oplus \left(\oplus^{\binom{r}{2}}\mathrm{H}^{0}_{\mathrm{et}}(\mathrm{B},\mu_{m}(-1))\right),$$

with the understanding that  $\binom{r}{2} = 0$  for r = 1, where <sub>m</sub>A denotes the m-torsion part of the abelian group A. Here  $\mu_m(0)$  is identified with the constant sheaf  $\mathbb{Z}/m$ .

**Proposition 2.2.** Let X denote a smooth scheme of finite type over B.

• Then the localization sequence

$$H_{M}^{*,\bullet}(X \times \mathbb{A}^{1}, \mathbb{Z}/m) \xrightarrow{J} H_{M}^{*,\bullet}(X \times \mathbb{G}_{m}, \mathbb{Z}/m) \xrightarrow{} H_{M,X \times \{0\}}^{*+1,\bullet}(X \times \mathbb{A}^{1}, \mathbb{Z}/m) \cong H_{M}^{*-1,\bullet-1}(X \times \{0\}, \mathbb{Z}/m)$$

(where  $j : X \times \mathbb{G}_m \to X \times \mathbb{A}^1$  denotes the obvious open immersion) breaks up into short exact sequences, thereby providing the isomorphism:

$$\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X}\times\mathbb{G}_{\mathrm{m}},\mathbb{Z}/\mathrm{m})\cong\mathrm{H}^{*,\bullet}_{\mathrm{M}}(\mathrm{X},\mathbb{Z}/\mathrm{m})\oplus\mathrm{H}^{*-1,\bullet-1}_{\mathrm{M}}(\mathrm{X},\mathbb{Z}/\mathrm{m}).$$

• A corresponding result holds in étale cohomology with coefficients in  $\mu_m(i)$ .

*Proof.* Let  $p: X \times \mathbb{A}^1 \to X \times \{1\}$  denote the obvious projection and let  $i: X \cong X \times \{1\} \to X \times \mathbb{G}_m$  denote the corresponding closed immersion.

Then it is clear that the induced map

$$p^*: \mathrm{H}^{*, \bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{*, \bullet}_{\mathrm{M}}(\mathrm{X} \times \mathbb{A}^1, \mathbb{Z}/\mathrm{m})$$

is an isomorphism and the composition of the maps

$$i^* \circ j^* \circ p^* : \mathrm{H}^{*, \bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m}) \to \mathrm{H}^{*, \bullet}_{\mathrm{M}}(\mathrm{X} \times \{1\}, \mathbb{Z}/\mathrm{m})$$

is the identity, thereby proving that the map  $j^* : H^{*,\bullet}_M(X \times \mathbb{A}^1, \mathbb{Z}/m) \to H^{*,\bullet}_M(X \times \mathbb{G}_m, \mathbb{Z}/m)$  is a split injection.  $\Box$ 

Making use of the last proposition, the following result computes the Brauer groups of schemes of the form  $X \times \mathbb{G}_{m}^{\times q}$ ,  $q \geq 1$ , over a base scheme B as in (1.1). This plays a key role in our computations.

Corollary 2.3. Let X denote a smooth scheme of finite type over B. Then the following hold:

(i) We obtain the isomorphism for any positive integer q:

$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m}) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{1,0}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})).$$

Moreover,  $H^{1,0}_M(X,\mathbb{Z}/m)\cong 0$  so that the last isomorphism becomes:

$$\mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mathbb{Z}/\mathrm{m}) \cong \mathrm{H}^{2,1}_{\mathrm{M}}(\mathrm{X}, \mathbb{Z}/\mathrm{m})$$

(ii) The corresponding result for étale cohomology with respect to  $\mu_m(i)$ , where  $\mu_m(0)$  is identified with the constant sheaf  $\mathbb{Z}/m$ , is:

$$\begin{split} \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mu_{m}(1)) &\cong \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(1)) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(0))) \oplus (\oplus_{i=1}^{\binom{q}{2}} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(-1))) \\ \text{with the understanding that for } q = 1, \ \binom{q}{2} = 0. \end{split}$$

(iii) One also obtains the following isomorphisms for étale cohomology with respect to  $\mu_m(i)$ , where  $\mu_m(0)$  is identified with the constant sheaf  $\mathbb{Z}/m$ :

$$\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X} \times \mathbb{G}_{\mathrm{m}}^{\times q}, \mu_{m}(1)) \cong \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(1)) \oplus (\oplus_{i=1}^{q} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{X}, \mu_{m}(0))).$$

(iv) Consequently  $_{m}Br(X \times \mathbb{G}_{m}^{\times q}) \cong {}_{m}Br(X) \oplus (\oplus^{q}H^{1}_{et}(X, \mu_{m}(0))) \oplus (\oplus^{\binom{q}{2}}H^{0}_{et}(X, \mu_{m}(-1))),$  with the same understanding that  $\binom{q}{2} = 0$  for q = 1.

*Proof.* The isomorphism in the first statement in (i) may be deduced from the last proposition by ascending induction on q. Here one may want to observe that the motivic complexes  $\mathbb{Z}(j)$  are defined only for  $j \ge 0$ , i.e.,  $\mathbb{Z}(j) = 0$ , for j < 0. Next we consider the remaining statements in (i).

Observe that the term  $H^{1,0}_M(X, \mathbb{Z}/m)$ 

$$\begin{split} H^{1,0}_M(X,\mathbb{Z}/m) &\cong H^{3,1}_{M,X\times\{0\}}(X\times\mathbb{A}^1,\mathbb{Z}/m) \\ &\cong H^3_{Zar}(X\times\mathbb{A}^1,i_*(\mathbb{Z}/m(0)[-2])) \end{split}$$

by (7.3). This is because the codimension of  $X \times \{0\}$  in  $X \times \mathbb{A}^1$  is 1. Now  $\mathbb{Z}/m(0)$ , which is the motivic complex of weight 0 identifies with the constant sheaf  $\mathbb{Z}/m$  (given by the ring of integers modulo m). The

 $\mathbf{5}$ 

shift [-2] shifts this sheaf to degree 2, so that  $\mathbb{Z}/m(0)[-2]$  is the complex of Zariski sheaves concentrated in degree 2, where it is  $\mathbb{Z}/m$ . Since constant sheaves are flabby on the Zariski site of any irreducible scheme, the Zariski cohomology of  $X \times \{0\}$  with respect to this complex is trivial in degree 3 or higher. This observation completes the proof that  $H^{1,0}_{M}(X, \mathbb{Z}/m) \cong 0$  and completes the proof of the remaining statements in (i).

The statements in (ii) and (iii) on étale cohomology may be proven using ascending induction on q. One may also want to observe that  $H^{i}_{et}(Y, \mu_{m}(j)) = 0$ , for all i < 0.

The last statement now follows from (i) and (ii), as well as the identification of the Brauer group  ${}_{m}Br(Y)$ for a smooth scheme Y with the cokernel of the cycle map  $H^{2,1}_M(Y,\mathbb{Z}/m) \to H^2_{et}(Y,\mu_m(1))$  as observed in the discussion following (1.4).

**Remark 2.4.** In [SGS, 2.8 Theorem, 2.10 Theorem], the authors prove a variant of the statement in Corollary 2.3(iv), under the assumption the base scheme B is a field. Moreover, their techniques involve the use of Milnor K-theory, and therefore seem to be restricted to the case where the base scheme B is a field and also a bit more involved than our arguments above, which involve largely Proposition 2.2, induction on the number of factors in a split torus and (1.4).

2.1. External product pairing and the computation of the Brauer group of a split torus. We proceed to provide an interpretation of the summands on the right in Corollary 2.3(iv) in terms of cyclic algebras: see [P, 1.5.7] or [GS, 2.5]. The following discussion is based on the one in [P, 1.5.7.4]. We will assume that X and Z are smooth schemes over B = Spec R as in 1.1; in addition to assuming the integer m is invertible in R, we will further assume R has a primitive m-th root of unity. We first consider the following external product pairing, where  $\mu_m(0)$  is identified with the constant sheaf  $\mathbb{Z}/m$ :

(2.2) 
$$\mathbf{x} : \mathrm{H}^{1}_{\mathrm{et}}(\mathbf{Z}, \mu_{m}(1)) \otimes \mathrm{H}^{1}_{\mathrm{et}}(\mathbf{X}, \mu_{m}(0)) \to \mathrm{H}^{2}_{\mathrm{et}}(\mathbf{Z} \times \mathbf{X}, \mu_{m}(1)).$$

One can also interpret the above pairing in terms of the following well-known construction of cyclic algebras. Observe that the boundary map

$$\delta : cokernel(\Gamma(\mathbf{Z}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \Gamma(\mathbf{Z}, \mathbb{G}_{\mathrm{m}})) \to \mathrm{H}^{1}_{\mathrm{et}}(\mathbf{Z}, \mu_{m}(1))$$

(obtained from the Kummer sequence) is always injective. (In case  $Pic(Z) \cong 0$ , for example, if Z is the spectrum of a local ring or a Noetherian unique factorization domain, then the above map is an isomorphism, but we do not need to assume this.)

Let  $a \in \Gamma(\mathbb{Z}, \mathbb{G}_m)$  and let  $\mathbb{Y} \to \mathbb{X}$  denote a  $\mathbb{Z}/m$ -torsor corresponding to a class in  $\mathrm{H}^1_{\mathrm{et}}(\mathbb{X}, \mu_m(0))$ . Let  $\sigma$  denote the generator of  $Aut_{\mathbb{X}}(\mathbb{Y}) \cong \mathbb{Z}/m$ . Associated to  $\mathbb{Y}$  and the class a (identified with  $a \otimes 1 \in \mathcal{O}_{\mathbb{Z}} \otimes \mathcal{O}_{\mathbb{Y}} \cong \mathcal{O}_{\mathbb{Z} \times \mathbb{Y}}$ ), one defines the cyclic algebra  $\mathcal{O}_{\mathbb{Z} \times \mathbb{Y}}[x]_{\sigma}/(x^m - a)$ , where  $x.y' = \sigma(y').x$ , for all  $y' \in \mathcal{O}_{\mathbb{Y} \times \mathbb{Z}}$ . This defines a class in  ${}_m\mathrm{Br}(\mathbb{Z} \times \mathbb{X})$  and identifies with the class defined as the image of  $\delta(a) \in \mathrm{H}^1_{\mathrm{et}}(\mathbb{Z}, \mu_m(1))$  and  $\mathbb{Y}$  under the external product pairing in (2.2).

Next we take  $Z = \mathbb{G}_m$ , the multiplicative group scheme defined over B. Now  $\mathcal{O}_{\mathbb{G}_m} = \mathcal{O}_{\mathrm{B}}[t, t^{-1}]$ . Let  $Y \to X$  denote a  $\mathbb{Z}/m$ -torsor corresponding to a class in  $\mathrm{H}^1_{\mathrm{et}}(X, \mu_m(0))$  as in the last paragraph. Then, one may verify that the mapping  $Y \mapsto \mathcal{O}_{Y \times \mathbb{G}_m}[x]_{\sigma}/(x^m - t)$ , is an injection  $\mathrm{H}^1_{\mathrm{et}}(X, \mathbb{Z}/\mathrm{m}) \to {}_m\mathrm{Br}(X \times \mathbb{G}_{\mathrm{m}})$ , with inverse defined by the residue map associated to the divisor obtained by setting t = 1 in  $\mathbb{G}_m$ : see [CTS, p. 32]. (To be able to invoke [CTS, p. 32], one needs to first pull back classes in \mathrm{H}^1\_{\mathrm{et}}(\mathbb{G}\_{\mathrm{m}}, \mu\_m(1)) and in

 $\begin{aligned} H^{1}_{\text{et}}(\mathbf{X},\mu_{m}(0)) \text{ to classes in } H^{1}_{\text{et}}(\mathbf{K}(\mathbb{G}_{m}\times\mathbf{X}),\mu_{m}(1)) \text{ and } H^{1}_{\text{et}}(\mathbf{K}(\mathbb{G}_{m}\times\mathbf{X}),\mu_{m}(0)). \text{ Observe that the composite map } p^{*}_{2}: H^{1}_{\text{et}}(\mathbf{X},\mu_{m}(0)) \to H^{1}_{\text{et}}(\mathbb{G}_{m}\times\mathbf{X},\mu_{m}(0)) \to H^{1}_{\text{et}}(\mathbf{K}(\mathbb{G}_{m}\times\mathbf{X}),\mu_{m}(0)) \text{ is an injection.}) \end{aligned}$ 

Next one may take  $X = \mathbb{G}_m$  (again multiplicative group scheme defined over B) to obtain the external product pairing:

(2.3) 
$$\mathbf{x}: \mathrm{H}^{1}_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_{m}(1)) \otimes \mathrm{H}^{1}_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}, \mu_{m}(0)) \to \mathrm{H}^{2}_{\mathrm{et}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}, \mu_{m}(1)).$$

We proceed to interpret this pairing also in terms of cyclic algebras, under the assumption the base scheme B has the property that m is invertible in  $\mathcal{O}_{\mathrm{B}}$  and that it has a primitive m-th root of unity  $\zeta$ . Therefore, the sheaf  $\mu_m$  identifies the with the constant sheaf  $\mathbb{Z}/m$ . Given a unit  $b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$ , let  $Y \to \mathbb{G}_m$  denote the  $\mathbb{Z}/m$ -torsor given by Spec  $(\mathcal{O}_{\mathbb{G}_m}[x]/(x^m - b)) \to \mathbb{G}_m$ : we equip this torsor with the automorphism  $\sigma$  given by sending  $x \mapsto x\zeta$ . Therefore, given two units  $a, b \in \Gamma(\mathbb{G}_m, \mathbb{G}_m)$ , one may define a cyclic algebra  $(a, b)_{\zeta}$ , by applying the construction in the last paragraph with  $X = \mathbb{G}_m$ , and the torsor  $Y \to X$  given by the torsor Spec  $(\mathcal{O}_X[x]/(x^m - b)) \to X = \mathbb{G}_m$ .

At this point if X is any smooth scheme of finite type over B, pre-composing the external product pairing in (2.3) with the cup-product with  $H^0_{et}(X, \mathbb{Z}/m)$  defines classes in  $H^2_{et}(X \times \mathbb{G}^2_m; \mu_m(1))$ , and hence classes in  ${}_mBr(X \times \mathbb{G}^2_m)$ . In terms of cyclic algebras this corresponds to letting  $a = t_1$  and  $b = t_2$  in the discussion in the last paragraph, and where  $\mathcal{O}_{\mathbb{G}^2_m} = \mathcal{O}_{\mathrm{B}}[t_1, t_2, t_1^{-1}, t_2^{-1}]$ . This defines the summand  $H^0_{\mathrm{et}}(X, \mu_{\mathrm{m}}(-1))$  in  ${}_m\mathrm{Br}(X \times \mathbb{G}^2_{\mathrm{m}})$  appearing on the right-hand-side of Corollary 2.3(iv) with q = 2.

Now we take  $Z = \mathbb{G}_m^{\times r}$ , where  $\mathbb{G}_m^{\times r} = \operatorname{Spec} \mathbb{R}[t_1, t_1^{-1}, \cdots, t_r, t_r^{-1}]$ . (Observe that when R is a DVR, the hypothesis  $\operatorname{Pic}(Z) = 0$  is clearly satisfied.) We may take  $t_i \in \Gamma(Z, \mathbb{G}_m) = \operatorname{Hom}(Z, \mathbb{G}_m)$ : observe that the latter identifies with the ring of characters of Z. We will also take  $X = B = \operatorname{Spec} \mathbb{R}$ . If  $b \in \Gamma(\operatorname{Spec} \mathbb{R}, \mathbb{G}_m) = \mathbb{R}^*$ , b corresponds to a  $\mathbb{Z}/m$ -torsor  $Y = \operatorname{Spec}(\mathbb{R}[x]/(x^m - b))$  over  $\operatorname{Spec} \mathbb{R}$ . Thus b corresponds to a class in  $\operatorname{H}^1_{\operatorname{et}}(\operatorname{Spec} \mathbb{R}, \mu_m(0))$ . Now we may form the cyclic algebra  $(b, t)_{\zeta}$ . These cyclic algebras as b varies over the units in the ring R and t varies among the characters  $t_i, i = 1, \cdots, r$  of the split torus  $T = \mathbb{G}_m^r$  correspond to the sum  $\oplus^r \operatorname{H}^1_{\operatorname{et}}(\operatorname{Spec} \mathbb{R}, \mu_m(0))$  in the right hand side of (2.1).

Next we take  $Z = \mathbb{G}_m$  and  $X = \mathbb{G}_m$ , so that we may take  $a = t_1$ , and  $b = t_2$ , where  $\mathbb{G}_m^2 = \operatorname{Spec} \mathbb{R}[t_1, t_2, t_1^{-1}, t_2^{-1}]$ We will view b as corresponding to the  $\mathbb{Z}/m$ -torsor  $\operatorname{Spec}(\mathcal{O}_{\mathbb{G}_m}[\mathbf{x}]/(\mathbf{x}^m - \mathbf{b}))$  over  $\mathbb{G}_m$ . Now, corresponding to the pair of coordinates  $t_1, t_2$ , we form the cyclic algebra  $(t_1, t_2)_{\zeta}$ . One may now repeat this construction taking two factors of  $\mathbb{G}_m^{\times r}$  corresponding to the coordinates  $t_i, t_j, i < j$  and form the cyclic algebras  $(t_i, t_j)_{\zeta}$ . As we vary  $t_i, t_j$  over all ordered pairs of coordinates, we obtain  $\binom{r}{2}$  such cyclic algebras. These will account for each of the  $\binom{r}{2}$  summands  $\operatorname{H}^0_{\text{et}}(\operatorname{Spec} \mathbb{R}, \mu_m(-1))$ . Consequently, we obtain the following Corollary to Theorem 2.1:

Corollary 2.5.

(2.4) 
$${}_{m}\mathrm{Br}(\mathbb{G}_{\mathrm{m}}^{r}) = {}_{m}\mathrm{Br}(\mathrm{B}) \bigoplus \left( \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (t_{i}, t_{j})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \in \mathrm{R}^{*}} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{\mathbb{Z}} \right) \bigoplus \left( \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j \leq r} \underbrace{\bigoplus_{i \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < j < r}}_{i \leq i < j < r} \underbrace{\bigoplus_{i \leq i < j < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r}}_{i \leq i < r} \underbrace{\bigoplus_{i \leq i < j < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r}}_{i \leq i < r} \underbrace{\bigoplus_{i \leq i < j < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < j < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \leq i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i \in i < r} \underbrace{\bigoplus_{i \in i < r} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i}, t_{i})_{\zeta}}_{i$$

where  $\mathcal{A}$  denotes the free  $\mathbb{Z}/m\mathbb{Z}$ -module with basis the cyclic algebras  $(t_i, t_j)_{\zeta}$  and  $\mathcal{B}$  denotes the  $\mathbb{Z}/m\mathbb{Z}$ module on the cyclic algebras  $(b_i, t_i)_{\zeta}$ .

2.2. Localization sequences and the residue map. Next we consider localization sequences for motivic and étale cohomology theories, which provide a convenient technique for computing Brauer groups. We begin with the following variant of the Snake Lemma: see [Iver, Snake Lemma 1.6].

Lemma 2.6. Consider the commutative diagram



in an abelian category with exact rows. Then the following hold:

- (i) If  $\alpha$  and  $\beta$  are isomorphisms and  $\delta$  is a monomorphism, then the map  $\gamma$  is also a monomorphism.
- (ii) If  $\alpha$  is an epimorphism and  $\eta$  is a monomorphism, then

 $kernel(\beta) \rightarrow kernel(\gamma) \rightarrow kernel(\delta) \rightarrow cokernel(\beta) \rightarrow cokernel(\gamma) \rightarrow cokernel(\delta)$ 

is exact. In particular, if  $\alpha$  is an epimorphism,  $\eta$  is a monomorphism and both  $\beta$  and  $\delta$  are isomorphisms, then so is  $\gamma$ .

**Proposition 2.7.** We start by considering the following situation. Consider X a smooth scheme of finite type over a base scheme B as in (1.1) which is of pure dimension over B. Fix a closed subscheme Z in X that is of pure codimension in X with open complement U. Then one obtains a commutative diagram

$$\begin{array}{c} \operatorname{H}^{1,1}_{M}(\mathrm{U},\mathbb{Z}/m) \longrightarrow \operatorname{H}^{2,1}_{M,Z}(\mathrm{X},\mathbb{Z}/m) \longrightarrow \operatorname{H}^{2,1}_{M}(\mathrm{X},\mathbb{Z}/m) \longrightarrow \operatorname{H}^{2,1}_{M}(\mathrm{U},\mathbb{Z}/m) \longrightarrow \\ a_{1} \downarrow \qquad a_{2} \downarrow \qquad a_{3} \downarrow \qquad \downarrow a_{4} \\ \operatorname{H}^{1}_{\mathrm{et}}(\mathrm{U},\mu_{m}(1)) \longrightarrow \operatorname{H}^{2}_{\mathrm{et},Z}(\mathrm{X},\mu_{m}(1)) \longrightarrow \operatorname{H}^{2}_{\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \longrightarrow \operatorname{H}^{2}_{\mathrm{et}}(\mathrm{U},\mu_{m}(1)) \longrightarrow \\ \end{array}$$

$$\begin{array}{c} \longrightarrow \operatorname{H}^{3,1}_{M,Z}(\mathrm{X},\mathbb{Z}/m) \longrightarrow \operatorname{H}^{3,1}_{M}(\mathrm{X},\mathbb{Z}/m) \\ a_{5} \downarrow \qquad a_{6} \downarrow \\ \longrightarrow \operatorname{H}^{3}_{\mathrm{et},Z}(\mathrm{X},\mu_{m}(1)) \longrightarrow \operatorname{H}^{3}_{\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \end{array}$$

so that the following hold:

- (1) The maps  $a_1$  and  $a_2$  are isomorphisms always (under our hypotheses as in (1.1).
- (2) The maps  $a_3$  and  $a_4$  are always injective.
- (3) Under the assumption that Z is also regular of pure codimension 1, we obtain isomorphisms  $H^{2,1}_{M,Z}(X, \mathbb{Z}/\ell^n)) \cong H^{0,0}_M(Z, \mathbb{Z}/\ell^n)$  and  $H^2_{et,Z}(X, \mu_m(1))) \cong H^0_{et}(Z, \mu_m(0)).$
- (4)  $H^{3,1}_{M,Z}(X, \mathbb{Z}/m) \cong 0.$
- (5)  $\mathrm{H}^{3,1}_{\mathrm{M}}(\mathbf{X}, \mathbb{Z}/m) = 0.$
- (6) If Z has pure codimension > 1, then  $H^3_{et,Z}(X, \mu_m(1)) \cong 0$  as may be seen from cohomological purity.

*Proof.* One can also see a proof of this result in [CTS, Theorems 3.7.1 and 3.7.2]. The proof we provide below is a bit different, as it also involves certain results on motivic cohomology over Dedekind domains as in the Appendix. The fact that the maps  $a_3$  and  $a_4$  are always injective follows from the Kummer sequence considered in (1.4). Observe that the scheme X is assumed to be smooth. The isomorphisms in the third statement are essentially Thom-isomorphisms, which exist as the schemes X and Z are assumed to be smooth.

### R. JOSHUA AND J. LEE

The isomorphism  $\mathrm{H}^{3,1}_{\mathrm{M},\mathbb{Z}}(\mathrm{X},\mathbb{Z}/m) \cong 0$  in statement (4) may be obtained as follows. The localization sequence in (7.2) shows that  $\mathrm{H}^{3,1}_{\mathrm{M},\mathbb{Z}}(\mathrm{X},\mathbb{Z}/m)$  identifies with the cohomology of the complex  $i_*(\mathbb{Z}/m^{\mathbb{Z}}(1-c)[-2c]$ computed on the Zariski site of the base scheme X. Here c is the codimension of Z in X. If c > 1, clearly this complex is trivial and hence conclusion follows in this case. If c = 1, one observes the isomorphism:  $\mathrm{H}^{3,1}_{\mathrm{M},\mathbb{Z}}(\mathrm{X},\mathbb{Z}/m) \cong \mathrm{H}^3_{\mathrm{Zar}}(\mathrm{X},\mathrm{i}_*(\mathbb{Z}/m^{\mathbb{Z}}(0)[-2]) \cong \mathrm{H}^1_{\mathrm{Zar}}(\mathbb{Z},\mathbb{Z}/m^{\mathbb{Z}}(0)) \cong 0$ , since  $\mathbb{Z}/m^{\mathbb{Z}}(0)$  is the constant sheaf  $\mathbb{Z}/m$  and Z may be assumed to be irreducible.

The vanishing of the cohomology in the statement (5) may be obtained as follows. First, when the base scheme B is a field, this follows from the identification of the motivic cohomology  $H^{3,1}_{M}(X, \mathbb{Z}/m)$  with  $CH^{1,2-3}(X, \mathbb{Z}/m)$  which is trivial for obvious reasons. In general, as is shown in [Geis, Corollary 3.3], one takes the motivic complex  $\mathbb{Z}/m(1)$ , viewed as a complex of sheaves on the (small) flat site of the given scheme X and takes its pushforward to the Zariski site of the base scheme B, and then computes the Zariski cohomology of the resulting complex on B. The complex  $\mathbb{Z}/m(1)$  is concentrated in degrees 0 and 1: in fact it is the complex  $\mathbb{G}_m \xrightarrow{m} \mathbb{G}_m$  concentrated in degrees 0 and 1. Since B has Zariski cohomological dimension at most 1, the resulting complex on the Zariski site of B has no cohomology in degree 3 or above, proving the vanishing of the cohomology in the last statement. Moreover the vanishing of the local cohomology in the last statement is clear.

Therefore, it suffices to prove the first statement, which we proceed to do presently. We will first consider the map  $a_2$ . In case the codimension of Z in X is higher than 1, both the source and target of the map  $a_2$  are trivial. Therefore, it suffices to consider the case when the codimension of Z in X is 1. First we will consider the case when Z is also regular. Now the localization sequence as in (7.2) readily provides the identification:

(2.5) 
$$\mathrm{H}^{2,1}_{\mathrm{M},\mathbb{Z}}(\mathrm{X},\mathbb{Z}/m) \cong \mathrm{H}^{0,0}_{\mathrm{M}}(\mathrm{Z},\mathbb{Z}/m), \text{ while}$$

Thom isomorphism (or purity) provides the isomorphism:

(2.6) 
$$\mathrm{H}^{2}_{\mathrm{et},Z}(\mathrm{X},\mu_{m}) \cong \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Z},\mu_{m}(0))$$

Clearly the right-hand-sides of both (2.5) and (2.6) are isomorphic to the sum  $\oplus \mathbb{Z}/m$  indexed by the connected components of Z and are therefore isomorphic, thereby proving that the map  $a_2$  is an isomorphism when Z is also regular.

In general, making use of the assumption the base scheme B is excellent, we may let  $Z_s$  denote the singular locus of Z which will be of codimension 1 or higher in Z. Then, in the localization sequence

$$\cdots \to \mathrm{H}^{2}_{\mathrm{Z}_{\mathrm{s}},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{2}_{\mathrm{Z},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{2}_{\mathrm{Z}-\mathrm{Z}_{\mathrm{s}},\mathrm{et}}(\mathrm{X}-\mathrm{Z}_{\mathrm{s}},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z}_{\mathrm{s}},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \cdots$$

the first and last terms are trivial since the codimension of  $Z_s$  in X is at least 2. One obtains a similar localization sequence in motivic cohomology, using which we reduce to the case where Z is regular.

Next we consider the map  $a_1$ . Then the Beilinson-Lichtenbaum conjecture (which follows as a consequence of the *motivic Bloch-Kato conjecture*) first shows that the map  $a_1$  is an isomorphism, when the base scheme B is the spectrum of a field. Observe that, this is the statement that the natural map

(2.7) 
$$\mathbb{Z}/m(i) \simeq \tau_{\leq i} R \epsilon_* \epsilon^* (\mathbb{Z}/m(i)) \simeq \tau_{\leq i} R \epsilon_* (\mu_m(i))$$

is a quasi-isomorphism, where  $\epsilon$  is the obvious map of sites from the big étale site of the scheme B to the corresponding big Nisnevich site. It follows from the discussion in [Geis, Theorem 1.2] that the quasiisomorphism in (2.7) extends to the case where B is a Dedekind domain. **Corollary 2.8.** Assume X is a smooth scheme of pure dimension over B, with Z a closed subscheme of pure codimension in X with open complement U.

(i) Then, we obtain the exact sequence:

$$0 \rightarrow {}_{m}\mathrm{Br}(\mathrm{X}) \rightarrow {}_{m}\mathrm{Br}(\mathrm{U}) \longrightarrow \mathrm{H}^{3}_{\mathrm{Z,et}}(\mathrm{X}, \mu_{m}(1)),$$

and in case Z has pure codimension 1 in X and is also regular, one has the identification  $H^3_{Z,et}(X, \mu_m(1)) \cong H^1_{et}(Z, \mu_m(0)).$ 

(ii) In case Z is of pure codimension 1, but only generically smooth, we obtain the exact sequence:

 $0 \to {}_m Br(X) \to {}_m Br(U) {\longrightarrow} H^1_{et}(Z - Z_s, \mu_m(0)),$ 

where  $Z_s$  denotes the singular locus of Z.

(iii) In case the Z has pure codimension > 1 in X, we obtain:

$${}_{m}\mathrm{Br}(\mathbf{X}) \stackrel{\cong}{\to} {}_{m}\mathrm{Br}(\mathbf{U}).$$

*Proof.* First we consider the case when Z has pure codimension 1 in X. Then one invokes Proposition 2.7 and Lemma 2.6 with  $\alpha = a_1$ ,  $\beta = a_2$ ,  $\gamma = a_3$ ,  $\delta = a_4$  and  $\eta = a_5$  to obtain the first exact sequence stated in the corollary. Next we consider the case when Z is also regular. Then one observes the Thom isomorphism (in view of the assumption that Z is also smooth):

$$\mathrm{H}^{3}_{\mathrm{et},\mathbf{Z}}(\mathbf{X},\mu_{m}(1))\cong\mathrm{H}^{1}_{\mathrm{et}}(\mathbf{Z},\mu_{m}(0)).$$

Observe that the resulting map

$$\operatorname{Br}(U)_m \longrightarrow \operatorname{H}^1_{\operatorname{et}}(Z, \mu_m(0))$$

identifies with the residue map discussed in section 2.3. (Hence we denote this map by res.)

Next we consider (ii). Recall that the base scheme B is assumed to be excellent. In this case one obtains the long exact sequence:

$$\cdots \to \mathrm{H}^{3}_{\mathrm{Z}_{\mathrm{s}},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z}-\mathrm{Z}_{\mathrm{s}},\mathrm{et}}(\mathrm{X}-\mathrm{Z}_{\mathrm{s}},\mu_{m}(1)) \cong \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Z}-\mathrm{Z}_{\mathrm{s}},\mu_{m}(0) \to \cdots$$

Observe that the codimension of  $Z_s$  in X is at least 2. Therefore, and making use of the assumption that B is excellent, one may prove readily that  $H^3_{Z_s,et}(X, \mu_m(1)) \cong 0$ , which will prove the last map in the long exact sequence above in an injection. (In fact, one may filter the scheme  $Z_s$  further by  $\{Z_{s,k} \subseteq Z_{s,k-1} \subseteq \cdots Z_{s,0} = Z_s\}$  so that the codimension of  $Z_{s,k}$  in X is at least k + 1. Then one may use a localization sequence

$$\cdots \to \mathrm{H}^{3}_{\mathrm{Z}_{\mathrm{s},\mathrm{k}},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z}_{\mathrm{s},\mathrm{k}-1},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z}_{\mathrm{s},\mathrm{k}-1}-\mathrm{Z}_{\mathrm{s},\mathrm{k}},\mathrm{et}}(\mathrm{X}-\mathrm{Z}_{\mathrm{s},\mathrm{k}},\mu_{m}(1)) \to \cdots$$

and use descending induction on k to prove the required vanishing.) This completes the proof of (ii).

In case the codimension of Z in X is larger than 1, then  $H^3_{Z,et}(X, \mu_m(1)) \cong 0$ , so that the statement (i) proves the last conclusion.

In order to proceed with the computation of the Brauer group of a toric variety, we also need localization sequences, which is discussed next.

**Proposition 2.9.** (i) The localization sequence in Corollary 2.8(i) is functorial in X, U and Z in the following sense: if  $X' \to X$  is a map of smooth schemes and  $U' = U \times_X X'$ ,  $Z' = Z \times_X X'$ , then one

obtains a commutative diagram of localization sequences:

(ii) Assume that X is a smooth scheme of finite type over the given base scheme B provided with the action of a smooth affine group scheme G with finitely many orbits. Let U denote the open G stable subscheme which is the (disjoint) union of the open orbits and let {O<sub>i</sub>|i = 1, · · · , m} denote the codimension 1-orbits. Let Z = ∪<sup>m</sup><sub>i=1</sub> O<sub>i</sub>. Then there exists a localization sequence:

$$0 \to {}_{m}\mathrm{Br}(\mathrm{X}) \to {}_{m}\mathrm{Br}(\mathrm{U}) \to \oplus_{\mathrm{i}=1}^{\mathrm{m}}\mathrm{H}^{1}_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}}, \mu_{m}(0)).$$

Moreover, the last map may then be identified with the residue map.

*Proof.* The first statement follows readily, since Z is also assumed to be smooth. In fact the localization sequence as in (i) holds more generally for any regular scheme X with U an open subscheme and Z the complement of U in X, which is assumed to be also regular as shown in [CTS, p. 91, (3.17)].

To see the last statement, one starts with the localization sequence:

$$0 \rightarrow {}_{m}Br(X) \rightarrow {}_{m}Br(U) \rightarrow H^{3}_{Z,et}(X, \mu_{m}(1))$$

whose existence should be clear from Corollary 2.8. Therefore, it suffices to prove that there is an injection

$$\mathrm{H}^{3}_{\mathrm{Z,et}}(\mathrm{X},\mu_{m}(1)) \to \oplus \mathrm{H}^{1}_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}},\mu_{m}(0)).$$

For this, first observe that if X is of dimension 1, then Z is of dimension 0 and hence smooth. Therefore, the map above would be an isomorphism in this case. Therefore, we may assume that X has dimension at least 2, and let Y denote the union of all the orbits of codimension 2 or more. Then one obtains the long-exact sequence:

(2.8) 
$$\cdots \to \mathrm{H}^{3}_{\mathrm{Y},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z},\mathrm{et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z}-\mathrm{Y},\mathrm{et}}(\mathrm{X}-\mathrm{Y},\mu_{m}(1)) \to \cdots$$

Now X is smooth, and therefore X - Y is also smooth. Z - Y is the disjoint union of orbits of codimension equal to 1, so it is also smooth. Therefore, by purity,

$$\mathrm{H}^{3}_{\mathrm{Z-Y,et}}(\mathrm{X}-\mathrm{Y},\mu_{m}(1)) \cong \bigoplus_{i=1}^{n} \mathrm{H}^{1}_{\mathrm{et}}(\mathcal{O}_{i},\mu_{m}(0)).$$

Since Y has codimension at least 2 in X,  $H^3_{Y,et}(X, \mu_m(1)) \cong 0$ . One may prove this statement using descending induction on the codimension of Y, since when Y has codimension = the dimension of X, Y consists of only 0-dimensional orbits, in which case the above statement is clear by purity. In general, one may use the localization sequence as in (2.8) and descending induction on the codimension of Y in X. This shows that the induced map

$$\mathrm{H}^{3}_{\mathrm{Z,et}}(\mathrm{X},\mu_{m}(1)) \to \mathrm{H}^{3}_{\mathrm{Z-Y,et}}(\mathrm{X-Y},\mu_{m}(1)) \cong \oplus_{i=1}^{n} \mathrm{H}^{1}_{\mathrm{et}}(\mathcal{O}_{i},\mu_{m}(0))$$

is injective. The identification of the map

$$_{m}\mathrm{Br}(\mathrm{U}) \rightarrow \oplus_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{H}^{1}_{\mathrm{et}}(\mathcal{O}_{\mathrm{i}},\mu_{m}(0))$$

with residue map follows from the discussion below. This completes the proof of the Proposition.  $\Box$ 

2.3. More on the residue map. Assume the situation of Proposition 2.9. Then the resulting map

(2.9) 
$${}_{m}\mathrm{Br}(\mathrm{U}) \to \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Z}, \mu_{m}(0))$$

has an explicit description in terms of *the residue map*, discussed in detail in [GS, Chapter 6] and also [CTS, Chapter I, 1.4], which we will summarize here.

First one observes that if p denotes the ideal sheaf in  $\mathcal{O}_X$  defining the divisor Z, then the localization of  $\mathcal{O}_X$  at p is a sheaf of discrete valuation rings. If K(X) denotes the corresponding function field of X, and k(Z) denotes the function field of Z, then one observes the following:

(i) the field  $k(\mathbf{Z}) \cong \mathcal{O}_{\mathbf{X},p}/p\mathcal{O}_{\mathbf{X},p}$  and the latter identifies with the residue field of the corresponding valuation on  $\mathcal{O}_{\mathbf{X}}$ .

Throughout the remaining statements let m denote a fixed positive integer invertible in  $\mathcal{O}_{\mathbf{X}}$ .

- (ii) As X is smooth, the natural map  ${}_{m}Br(X) \rightarrow {}_{m}Br(K(X))$  is an injection: see [CTS, Theorem 3.5.5].
- (iii) As Z (the generic point of Z) is closed and of dimension < the dimension of Z, the natural map  $H^1_{et}(Z, \mu_m) \to H^1_{et}(\operatorname{Spec} k(Z), \mu_m)$  is also an injection.
- (iv) A key result due to Merkurjev-Suslin identifies  $K_2^{Milnor}(K(X))/mK_2^{Milnor}(K(X))$  with  $H^2_{et}(K(X), \mu_m(2))$ and  $K_1^{Milnor}(k(Z))/mK_1^{Milnor}(k(Z))$  with  $H^1_{et}(k(Z), \mu_m(1))$ . (See [MS], [GS, Chapter 8] and [CTS, Chapter 1, 1.4].)
- (v) Another key result due to Merkurjev-Suslin shows that if the field K contains a primitive *m*-th root of unity  $\zeta$ , then any class in the Brauer group  ${}_{m}Br(K)$  is *Brauer equivalent* to finite product of cyclic algebras of the form  $(a_1, b_1)_{\zeta} \otimes_{\mathrm{K}} \cdots \otimes_{\mathrm{K}} (a_m, b_m), a_i, b_i \in \mathrm{K}^*, i = 1, \cdots, m$ . (See [GS, Theorem 2.5.7].)
- (vi) One also has the following commutative square (see [GS, Proposition 7.5.1]), which proves the compatibility of the *residue map* in étale cohomology with the *tame symbol map* on Milnor K-theory as in:

where the map denoted Tame.sym is the tame symbol defined as follows:

(2.11) 
$$\delta_v : \mathbf{K}(\mathbf{X})^* \times \mathbf{K}(\mathbf{X})^* \to k(\mathbf{Z})^*, \\ \delta_v(\mathbf{a}, \mathbf{b}) = (-1)^{v(\mathbf{a}) \cdot v(\mathbf{b})} \frac{\mathbf{a}^{v(\mathbf{b})}}{\mathbf{b}^{v(\mathbf{a})}} \operatorname{mod}(\mathbf{p})$$

where p denotes the ideal defining Z in X. Moreover,  $h^2$  and  $h^1$  are the *Galois symbol maps*: see [GS, Definition 4.6.4].

(vii) Finally we also make the observation that, assuming K(X) contains a primitive *m*-th root of unity  $\zeta$ , multiplication by  $\zeta$  defines an isomorphism  $\mu_m(i-1) \rightarrow \mu_m(i)$ . Therefore, under this assumption and with the choice of a primitive *m*-th root of unity of  $\zeta$ , the residue map above may be also viewed as a map

(2.12) 
$$\operatorname{H}^{2}_{\mathrm{et}}(\mathrm{K}(\mathrm{X}), \mu_{m}(1)) \xrightarrow{\operatorname{res}} \operatorname{H}^{1}_{\mathrm{et}}(k(\mathrm{Z}), \mu_{m}(0))$$

Moreover, in this case, the residue map has the following concrete interpretation:

(2.13)  $residue((a,b)_{\zeta}) =$  the Galois extension of  $k(\mathbf{Z})$  given by adjoining  $(\frac{a^{v_{\mathbf{Z}}(b)}}{h^{v_{\mathbf{Z}}(a)}})^{1/m}$ .

#### R. JOSHUA AND J. LEE

### 3. Brauer groups of smooth split toric varieties and their fans

In this section we will compute the Brauer groups of all smooth split toric varieties over any field K. Let X denote a smooth split toric variety over the given field K and let m denote a fixed positive integer invertible in K. Assume r > 0 is a fixed integer for which K has a primitive m-th root of unity  $\zeta$ . Let T denote the open dense split torus in X of rank r. Next observe in view of Theorem 2.1 and Proposition 2.9, there are two types of contributions to the Brauer group of X:

- (i) coming from the part generated by the  $\binom{r}{2}$  cyclic algebras of the form  $(t_i, t_j)_{\zeta}$ , where  $t_i$  and  $t_j$  are two coordinates of T, with i < j, and
- (ii) coming from the part generated by the r cyclic algebras of the form  $(b, t_i)_{\zeta}$ , where  $b \in K^*$  and  $t_i$  is a coordinate of T.

We will denote the subgroup of  ${}_{m}Br(T)$  generated by  $\{\Pi_{1 \leq i < j \leq r}(t_{i}, t_{j})_{\zeta}^{e_{i,j}} | m > e_{i,j} \geq 0\}$  by  $\mathcal{A}$ , and the subgroup generated by  $\{\Pi_{1 \leq i \leq r}(b_{i}, t_{i})_{\zeta}^{e_{i}} | m > e_{i} \geq 0, b_{i} \in K^{*}\}$  by  $\mathcal{B}$ . Observe that any Azumaya algebra generated by the cyclic algebras  $(t_{i}, t_{j})_{\zeta}$ ,  $1 \leq i < j \leq r$  will be of the form  $\Pi_{i < j}(t_{i}, t_{j})_{\zeta}^{e_{i,j}}$ , for some choice of integers  $0 \leq e_{i,j} < m$  and that any Azumaya algebra generated by the cyclic algebras  $(b_{i}, t_{i})_{\zeta}^{e_{i}}$ , for some choice of integers  $0 \leq e_{i,j} < m$  and that any Azumaya algebra generated by the cyclic algebras  $(b_{i}, t_{i})_{\zeta}^{e_{i}}$ , for some choice of integers  $0 \leq e_{i} < m$ .

**Definition 3.1.** (i) Let **M** (**N**) denote the lattice of characters (co-characters or 1-parameter subgroups, respectively) associated to the split torus T. For each  $i = 1, \dots, r$ ,  $\mathbf{m}_i$  will denote the character of T defined by

$$\mathbf{m}_i(t_1,\cdots,t_r)=t_i$$

Moreover, in this setting we let  $\chi^{\mathbf{m}_i} = t_i$ .

- (ii) Let Δ denote the fan associated to X and let N' denote the subgroup generated by U<sub>σ∈Δ</sub> σ∩N. (Observe that the generators of the rays of the fan {ρ<sub>k</sub>|k = 1,...,n} in N generate N'. ) Then N' = Za<sub>1</sub>n<sub>1</sub> ⊕ ...Za<sub>u</sub>n<sub>u</sub> where n<sub>1</sub>,..., n<sub>u</sub>, n<sub>u+1</sub>,..., n<sub>r</sub> is a basis for N dual to the basis {m<sub>i</sub>|i = 1,...,r} for M, and a<sub>i</sub> ≥ 1 are integers with a<sub>i</sub>|a<sub>i+1</sub>, for i = 1,..., u − 1. (This follows from the structure theorem for the finitely generated abelian group N/N' or equivalently, the structure theorem for finitely generated free abelian subgroups. ) We call the set {a<sub>1</sub>,..., a<sub>u</sub>} the set of invariant factors of the given fan Δ.
- (iii) For an element  $b \in K^*$ , we let  $\overline{b}$  denote the image of b in  $K^*/(K^*)^m$ . Let  $ord_m(b)$  denote the order of  $\overline{b}$ , which is the least positive integer so that  $b^{ord_m(b)} \in (K^*)^m$ .

Then our main result is the following theorem. (One may want to observe that in the following theorem, the exponents  $e_{i,j}$  and  $e_i$  are all strictly less than m, since m is the order of all the cyclic algebras appearing there.)

**Theorem 3.2.** For a positive integer m invertible in K, the following hold, where  $\zeta \in K$  is a primitive m-th root of unity:

(i)  ${}_{m}Br(X) \cap \mathcal{A} = the subgroup generated by \{\Lambda = \Pi_{i < j}(t_{i}, t_{j})_{\zeta}^{e_{i,j}} | m > e_{i,j} \ge 0\}$  satisfying the following conditions: for each  $s = 1, \ldots, min\{u, r - 1\}$ , if  $m_{s} = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}$ , then  $(\frac{m}{m_{s}})|a_{s}$ . Moreover, we also obtain the isomorphism

 ${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} = \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z}),$ 

which shows this part of the Brauer group may be computed directly from the fan.

(ii)  ${}_{m}\operatorname{Br}(X) \cap \mathcal{B}$  is the subgroup generated by  $\{\Lambda = \prod_{i=1}^{r} (b_{i}, t_{i})_{\zeta}^{e_{i}} | m > e_{i} \geq 0\}$ , as  $b_{i} \in \operatorname{K}^{*}$  varies among the corresponding classes in  $\operatorname{H}^{1}_{\operatorname{et}}(\operatorname{Spec} \operatorname{K}, \mu_{m}(0))$  so that the following conditions are satisfied: for each  $s = 1, \cdots, u$ , if  $m_{s} = hcf\{m, e_{s}, ord_{m}(b_{s})\}$ , then  $(\frac{ord_{m}(b_{s})}{m_{s}})|a_{s}$ . Moreover,

 ${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} = \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N}', \mathrm{K}^{*}/\mathrm{K}^{*m}),$ 

which also shows this part of the Brauer group may be also computed directly from the fan. (iii) Now,

$${}_{m}\mathrm{Br}(\mathrm{X}) \cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \oplus ({}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A}) \oplus ({}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B})$$

 $\cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \oplus \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'),\mathbb{Z}/m\mathbb{Z}) \oplus \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\mathbf{N}/\mathbf{N}',\mathrm{K}^{*}/\mathrm{K}^{*m}).$ 

(iv) Therefore, if  ${}_{m}\mathrm{Br}_{1}(\mathrm{X})^{3} = \ker({}_{m}\mathrm{Br}(\mathrm{X}) \to {}_{m}\mathrm{Br}(\bar{\mathrm{X}}))$ , where  $\bar{\mathrm{X}}$  denotes the pull-back of X to the algebraic closure of K, then  ${}_{m}\mathrm{Br}_{1}(\mathrm{X}) \cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \oplus ({}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B}) \cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \oplus \mathrm{Hom}(\mathbf{N}/\mathbf{N}', \mathrm{K}^{*}/\mathrm{K}^{*m}).$ 

**Corollary 3.3.** In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension $(\sigma) \geq r-1$ , or if u = r-1, and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, r-1$ , then  ${}_m\text{Br}(X) \cap \mathcal{A}$  is trivial. In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension $(\sigma) \geq r$ , or if u = r and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, r$ , then both  ${}_m\text{Br}(X) \cap \mathcal{B}$  and  ${}_m\text{Br}(X) \cap \mathcal{A}$  are trivial, so that  ${}_m\text{Br}(X) \cong {}_m\text{Br}(\text{Spec K})$ .

**Remark 3.4.** Observe that the results in above theorem show that the m-torsion part of the Brauer group of any split toric variety is determined by the m-torsion part of the Brauer group of the base field, the fan and the lattice  $\mathbf{N}$  in an explicit manner, when m is invertible in the base field and the base field has an m-th root of unity. In addition, we also provide an alternate explicit description in terms of cyclic algebras as the first statements in (i) and (ii) show.

The rest of this section will be devoted to a proof of the above theorem, when X is a smooth toric variety defined over the field K, the integer m is invertible in K and  $\zeta$  is a primitive m-th root of unity in K. As one can see, at least for (i), we follow roughly the arguments in [DF](where they only considered the case of complex toric varieties), supplemented by Theorem 2.1.

We start with the following variant of the localization sequence in Proposition 2.9(ii), where  $\rho_k$ ,  $k = 1, \dots, n$  denote the rays in the fan  $\Delta$ , and  $\operatorname{orb}(\rho_i)$  denotes the corresponding codimension 1-orbit of T in X:

(3.1) 
$$0 \to {}_{m}\mathrm{Br}(\mathrm{X}) \to {}_{m}\mathrm{Br}(\mathrm{T}) \xrightarrow{r_{\mathrm{cs}}} \oplus_{i=1}^{n} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{orb}(\rho_{i}), \mu_{m}(0))$$

Let  $\rho_k$  denote a ray in the fan associated to the given toric variety X. Given a cyclic algebra of the form  $(\alpha, \beta)_{\zeta}$ , where  $\alpha, \beta$  belong to  $\Gamma(T, \mathbb{G}_m) = \mathbf{M}$ , the class of the residue,  $res((\alpha, \beta)_{\zeta})$  in  $\mathrm{H}^1_{\mathrm{et}}(\mathrm{orb}(\rho_k), \mu_m(0))$  corresponds to the cyclic Galois extension of the function field  $K(\mathrm{orb}(\rho_k))$  obtained by adjoining the *m*-th root of  $\alpha^{v_k(\beta)}/\beta^{v_k(\alpha)}$ , where  $v_k$  denotes the valuation associated with the toric divisor  $\rho_k$ . In case,  $\beta = b \in \mathrm{K}^*$ ,  $v_k(\beta) = 0$ , so that it suffices to determine  $v_k$  on the characters of T.

We next consider the following Lemma.

**Lemma 3.5.** (See [DF, Lemma 1.5]) Let  $\eta_k$  denote the primitive vector in  $\mathbf{N} \cap \rho_k$  and  $\langle , \rangle : \mathbf{M} \times \mathbf{N} \to \mathbb{Z}$ denotes the natural perfect pairing. Let  $\eta_k^{\perp} = \{\mathbf{m} \in \mathbf{M} | \langle \mathbf{m}, \eta_k \rangle = 0\}$ . Then

(i)  $K(\overline{orb}(\rho_k))$  is the quotient field of  $k[\eta_k^{\perp}]$ , and

<sup>&</sup>lt;sup>3</sup>This is the *m*-torsion part of the *algebraic* Brauer group of X.

(ii)  $v_k(t_i) = \langle \mathbf{m}_i, \eta_k \rangle$ ,  $t_i \in \mathbf{M}$ , where the relationship between  $\mathbf{m}_i$  and  $t_i$  is as in Definition 3.1(i).

Again following [DF], we define the ramification map along  $\rho_k$ :

$$ram_{\rho_k}: \mathcal{A} \to \mathbf{M}/\mathbf{m}\mathbf{M}$$

by:

(3.3) 
$$ram_{\rho_k}((t_i, t_j)_{\zeta}) = v_k(t_j)\mathbf{m}_i - v_k(t_i)\mathbf{m}_j + m\mathbf{M}.$$

Let  $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$  denote the  $\mathbb{Z}$ -basis for  $\mathbf{N}$  and  $\{\mathbf{m}_1, \dots, \mathbf{m}_r\}$  the dual basis for  $\mathbf{M}$  chosen as in Definition 3.1. We will identify  $t_i$  with  $\mathbf{m}_i$ ,  $i = 1, \dots, r$ . Then we define the map

$$(3.4) \qquad \phi: \mathcal{A} \to \operatorname{Hom}_{\mathbb{Z}}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$$

sending the class of the Azumaya algebra  $\Lambda = \prod_{i < j} (t_i, t_j)_{\zeta}^{e_{i,j}}$  (where  $e_{i,j}$  are some non-negative integers) to the  $\mathbb{Z}$ -linear map  $T_{\Lambda} : \mathbf{N} \to \mathbf{M}/(m\mathbf{M})$  whose matrix representation with respect to the chosen bases  $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$  and  $\{\mathbf{m}_1, \dots, \mathbf{m}_r\}$  is given by the matrix

(3.5) 
$$M_{\Lambda} = \begin{pmatrix} 0 & e_{1,2} & e_{1,3} & \cdots & & & e_{1,r} \\ -e_{1,2} & 0 & e_{23} & \cdots & & & e_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -e_{1,r} & -e_{2,r} & \vdots & \cdots & -e_{r-1,r} & 0 \end{pmatrix}$$

In fact  $T_{\Lambda}$  is the Z-linear map so that

$$T_{\Lambda}(\mathbf{n}_k) = \Sigma_{i < j}(e_{ij} < \mathbf{m}_j, \mathbf{n}_k > \mathbf{m}_i - e_{ij} < m_i, \mathbf{n}_k > \mathbf{m}_j) + m\mathbf{M}$$

Now it is easy to see (see for example, [DF, Proof of Lemma 1.7]) that the map  $\phi : \mathcal{A} \to \text{Hom}_{\mathbb{Z}}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$  is *injective*. Moreover, for each  $s = 1, \dots, u$ ,

(3.6) 
$$\mathbf{T}_{\Lambda}(a_{\mathbf{s}}\mathbf{n}_{\mathbf{s}}) = a_{\mathbf{s}} * \begin{pmatrix} e_{1,s} \\ e_{2,s} \\ \vdots \\ e_{s-1,s} \\ 0 \\ -e_{s,s+1} \\ \vdots \\ -e_{s,r} \end{pmatrix}$$

where the vector on the right is the s-th column of the matrix in (3.5). Therefore, we obtain the following criterion.

# Criterion for being unramified along $a_s \mathbf{n}_s : ram_{a_i \mathbf{n}_s} : \mathcal{A} \to \mathbf{M}/m\mathbf{M}, s = 1, \cdots, min\{u, r-1\}$ :

We say the cyclic algebra  $\Lambda = \prod_{1 \le i < j \le n} (t_i, t_j)_{\zeta}^{e_{i,j}}$  is unramified along  $a_s \mathbf{n}_s$  if  $T_{\Lambda}(a_s \mathbf{n}_s)$  is in  $m\mathbf{M}$ : this is true if and only if the following condition is satisfied:<sup>4</sup>

for each  $s = 1, \cdots, min\{u, r-1\},\$ 

<sup>&</sup>lt;sup>4</sup>One may verify that this is exactly the same condition discussed in [DF, Lemma 1.7(b)]. Any apparent differences between what appears in [DF] and in our formulation above is superificial and caused by the fact we concentrate on the *m*-torsion part of the Brauer group for a fixed m.

We will next define the ramification map along  $\rho_k$ ,  $k = 1, \dots, u$ :

(3.8) 
$$ram_{\rho_k}: \mathcal{B} \to \mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}$$

by:

(3.9) 
$$ram_{\rho_k}((b_j, t_j)_{\zeta}^{e_j}) = b_j^{e_j v_k(t_j)} * (\mathbf{K}^*)^{\mathbf{m}}, j = 1, \cdots, r,$$

where \* denotes multiplication. In view of the above definitions, we will next define a map

(3.10) 
$$\psi: \mathcal{B} \to \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^*/(\mathbf{K}^*)^{\mathrm{m}})$$

as follows. Let  $b_i \in K^*$  viewed as a class in  $H^1_{\text{et}}(\operatorname{Spec} k, \mu_m(0))$ . Let  $\Lambda = \prod_{i=1}^r (b_i, t_i)_{\zeta}^{e_i}$  denote an Azumaya algebra in  $\mathcal{B}$ . Then we let  $\psi(\Lambda) : \mathbf{N} \to K^*/(K^*)^m$  denote the map of abelian groups  $f_{\Lambda}$  defined by

(3.11) 
$$f_{\Lambda}(a_s \mathbf{n}_s) = b_s^{a_s * e_s} = \prod_{j=1}^r b_j^{e_j * a_s < \mathbf{m}_j, \mathbf{n}_s >}.$$

**Lemma 3.6.** The  $\mathbb{Z}$ -linear map  $\psi$  is injective.

Proof. One may first check that  $f_{\Lambda} : \mathbf{N} \to \mathbf{K}^*/(\mathbf{K}^*)^m$  is a group homomorphism. Therefore, it is enough to show that if  $b_s^{e_s}$  belongs to  $(\mathbf{K}^*)^m$ , then the class of the Azumaya algebra  $((b_s, t_s)_{\zeta})^{e_s}$  in B is trivial. Therefore, suppose  $b_s^{e_s} = c^m$  for some  $c \in \mathbf{K}^*$ . Now one observes from [Miln, Theorem 15.1, and also p. 146] that as a class in  $\mathcal{B}$ ,

$$((b_s, t_s)_{\zeta})^{e_s} = (b_s^{e_s}, t_s)_{\zeta} = (c^m, t_s)_{\zeta} = ((c, t_s)_{\zeta})^m.$$

Since the order of the cyclic algebra  $(c, t_s)_{\zeta}$  is m, it follows that the class  $((c, t_s)_{\zeta})^m$  is trivial.

Criterion for being unramified along  $ram_{a_s\mathbf{n}_s}: \mathcal{B} \to \mathrm{K}^*/\mathrm{K}^{*\mathrm{m}}$ . For an element  $b \in \mathrm{K}^*$ , we let  $\bar{b}$  denote the image of b in  $\mathrm{K}^*/(\mathrm{K}^*)^{\mathrm{m}}$ . Let  $ord_m(b)$  denote the order of  $\bar{b}$  as in Definition 3.1(iv). Then we obtain the following criterion for  $ram_{a_s\mathbf{n}_s}(\prod_{i=1}^r (b_i, t_i)_{\zeta}^{e_i}) = 0$ ,  $s = 1, \cdots, u$  (i.e., for the cyclic algebra  $\prod_{i=1}^r (b_i, t_i)_{\zeta}^{e_i}$  to be in the kernel of  $ram_{a_s\mathbf{n}_s}$ ):

(3.12) if 
$$m_s = hcf\{m, e_s, ord_m(b_s)\}$$
, then  $\frac{ord_m(b_s)}{m_s}$  must divide  $a_s$ ,  
where  $ord_m(b_s)$  is defined as in Definition 3.1(iv).

**Proof of Theorem 3.2.** We begin with the commutative diagram (see [DF, Proof of Theorem 1.8]):

Since  $\mathbf{N}/\mathbf{N}' \cong \bigoplus_{i=1}^{u} (\mathbb{Z}/\mathbb{Z}a_i)\mathbf{n}_i \oplus \bigoplus_{i=u+1}^{r} \mathbb{Z}\mathbf{n}_i$ , the first conclusion in (i) follows.

Next we consider the commutative diagram:

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

$$(3.14) \qquad 0 \xrightarrow{\qquad \qquad } \mathcal{B} \xrightarrow{\qquad \qquad } \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathrm{m}})$$

Since  $\mathbf{N}/\mathbf{N}' \cong \bigoplus_{i=1}^{u} (\mathbb{Z}/\mathbb{Z}a_i)\mathbf{n}_i \oplus \bigoplus_{i=u+1}^{r} \mathbb{Z}\mathbf{n}_i$ , the first conclusion in (ii) follows.

One may observe from the discussion of the ramification in (3.2) that the image of the residue map *res* on  $\mathcal{A}$  is contained  $\mathbf{M}/\mathbf{m}\mathbf{M}$  and therefore is a transcendental over the field K. Similarly one may observe from the discussion in (3.8) that the image of the residue map *res* on  $\mathcal{B}$  is in the field K. Therefore, no cancellation is possible between these. Therefore, one may observe that the product  $(\prod_{i < j} (t_i, t_j)_{\zeta}^{e_{i,j}})(\prod_i (b, t_i)_{\zeta}^{e_i})$  belongs to  ${}_m \mathrm{Br}(\mathrm{X})$  if and only if both factors  $\prod_{i < j} (t_i, t_j)_{\zeta}^{e_{i,j}}$  and  $\prod_i (b, t_i)_{\zeta}^{e_i}$  belong to  ${}_m \mathrm{Br}(\mathrm{X})$ . This proves the first statement in (iii).

One may prove the isomorphism

(3.15) 
$${}_{m}\mathrm{Br}(\mathbf{X}) \cap \mathcal{A} \cong \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z})$$

in the second statement in (i) as follows.

Let  $A = \bigoplus_{1 \leq i < j \leq r} \mathbb{Z}/m\mathbb{Z}(t_i, t_j)_{\zeta}$ , which denotes the direct sum of the free  $\mathbb{Z}/m\mathbb{Z}$ -module with basis the cyclic algebras  $(t_i, t_j)_{\zeta}, 1 \leq i < j \leq r$ . (Observe that now  $\sum_{i,j} e_{i,j}(t_i, t_j)_{\zeta}$  corresponds to the product  $\prod_{i,j} (t_i, t_j)_{\zeta}^{e_{i,j}}$  of cyclic algebras. See, for example, [Mag, top line of p. 167].)) Define a map

(3.16) 
$$\Phi: \operatorname{Hom}_{(Ab.groups)}(\wedge^2 \mathbf{N}, \mathbb{Z}/m\mathbb{Z}) \to A$$

as follows: Given a homomorphism  $\varphi : \wedge^2 \mathbf{N} \to \mathbb{Z}/m\mathbb{Z}$ , let  $e_{ij} \in \mathbb{Z}/m\mathbb{Z}$  be the image of  $n_i \wedge n_j$   $(1 \le i < j \le r)$ under  $\varphi$ . The map  $\Phi$  then maps  $\varphi$  to  $\sum_{1 \le i < j \le r} e_{ij}(t_i, t_j)\zeta$ . Note that this is a homomorphism of groups: if  $\varphi, \psi \in \operatorname{Hom}(\wedge^2(N), \mathbb{Z}/m\mathbb{Z})$  with  $e_{ij} = \varphi(n_i \wedge n_j)$  and  $f_{ij} = \psi(n_i \wedge n_j)$ , then

$$(3.17) \ \Phi(\varphi+\psi) = \sum_{1 \le i < j \le r} (e_{ij} + f_{ij})(t_i, t_j)_{\zeta} = \sum_{1 \le i < j \le r} e_{ij}(t_i, t_j)_{\zeta} + \sum_{1 \le i < j \le r} f_{ij}(t_i, t_j)_{\zeta} = \Phi(\varphi) + \Phi(\psi).$$

Conversely, given an element  $\prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}} \in A$ , we can define the inverse of  $\Phi$  by mapping A to the homomorphism  $\varphi : n_i \wedge n_j \mapsto e_{ij}$ . Thus, we have an isomorphism of groups

(3.18) 
$$\operatorname{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}), \mathbb{Z}/m\mathbb{Z}) \cong \mathbf{A}.$$

We will now explain the relation between the group  $\operatorname{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}), \mathbb{Z}/m\mathbb{Z})$  and the injective homomorphism  $\phi : \mathcal{A} \to \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$  given in (3.4). Recall each  $\Lambda = \prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$ is associated with the matrix  $M_{\Lambda}$  as in (3.5). The injective homomorphism  $\phi$  is then defined by sending  $\Lambda$ to the homomorphism  $\phi(\Lambda) \in \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$  which maps  $\mathbf{n}_i$  to

(3.19) 
$$\phi(\Lambda)(\mathbf{n}_i) = \sum_{k < i} e_{ki} \mathbf{m}_k - \sum_{k > i} e_{ik} \mathbf{m}_k + m \mathbf{M}.$$

In other words,  $\phi(\Lambda)$  maps each  $\mathbf{n}_i$  to the *i*th column of the matrix  $M_\Lambda$  viewed as an element in  $\mathbf{M}/\mathbf{m}\mathbf{M}$ . To relate this construction with  $\operatorname{Hom}_{(Ab.groups)}(\wedge^2 \mathbf{N}, \mathbb{Z}/m\mathbb{Z})$ , we identify the group  $\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$ 

using the following isomorphisms:

(3.20) 
$$\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M}) \cong \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z})$$
$$\cong \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \operatorname{Hom}_{(\mathbf{Ab.groups})}(\mathbf{N}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z})$$
$$\cong \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \operatorname{Hom}_{(\mathbf{Ab.groups})}(\mathbf{N}, \mathbb{Z}/m\mathbb{Z}))$$
$$\cong \operatorname{Hom}_{(Ab.groups)}(\mathbf{N} \otimes \mathbf{N}, \mathbb{Z}/m\mathbb{Z})$$

where the last isomorphism is the tensor-hom adjunction and the penultimate isomorphism uses the freeness of the lattice **N**. Let  $\overline{\phi(\Lambda)}$  denote the image of  $\phi(\Lambda)$  in  $\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}\otimes\mathbf{N},\mathbb{Z}/m\mathbb{Z})$  under this identification. Then,  $\overline{\phi(\Lambda)}$  can be described by the following rule:

(3.21) 
$$\overline{\phi(\Lambda)}(\mathbf{n}_i \otimes \mathbf{n}_j) = (\phi(\Lambda)(\mathbf{n}_i))(\mathbf{n}_j) = \begin{cases} -e_{ji} & j < i \\ 0 & j = i \\ e_{ij} & j > i. \end{cases}$$

Suppose  $1 \le i < j \le r$  and consider the images of  $\mathbf{n}_i \otimes \mathbf{n}_j$  and  $\mathbf{n}_j \otimes \mathbf{n}_i$  under the map  $\overline{\phi(\Lambda)}$ ; we have

(3.22) 
$$\phi(\Lambda)(\mathbf{n}_i \otimes \mathbf{n}_j) = e_{ij} \quad \text{because } i < j,$$

and

(3.23) 
$$\overline{\phi(\Lambda)}(\mathbf{n}_j \otimes \mathbf{n}_i) = -e_{ij} \quad \text{because } i < j.$$

On the other hand, if i = j, then we have  $\overline{\phi(\Lambda)}(\mathbf{n}_i \otimes \mathbf{n}_j) = 0$ . In other words,  $\overline{\phi(\Lambda)}$  induces a unique homomorphism  $\wedge^2 \mathbf{N} \to \mathbb{Z}/m\mathbb{Z}$ . This homomorphism is the image of  $\prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$  under the inverse of  $\Phi$  defined in (3.16). In summary, we have shown that

Lemma 3.7. The image of the homomorphism  $\phi : \mathcal{A} \to \operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$  is  $\operatorname{Hom}_{(Ab.groups)}(\wedge^{2}\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$  when  $\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}, \mathbf{M}/m\mathbf{M})$  is identified with  $\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}\otimes\mathbf{N}, \mathbb{Z}/m\mathbb{Z})$ .

In (3.7), we already showed that  $\Lambda$  is unramified along  $a_s \mathbf{n}_s$  if and only if  $\mathbf{M}_{\Lambda} \cdot \mathbf{a}_s \mathbf{n}_s = 0$ . As  $\mathbf{M}_{\Lambda} \cdot \mathbf{a}_s \mathbf{n}_s$  is identified with  $\overline{\phi(\Lambda)}(a_s \mathbf{n}_s \otimes -)$  using the above identification,  $\Lambda$  is unramified along all rays  $\rho_k$  in the fan if and only if

(3.24) 
$$\overline{\phi(\Lambda)}(a_s\mathbf{n}_s\otimes\mathbf{n}_j)=0 \text{ for all } j=1,2,\ldots,r.$$

It follows that the class represented by  $\Lambda$  lies in the kernel of the residue map r if and only if the condition (3.24) holds for all  $s = 1, \ldots, u$ , i.e., for each ray  $\mathbf{n}_s$ .

At this point, we recall that  $\mathbf{N}'$  is the sublattice generated by the rays  $\mathbf{n}_s$  and thus the condition (3.24) is equivalent to saying that

(3.25) 
$$\overline{\phi(\Lambda)}(a_i \mathbf{n}_i \otimes \mathbf{n}_j) = 0 \text{ for all } i = 1, \cdots u, j = 1, \cdots r.$$

Finally we observe that there is an exact sequence <sup>5</sup>

$$(3.26) \mathbf{N}' \otimes \mathbf{N} \to \wedge^2 \mathbf{N} \to \wedge^2 (\mathbf{N}/\mathbf{N}') \to \mathbf{0}$$

where the first arrow maps  $\mathbf{n}' \otimes \mathbf{n}$  to  $\mathbf{n}' \wedge \mathbf{n}$ . Therefore,  $\overline{\phi(\Lambda)}$  corresponds to an element in the kernel of the residue map if and only if it corresponds to an element of  $\operatorname{Hom}_{(Ab.groups)}(\wedge^2(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z})$ . This completes the proof of the isomorphism in (3.15).

<sup>&</sup>lt;sup>5</sup>One may infact use this short exact sequence to define  $\wedge^2(\mathbf{N}/\mathbf{N}')$ .

Next we proceed to prove the isomorphism

(3.27) 
$${}_{m}\mathrm{Br}(\mathbf{X}) \cap \mathcal{B} \cong \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N}', \mathbf{K}^*/(\mathbf{K}^*)^{\mathrm{m}})$$

in the second statement of (ii). First observe the injective map  $\psi$  defined in (3.10) defines an injective map

$${}_{m}\mathrm{Br}(\mathbf{X}) \cap \mathcal{B} \to \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N}', \mathbf{K}^{*}/(\mathbf{K}^{*})^{\mathbf{m}})$$

which takes an element  $\Lambda = \prod_{j=1}^{r} (b_j, t_j)_{\zeta}^{e_i}$  to the map

(3.28) 
$$f_{\Lambda}(n_i) = \prod_{j=1}^r b_j^{e_j \langle m_j, n_i \rangle} \cdot (\mathbf{K}^*)^{\mathbf{m}} = b_i^{e_i} \cdot (\mathbf{K}^*)^{\mathbf{m}}.$$

This map can be shown to be also surjective as follows. A homomorphism  $\varphi : \mathbf{N}/\mathbf{N}' \to \mathbf{K}^*/(\mathbf{K}^*)^{\mathrm{m}}$  corresponds to a choice of r elements  $b_i \in \mathbf{K}^*$  such that  $b_i^{a_i} \in (\mathbf{K}^*)^{\mathrm{m}}$ . Given such collection  $\{b_i | i = 1, \dots, r\}$  of elements, define a class

(3.29) 
$$\Lambda_{\varphi} = \prod_{j=1}^{r} (b_j, t_j)_{\zeta} \in {}_m \mathrm{Br}(\mathrm{T}_{\mathrm{K}}) \cap \mathcal{B}.$$

Then, under the map  $\psi: \mathcal{B} \to \operatorname{Hom}_{(Ab,groups)}(\mathbf{N}, \mathbf{K}^*/(\mathbf{K}^*)^m), \Lambda_{\varphi}$  is mapped to the homomorphism

(3.30) 
$$f_{\Lambda_{\varphi}}: \mathbf{N} \to \mathbf{K}^* / (\mathbf{K}^*)^{\mathbf{m}}, \quad \mathbf{n}_{\mathbf{i}} \mapsto \prod_{\mathbf{j}=1}^r b_j \langle \mathbf{m}_{\mathbf{j}}, \mathbf{n}_{\mathbf{i}} \rangle \cdot (\mathbf{K}^*)^{\mathbf{m}} = b_i \cdot (\mathbf{K}^*)^{\mathbf{m}}$$

and thus

(3.31) 
$$f_{\Lambda_{\varphi}}(a_{i}\mathbf{n}_{i}) = b_{i}^{a_{i}} \cdot (\mathbf{K}^{*})^{m} = (\mathbf{K}^{*})^{m} \text{ for all } i = 1, 2, \dots, r,$$

with the last equality holding because of our assumption that  $b_i^{a_i}$  belongs to  $(\mathbf{K}^*)^{\mathbf{m}}$ . Thus,  $f_{\Lambda_{\varphi}}$  is an element of  $\operatorname{Hom}_{(Ab.groups)}(\mathbf{N}/\mathbf{N}', \mathbf{K}^*/(\mathbf{K}^*)^{\mathbf{m}})$  and the image of  $\Lambda_{\varphi}$  under the map is  $f_{\Lambda_{\varphi}}$ . This completes the proof of statements (i) and (ii) and hence also of (iii) of theorem 3.2. Statement (iv) is clear at this point.  $\Box$ 

**Proof of Corollary 3.3.** In view of the fact that the toric variety is regular, the invariant factor  $a_i = 1$  for all  $i = 1, \dots, r-1$  when there is a cone  $\sigma$  in the fan of dimension  $\geq r-1$ . Then the condition that  $\left(\frac{m}{m_s}\right)|a_s$ , for all  $s = 1, \dots, u$  in Theorem 3.2(i), proves the first statement in Corollary 3.3. Similarly, in case there is a cone  $\sigma$  in the fan of dimension  $\geq r$ , or if u = r, and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, r$ , the first statement in Theorem 3.2(ii) proves the second statement in Corollary 3.3.

# 4. Extension of the above results to toric schemes defined over discrete valuation rings and Dedekind domains

In this section we will extend the above results to toric schemes defined over more general bases. There will be two basic contexts we consider here:

- (i) where the entire toric structure is defined over a Dedekind domain R satisfying the simplifying assumption in (4.1), or
- (ii) where the base is the spectrum of a discrete valuation ring R defined as in [KKMSD, Chapter 4, section 3], and also where the toric scheme satisfies the simplifying assumption in (4.9).

Toric schemes over the spectra of any commutative ring R are defined in [Ro] as obtained by gluing affine toric schemes which are defined as the spectra associated to commutative rings defined using monoids. Moreover

the following hypothesis is required to hold:

(4.1)

the toric scheme X is smooth, contains as an open dense subscheme the split torus  $T \cong \mathbb{G}_m^{\times r}$  (defined over Spec R), and that all the orbits of T on X are schemes that are smooth and faithfully flat over Spec R.

The first class of toric schemes we consider are such toric schemes defined over the spectra of Dedekind domains, which are excellent. We will assume that R denotes a Dedekind domain which is an *excellent* ring, the positive integer m is a unit in R, and R contains a primitive m-th root of unity which will be denoted  $\zeta$ . In view of the above assumptions, the fan for the toric scheme X over Spec R is defined just as in Definition 3.1, and it corresponds to the fan of the toric scheme which is the fiber over the generic point of R = K.

Next we recall from the discussion at the end of section 1 that the results of Theorem 2.1, the pairings discussed following it, as well as Proposition 2.9, all extend to the case where the base field is replaced by a Dedekind domain R. As a result,

(4.2) 
$${}_{m}\mathrm{Br}(\mathbb{G}_{\mathrm{m}}^{\times r}) \cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus (\oplus^{r}\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R},\mu_{m}(0))) \oplus (\oplus^{\binom{r}{2}}\mathrm{H}^{0}_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{R},\mu_{m}(-1)))$$

Moreover, the  $\binom{r}{2}$  summands  $\operatorname{H}^{0}_{\operatorname{et}}(\operatorname{Spec} \mathbb{R}, \mu_{m}(-1)))$  correspond to the cyclic algebras of the form  $(t_{i}, t_{j})_{\zeta}$ , for  $1 \leq i < j \leq r$  as  $t_{i}, t_{j}$  vary among the characters of T, while the *r*-summands  $\operatorname{H}^{1}_{\operatorname{et}}(\operatorname{Spec} \mathbb{R}, \mu_{m}(0))$ correspond to cyclic algebras of the form  $(b_{i}, t_{i})_{\zeta}$ , where  $b_{i} \in \mathbb{R}^{*}$  (which denotes the units in  $\mathbb{R}$ ) and  $t_{i}$  is a character of T.

We will denote the subgroup of  ${}_{m}Br(T)$  generated by  $\{\Pi_{1 \leq i < j \leq r}(t_{i},t_{j})_{\zeta}^{e_{i,j}} | m > e_{i,j} \geq 0\}$  as  $t_{i},t_{j}$  vary among the characters of the split torus T by  $\mathcal{A}$ , and the subgroup generated by  $\{\Pi_{1 \leq i \leq r}(b_{i},t_{i})_{\zeta}^{e_{i}} | m > e_{i} \geq 0,$ and  $b_{i} \in \mathbb{R}^{*}\}$  by  $\mathcal{B}$ . Observe that any Azumaya algebra generated by the cyclic algebras  $(t_{i},t_{j})_{\zeta}, 1 \leq i < j \leq r$ will be of the form  $\Pi_{i < j}(t_{i},t_{j})_{\zeta}^{e_{i,j}}$ , for some choice of integers  $0 \leq e_{i,j} < m$  and that any Azumaya algebra generated by the cyclic algebras  $(b_{i},t_{i})_{\zeta}, i = 1, \cdots, r$  will be of the form  $\Pi_{i=1}^{r}(b_{i},t_{i})_{\zeta}^{e_{i}}$ , for some choice of integers  $0 \leq e_{i} < m$ . Then we obtain the following theorem.

**Theorem 4.1.** Let R denote a Dedekind domain, which we assume is an excellent ring. We will also assume that the positive integer m is a unit in R and that R contains a primitive m-th root of unity  $\zeta$ . Then, under the assumption (4.1) the following hold:

- (i)  ${}_{m}\operatorname{Br}(X) \cap \mathcal{A} = the subgroup generated by \{\Lambda = \Pi_{i < j}(t_{i}, t_{j})_{\zeta}^{e_{i,j}} | m > e_{i,j} \geq 0\}$  satisfying the following conditions: for each  $s = 1, \ldots, min\{u, r-1\}$ , if  $m_{s} = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}$ , then  $(\frac{m}{m_{s}})|a_{s}$ .
- (ii)  ${}_{m}Br(X) \cap \mathcal{B}$  is the subgroup generated by  $\{\Lambda = \Pi_{i=1}^{r}(b_{i}, t_{i})_{\zeta}^{e_{i}} | m > e_{i} \geq 0\}$ , as  $b_{i} \in \mathbb{R}^{*}$  varies among the corresponding classes in  $\mathrm{H}^{1}_{\mathrm{et}}(\operatorname{Spec} \mathbb{R}, \mu_{m}(0))$  so that the following conditions are satisfied: for each  $s = 1, \dots, u$ , if  $m_{s} = hcf\{m, e_{s}, ord_{m}(b_{s})\}$ , then  $(\frac{ord_{m}(b_{s})}{m_{s}})|a_{s}$ .
- (iii) Moreover,  ${}_{m}Br(X) \cong {}_{m}Br(Spec R) \oplus ({}_{m}Br(X) \cap \mathcal{A}) \oplus ({}_{m}Br(X) \cap \mathcal{B}).$
- (iv) In addition,

 ${}_{m}\mathrm{Br}(\mathbf{X}) \cap \mathcal{A} \cong \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z}), and$ 

 ${}_{m}\mathrm{Br}(\mathbf{X}) \cap \mathcal{B} \cong \mathrm{Hom}_{(\mathrm{Ab.groups})}(\mathbf{N}/\mathbf{N}', \mathbf{R}^{*}/\mathbf{R}^{*m}).$ 

#### R. JOSHUA AND J. LEE

*Proof.* Observe that the localization sequence in Proposition 2.9 holds also over the spectra of Dedekind domains. Therefore, in view of (4.2), one may observe that when X is a toric scheme defined over the Dedekind domain R, the Brauer group  ${}_{m}Br(X)$  is determined by the same exact sequence as in (3.1). Moreover, observe that  $\zeta$  remains a primitive *m*-th root of unity in the fraction field K. But then we obtain the commutative diagram:

where the first two vertical maps are clearly injective, and the last sum in the top row (the bottom row) varies over the codimension 1 orbits of T in X (of  $T_K$  in  $X_K$ , respectively). The assumptions in (4.1) ensure that the each codimension 1-orbit in X corresponds 1 - 1 to a codimension 1 orbit in  $X_K$ . Therefore, the last vertical map is also injective. Thus to determine the kernel of the residue map in the top row, we reduce to determining the kernel of the residue map in the second row: but the toric scheme now is the toric variety  $X_K$  defined over the field of fractions K, with  $T_K$  denoting the open orbit and  $\operatorname{orb}(\rho_i)_K$  denoting the corresponding codimension 1-orbits. Therefore, under the assumption (4.1), we obtain the first three statements in the theorem by invoking Theorem 3.2. The isomorphism:

$${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} \cong \mathrm{Hom}_{(\mathrm{Ab},\mathrm{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z}), \text{ and}$$

follows as in the proof of (3.15) while the identification

$$_m Br(X) \cap \mathcal{B} \cong Hom_{(Ab,groups)}(\mathbf{N}/\mathbf{N}', \mathbf{R}^*/\mathbf{R}^{*m})$$

follows as in the discussions following (3.27).

# Corollary 4.2. Assume the hypotheses of Theorem 4.1.

In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension $(\sigma) \geq r-1$ , or if u = r-1 and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, r-1$ , then  ${}_{m}Br(X) \cap \mathcal{A}$  is trivial.

In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension( $\sigma$ )  $\geq r$  or if u = r, and the invariant factors  $u_i = 1$ , for all  $i = 1, \dots, r$ , then both  ${}_m Br(X) \cap \mathcal{B}$  and  ${}_m Br(X) \cap \mathcal{A}$  are trivial, so that  ${}_m Br(X) \cong {}_m Br(Spec R)$ .

*Proof.* This follows by a reasoning as in the proof of the above theorem, and by invoking Corollary 3.3 with the field K there replaced by the field K in the present set-up, which is  $R_{(0)}$ .

Next we consider the second class of toric schemes, which are defined over the spectra of discrete valuation rings R which are also assumed to be excellent and defined as in [KKMSD, Chapter 4, section 3]: see also [GPS], [Qu] for similar discussions. We recall the following from [KKMSD, Chapter 4, section 3]. Throughout the following discussion, we will assume that the integer m is a unit in R and R contains a primitive m-th root of unity which will be denoted  $\zeta$ . We will also assume that the above primitive m-th root of unity in R is in fact the lift of a primitive m-th root of unity in the residue field k.

Let T denote a split torus of rank r over R, and let  $\mathbf{M}$  ( $\mathbf{N}$ ) denote the lattice of characters (co-characters, respectively) of T. Let  $\widetilde{\mathbf{M}} = \mathbf{M} \times \mathbb{Z}$  and  $\widetilde{\mathbf{N}} = \mathbf{N} \times \mathbb{Z}$ . One adopts the definition of *split toric schemes* over R as in [KKMSD, Chapter 4, section 3]. These are irreducible normal schemes X over R on which a split torus

21

T acts so that over the generic point of Spec R, X is in fact a toric variety defined over the field of fractions K of R. It is shown that there is a bijection between such affine toric schemes over Spec R and rational polyhedral cones  $\sigma \subseteq \mathbf{N}(T) \otimes \mathbb{R} \times \mathbb{R}_{\geq 0}$ , where  $\mathbb{R}_{+} = \{z \in \mathbb{R} | z \geq 0\}$ . Under this correspondence, it is shown there that the affine toric scheme  $X_{\sigma}$  corresponding to the cone  $\sigma$  is regular if and only if  $\sigma \cap (\mathbf{N}(T) \times \mathbb{Z})$  can be generated by a subset of a basis of  $\mathbf{N}(T) \times \mathbb{Z}$ .

Moreover any such toric scheme over R is obtained by gluing affine toric schemes and such toric schemes are classified by fans in  $\mathbf{N}(T) \otimes \mathbb{R} \times \mathbb{R}_{\geq 0}$ . Over the generic point of Spec R, such toric schemes correspond to fans in  $\mathbf{N}(T) \otimes \mathbb{R} \cong \mathbf{N}(T) \otimes \mathbb{R} \times \{0\}$ .

We will let

$$(4.4) p: \mathbf{N} \to \mathbb{Z} \text{ and } \pi: \mathbf{N} \to \mathbf{N}$$

denote the obvious projections. Then we start with the following observations:

- (i) Since the toric scheme X is assumed to be regular, and if the special fiber is reduced, it follows that  $p(\rho) = 1$  for each ray  $\rho$  not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ : see [Qu, Corollary 2.1.21].
- (ii) the irreducible components of the special fiber correspond to the rays of the fan not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ : see [Qu, 2.1.20].

Observe that

$$\mathbf{X} - \mathbf{T}_{\mathbf{K}} = \left(\bigcup_{\rho \in \Delta(1) \cap (\mathbf{N}_{\mathbb{R}} \times 0)} \mathbf{V}(\rho)\right) \bigcup \mathbf{X}_{k}$$

where  $X_k$  is the special fiber of X over the closed point of Spec R and  $V(\rho)$  denotes the divisor corresponding to  $\rho$ . Note that  $X_k$  has pure codimension one in X because  $X_k = X \times_{\text{Spec R}} \text{Spec }(R/(\pi))$ .

Let  $\tilde{\Delta}$  denote the fan in  $\tilde{\mathbf{N}}_{\mathbb{R}}$  associated to the given toric scheme. Since we assume that the special fiber is reduced, it follows that the irreducible components of  $X_k$  correspond to the rays of the fan  $\tilde{\Delta}$  not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ . We will let  $\{\rho_1, \ldots, \rho_{\nu}\}$  be the set of rays of  $\tilde{\Delta}$  contained in  $\mathbf{N}_{\mathbb{R}} \times 0$  and let  $\{\tau_1, \ldots, \tau_{\mu}\}$  be the set of rays of  $\tilde{\Delta}$  not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ . We define  $\mathbf{N}'$  to be the sublattice generated by  $\bigcup_{i=1}^{\nu} (\rho_i \cap \mathbf{N})$  as before. We choose a basis  $\{\mathbf{n}_1, \ldots, \mathbf{n}_u, \mathbf{n}_{u+1}, \cdots, \mathbf{n}_r\}$  of  $\mathbf{N}$  as before, i.e., in such a way that

(4.5) 
$$\mathbf{N}' = \bigoplus_{i=1}^{u} \mathbb{Z}a_i \mathbf{n}_i$$

with positive integers  $a_1|a_2|\cdots|a_u$ . Let  $\{\mathbf{m}_i|i=1,\cdots,r\}$  be the dual basis of **M**. Hence,

$$\widetilde{\mathbf{M}} = \bigoplus_{i=1}^{\prime} \mathbb{Z} \cdot \mathbf{m}_i \bigoplus \mathbb{Z} \mathbf{e}_i$$

where e = (0, 1).

In view of the localization sequence as in Proposition 2.9 (see also [CTS, Theorem 3.7.2]) one has the exact sequence

(4.6) 
$$0 \to {}_{m}\mathrm{Br}(\mathrm{X}) \to {}_{m}\mathrm{Br}(\mathrm{T}_{\mathrm{K}}) \xrightarrow{res} \overbrace{\left(\bigoplus_{i=1}^{\nu} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{K}(\mathrm{V}(\rho_{i})), \mu_{\mathrm{m}}(0))\right)}^{\mathcal{C}} \bigoplus \overbrace{\left(\bigoplus_{j=1}^{\mu} \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{K}(\mathrm{V}(\tau_{j})), \mu_{\mathrm{m}}(0))\right)}^{\mathcal{D}}$$

Thus, as in the case of the toric varieties over a field, the Brauer group of the toric scheme X is identified with the kernel of the map res, which is the residue map. As before, we have the complete description of

 $_m Br(T_K)$  as follows:

(4.7) 
$${}_{m}\mathrm{Br}(\mathrm{T}_{\mathrm{K}}) = {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{K}) \bigoplus \underbrace{\left(\bigoplus_{i\leq i< j\leq r} \mathbb{Z}/m\mathbb{Z} \cdot (t_{i},t_{j})_{\zeta}\right)}_{i\leq i< j\leq r} \bigoplus \underbrace{\left(\bigoplus_{i=1}^{r} \sum_{b_{i}\in K^{*}} \mathbb{Z}/m\mathbb{Z} \cdot (b_{i},t_{i})_{\zeta}\right)}_{\mathcal{B}}$$

where we use the same convention on the cyclic algebras  $(b_i, t_i)_{\zeta}$  and  $(t_i, t_j)_{\zeta}$  as before.

One may observe from the discussion of the ramification in (3.2) that the image of the residue map *res* on  $\mathcal{A}$  is contained  $\mathbf{M}/m\mathbf{M}$  and therefore is transcendental over the field K. Similarly one may observe from the discussion in (3.8) that the image of the residue map *res* on  $\mathcal{B}$  is in the field K. Therefore, no cancellation is possible between these. Therefore, in view of the decompositions in (4.6) and (4.7), and the above observation, determining the kernel of the residue map *res* on  $_m \mathrm{Br}(\mathrm{T}_{\mathrm{K}})$  amounts to determining the kernel of the residue map:

- (i)  $res_{\mathcal{A},\mathcal{C}}: \mathcal{A} \to \mathcal{C},$
- (ii)  $res_{\mathcal{A},\mathcal{D}}: \mathcal{A} \to \mathcal{D},$
- (iii)  $res_{\mathcal{B},\mathcal{C}}: \mathcal{B} \to \mathcal{C},$
- (iv)  $res_{\mathcal{B},\mathcal{D}}: \mathcal{B} \to \mathcal{D}$  as well as
- (v)  $res \circ \pi^* : {}_m Br(Spec K) \xrightarrow{\pi^*}{\longrightarrow} {}_m Br(T_K) \xrightarrow{res}{\longrightarrow} \mathcal{C} \oplus \mathcal{D}.$

**Proposition 4.3.** Let  $\rho \in \widetilde{\Delta}(1)$  be a ray and let  $\eta$  be the primitive vector of  $\rho$ . Let  $\eta^{\perp} = \{ \widetilde{\mathbf{m}} \in \widetilde{\mathbf{M}} | < \widetilde{\mathbf{m}}, \eta >= 0 \}.$ 

- (i) The function field  $K(V(\rho))$  is the quotient field of  $k[\boldsymbol{\eta}^{\perp}]$  if  $\rho \not\subseteq \mathbf{N}_{\mathbb{R}} \times 0$  and the quotient field of  $K[\boldsymbol{\eta}^{\perp} \cap \mathbf{M}]$ if  $\rho \subseteq \mathbf{N}_{\mathbb{R}} \times 0$ .
- (ii) The valuation of  $\chi^{\mathbf{m}}\pi^r \in K(\mathbf{X})$  along  $V(\rho)$  is

$$v_{\rho}(\chi^{\mathbf{m}}\pi^{r}) = < \widetilde{\mathbf{m}}, \boldsymbol{\eta} > \quad where \ \widetilde{\mathbf{m}} = (\mathbf{m}, r).$$

*Proof.* Let  $\mu \in \widetilde{\mathbf{M}}$  be such that  $\langle \mu, \eta \rangle = 1$ . Then we have a split exact sequence

$$0 \to \mathbb{Z}\boldsymbol{\eta} \to \mathbf{N} \to \mathbf{N}/\mathbb{Z}\boldsymbol{\eta} \to 0$$

and its dual

$$0 \to \boldsymbol{\eta}^{\perp} \to \widetilde{\mathbf{M}}^{\mathbf{m} \to <\mathbf{m}, \boldsymbol{\eta} >} \mathbb{Z}\boldsymbol{\mu} \to 0,$$

so that  $\widetilde{\mathbf{M}} = \boldsymbol{\eta}^{\perp} \oplus \mathbb{Z}\boldsymbol{\mu}$ .

Recall  $\mathbf{e} = (0,1) \in \widetilde{\mathbf{N}}$ . If  $\rho \subseteq \mathbf{N}_{\mathbb{R}} \times 0$ , then the orbit closure  $V(\rho)$  embeds into  $U_{\rho} = \operatorname{Spec} R([\boldsymbol{\eta}^{\perp} \oplus \mathbf{N}\boldsymbol{\mu}]/(\chi^{e} - \pi))$  via the ring homomorphism

$$\operatorname{R}[\boldsymbol{\eta}^{\perp} \oplus \mathbf{N}\boldsymbol{\mu}]/(\chi^{\mathbf{e}} - \pi) \to \operatorname{R}[\boldsymbol{\eta}^{\perp}]/(\chi^{\mathbf{e}} - \pi)$$

Hence, the prime ideal corresponding to  $V(\rho)$  is  $(\chi^{\mu})$ .

If  $\rho \not\subseteq \mathbf{N}_{\mathbb{R}} \times 0$ , then the orbit closure  $V(\rho)$  embeds into  $U_{\rho}$  via the ring homomorphism

$$R[\boldsymbol{\eta}^{\perp} \oplus \mathbf{N}\boldsymbol{\mu}]/(\chi^{\mathbf{e}} - \pi) \to k[\boldsymbol{\eta}^{\perp}].$$

Note that since  $\rho \not\subseteq \mathbf{N}_{\mathbb{R}} \times 0$ ,  $\mathbf{e} \notin \boldsymbol{\eta}^{\perp}$ . Thus, this homomorphism is well-defined and its kernel is again given by  $(\chi^{\boldsymbol{\mu}})$ .

If  $\rho \subseteq \mathbf{N}_{\mathbb{R}} \times 0$ , then  $-\mathbf{e} \in \boldsymbol{\eta}^{\perp}$  and so  $\pi$  is invertible in  $\mathbb{R}[\boldsymbol{\eta}^{\perp}]/(\chi^{\mathbf{e}} - \pi)$  which shows that  $\mathbb{R}[\boldsymbol{\eta}^{\perp}]/(\chi^{\mathbf{e}} - \pi) = K[\boldsymbol{\eta}^{\perp} \cap \mathbf{M}].$ 

If  $\rho \not\subseteq \mathbf{N}_{\mathbb{R}} \times 0$ , then  $V(\rho)$  is a toric variety over k whose coordinate ring is given by  $k[\boldsymbol{\eta}^{\perp}]$ . Hence, the function field of  $V(\rho)$  is the quotient field of  $k[\boldsymbol{\eta}^{\perp}]$ . This proves (i).

For (ii), let  $\chi^{\mathbf{m}}\pi^r \in \mathcal{K}(\mathcal{X})$  and write  $\widetilde{\mathbf{m}} = (\mathbf{m}, r)$ . Since  $\widetilde{\mathbf{M}} = \boldsymbol{\eta}^{\perp} \oplus \mathbb{Z}\boldsymbol{\mu}$ , we can find  $\mathbf{x} \in \boldsymbol{\eta}^{\perp}$  and  $n \in \mathbb{Z}$  such that  $\widetilde{\mathbf{m}} = \mathbf{x} + n\boldsymbol{\mu}$ . Since the prime ideal corresponding to  $\mathcal{V}(\rho)$  is  $(\chi^{\mu})$  in either case, the valuation of  $\chi^{\mathbf{m}}\pi^r = \chi^{\widetilde{\mathbf{m}}} = \chi^x(\chi^{\boldsymbol{\mu}})^n \in \mathcal{R}[\boldsymbol{\eta}^{\perp} \oplus \mathbf{N}\boldsymbol{\mu}]/(\chi^{\mathbf{e}} - \pi)$  is given by n. This integer can also be obtained as

$$<\widetilde{\mathbf{m}}, oldsymbol{\eta}>=<\mathbf{x}+\mathbf{n}oldsymbol{\mu}, oldsymbol{\eta}>=<\mathbf{x}, oldsymbol{\eta}>+n=n.$$

This shows that  $v_{\rho}(\chi^{\mathbf{m}}\pi^{r}) = < \widetilde{\mathbf{m}}, \eta > .$ 

The above results enable us to obtain an explicit description of the map *res* as follows. For  $\rho \in \Delta(1)$ , let  $\eta$  denote the primitive vector of the ray  $\rho$  so that  $\rho_i = \mathbb{R}_{\geq 0} \eta_i$ , for  $i = 1, \ldots, \nu$ . Similarly let  $\tau_j = \mathbb{R}_{\geq 0} \tilde{\eta}_j$ , with  $\tilde{\eta}_j$  denoting a primitive vector along  $\tau_j$ , for  $j = 1, \ldots, \mu$ . Recall that  $\rho_i$ ,  $i = 1, \cdots, \nu$  denote the rays in the fan  $\tilde{\Delta}$  that are contained in  $\mathbb{N}_{\mathbb{R}} \times 0$  and that  $\tau_j$ ,  $j = 1, \cdots, \mu$  denote the rays in  $\tilde{\Delta}$  that are *not* contained in  $\mathbb{N}_{\mathbb{R}} \times 0$ . For  $\mathbf{m}_i \in \mathbb{M}$ , we will also denote the image of  $\mathbf{m}_i$  in  $\widetilde{\mathbb{M}}$  by the same symbol  $\mathbf{m}_i$ . Then, observing that  $t_i = \chi^{\mathbf{m}_i}$ ,  $i = 1, \cdots, r$ , we obtain for  $1 \leq i < j \leq r$ :

$$(4.8) \qquad res((t_i, t_j)_{\zeta}) = \left( \left( \chi^{<\mathbf{m}_j, \boldsymbol{\eta}_v > \mathbf{m}_i - <\mathbf{m}_i, \boldsymbol{\eta}_v > \mathbf{m}_j} \right)_v, \left( \chi^{<\mathbf{m}_j, \boldsymbol{\widetilde{\eta}}_l > \mathbf{m}_i - <\mathbf{m}_i, \boldsymbol{\widetilde{\eta}}_l > \mathbf{m}_j} \right)_l \right)_{v=1, \dots, \nu, l=1\dots, \mu}$$

**Definition 4.4.** We say that the algebra  $\Lambda = \prod_{1 \leq i < j \leq r}^{r} (t_i, t_j)_{\zeta}^{e_{i,j}}$  is unramified along  $\rho_i, i = 1, \dots, \nu$  ( $\tau_j, j = 1, \dots, \mu$ ), if the  $\rho_i$ -th ( $\tau_j$ -th) coordinate of res( $\Lambda$ ) is zero. One defines the algebra  $\Lambda = \prod_{i=1}^{r} (b_i, t_i)^{e_i}$  with  $b_i \in \mathbf{K}^*$  to be unramified along  $\rho_i, i = 1, \dots, \nu$  ( $\tau_j, j = 1, \dots, \mu$ ) similarly.

Computation of kernel(res|\_A). From the formula (4.8), we deduce that a class  $\Lambda = \prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}} \in {}_{m} \operatorname{Br}(T_K)$  lies in  ${}_{m} \operatorname{Br}(X)$  if and only if  $\langle \mathbf{m}_j, \boldsymbol{\eta}_v \rangle \mathbf{m}_i - \langle \mathbf{m}_i, \boldsymbol{\eta}_v \rangle \mathbf{m}_j \in m\mathbf{M}$  for all  $v = 1, \dots v$ . Moreover, since  $\mathbf{m}_i$  viewed as an element of  $\widetilde{\mathbf{M}}$  has the last coordinate 0 and  $\boldsymbol{\eta}_v \in \mathbf{N}_{\mathbb{R}} \times 0$ , the pairing  $\langle \mathbf{m}_i, \boldsymbol{\eta}_v \rangle$  is equal to the pairing  $\langle \mathbf{m}_i, \boldsymbol{\eta}_v \rangle$  of two elements  $\mathbf{m}_i \in \mathbf{M}$  and  $\boldsymbol{\eta}_v \in \mathbf{N}$ . Hence, we deduce the following:

The kernel of  $res_{\mathcal{A},\mathcal{C}}: \mathcal{A} \to \mathcal{C}$  is given by those cyclic algebras  $\Lambda = \prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$  satisfying exactly the condition in Theorem 4.1(i).

At this point we will make the following simplifying assumption<sup>6</sup>:

(4.9) the toric scheme X is regular, the special fiber of X over the residue field k is reduced, and that the subgroup generated by  $\pi(\tilde{\eta}_i), j = 1, \dots, \mu$ , is contained in N'.

Next recall from the simplifying assumption (4.9) that  $\pi(\tilde{\eta}_j) \in \mathbf{N}'$  for all  $j = 1, \dots, \mu$ . Therefore, any cyclic algebra  $\Lambda$  in the kernel of  $res_{\mathcal{A},\mathcal{C}} : \mathcal{A} \to \mathcal{C}$  is unramified along  $\pi(\tilde{\eta}_j)$ , for any  $j = 1, \dots, \mu$  showing that the kernel of  $res_{\mathcal{A},\mathcal{C}} : \mathcal{A} \to \mathcal{C}$  is contained in the kernel of  $res_{\mathcal{A},\mathcal{D}} : \mathcal{A} \to \mathcal{D}$ . To see this in more detail, fix a ray  $\tau \in \tilde{\Delta}(1)$  that is not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$  and let  $\tilde{\eta}$  denote the primitive vector along  $\tau$  in  $\tilde{\mathbf{N}}$ . Since  $\pi(\tilde{\eta})$  lies in  $\mathbf{N}'$  (by our assumption in (4.9), there exist some integers  $\alpha_1, \dots, \alpha_u$  such that  $\pi(\tilde{\eta}) = \sum_{i=1}^u \alpha_i a_i \mathbf{n}_j$ .

<sup>&</sup>lt;sup>6</sup>Observe that the condition in (4.9) is simply saying the projections of the rays of the fan in the special fiber to  $\mathbf{N} \times \mathbf{0}$  are linear combinations of the rays in the generic fiber. One should view the above condition as making sure the toric structure of the special fiber is not too different from the toric structure of the generic fiber. In section 6, we will explore this condition through examples.

On the other hand, because  $\mathbf{m}_i \in \mathbf{M}$ , we have

(4.10) 
$$\langle \mathbf{m}_i, \widetilde{\boldsymbol{\eta}} \rangle = \langle \mathbf{m}_i, \pi(\widetilde{\boldsymbol{\eta}}) \rangle = \sum_{j=1}^u \langle \mathbf{m}_i, \alpha_j a_j \mathbf{n}_j \rangle = \alpha_i a_i \quad \text{for each } i = 1, \dots, r.$$

Now,  $\Lambda = \prod_{1 \leq i < j \leq r} (t_i, t_j)_{\zeta}^{e_{ij}}$  is unramified along  $\tau$  if and only if  $m |\alpha_i a_i e_{i,j}$ , for all  $1 \leq i < j \leq r$ . But, using the assumption that  $\Lambda$  is unramified along each  $\rho_i$ , we see from Theorem 4.1(i) that  $m |a_i e_{i,j}$ , for all  $1 \leq i < j \leq n$ . Therefore,  $m |\alpha_i a_i e_{i,j}$ , for all  $1 \leq i < j \leq n$ , proving the required conclusion. These observations therefore prove the following result.

**Proposition 4.5.** Under the assumption (4.9), the subgroup  ${}_{m}Br(X) \cap \mathcal{A}$  is isomorphic to a subgroup of  ${}_{m}Br(X_{K})$ , where  $X_{K}$  is the toric variety associated to the fan  $\Delta \cap (\mathbf{N}_{\mathbb{R}} \times 0)$ . Moreover,  ${}_{m}Br(X) \cap \mathcal{A} \cong \operatorname{Hom}_{(Ab, \operatorname{groups})}(\wedge^{2}(\mathbf{N}/\mathbf{N}'), \mathbb{Z}/m\mathbb{Z})$ .

Computation of kernel(res|\_{\mathcal{B}}). Next we proceed to determine the kernel of the residue map  $res_{\mathcal{B},\mathcal{C}}: \mathcal{B} \to \mathcal{C}$ . Therefore, let  $b_i \in K^*$  and i = 1, ..., r. Let the cyclic algebra  $(b_i, t_i)_{\zeta}$  represent a class in  $\mathcal{B} \subseteq {}_m Br(T_K)$ . For  $\rho \in \tilde{\Delta}(1)$  contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ , and  $b_i \in K^*$ ,  $v_{\rho}(b_i) = 0$ . Therefore, the image of  $(b_i, t_i)_{\zeta}^{e_i}$  under the residue map  ${}_m Br(T_K) \to H^1(K(V(\rho)), \mu_m(0))$  is the element

$$b_i^{e_i v_{\rho}(t_i)} / \chi^{e_i v_{\rho}(b_i) \mathbf{m}_i} = b_i^{e_i v_{\rho}(t_i)} = b_i^{e_i < \mathbf{m}_i, \rho >} \in \mathbf{K}^*$$

Choosing  $\rho = a_j \mathbf{n}_j$ ,  $j = 1, \dots, n$ , we see that the algebra  $\Lambda = \prod_{i=1}^r (b_i, t_i)_{\zeta}^{e_i}$  belongs to the kernel of the residue map  $res_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}$  precisely when the condition in Theorem 4.1(ii) holds.

Next let  $b_i = u_i \pi^{n_i}$  where  $u_i \in \mathbb{R}^*$  and  $n_i \in \mathbb{Z}$ . Then for any ray  $\rho \in \tilde{\Delta}$  which is contained in  $\mathbb{N}_{\mathbb{R}} \times 0$ ,

(4.11) 
$$b_i^{e_i v_{\rho}(t_i)} = (u_i \pi^{n_i})^{e_i v_{\rho}(t_i)} = (u_i)^{e_i v_{\rho}(t_i)} (\pi)^{e_i n_i v_{\rho}(t_i)} = u_i^{e_i < \mathbf{m}_i, \rho >} \cdot \chi^{\mathbf{e} e_i n_i < \mathbf{m}_i, \rho >}$$

since  $\pi = \chi^{\mathbf{e}}$ . Therefore, the class of the algebra  $\Pi_{i=1}^{r}(b_{i},t_{i})_{\zeta}^{e_{i}}$  in the subgroup  $\mathcal{B}$  identifies with the class of the algebra  $\Pi_{i=1}^{r}(u_{i},t_{i})_{\zeta}^{e_{i}}$  in  $\mathcal{B}$  provided

(4.12) 
$$\Sigma_{i=1}^{r} e_i n_i \text{ is divisible by } m.$$

Next we proceed to determine the kernel of the residue map  $res_{\mathcal{B},\mathcal{D}}: \mathcal{B} \to \mathcal{D}$ . Let  $b_i \in \mathrm{K}^*$ . Let  $\tau \in \tilde{\Delta}(1)$  be not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ , with  $\tilde{\boldsymbol{\eta}}$  denoting the primitive vector along  $\tau$ .

Then the image of the cyclic algebra  $(b_i, t_i)^{e_i}_{\zeta}$  under the residue map  $\mathcal{B} \to H^1(K(V(\tau)), \mu_m(0)) \subseteq \mathcal{D}$  is the image of the element

$$x = b_i^{e_i v_\tau(t_i)} / \chi^{e_i v_\tau(b_i) \mathbf{m}_i} \in \mathcal{O}_{X, V(\tau)}^*$$

in  $K(V(\tau))/(K(V(\tau)))^{*m}$  and  $K(V(\tau))$  is the quotient field of  $k[\tilde{\eta}^{\perp}]$ . (Recall  $t_i = \chi^{\mathbf{m}_i}, i = 1, \cdots, r$ .) Write  $b_i = u_i \pi^{n_i}$  where  $u_i \in \mathbb{R}^*$  and  $n_i \in \mathbb{Z}$ . Noting that  $\chi^{\mathbf{e}} = \pi$ , we can rewrite x as

(4.13) 
$$u_i^{e_i < \mathbf{m}_i, \widetilde{\boldsymbol{\eta}} >} \cdot \left( \frac{\chi^{\mathbf{e}e_i < \mathbf{m}_i, \widetilde{\boldsymbol{\eta}} >}}{\chi^{e_i \mathbf{m}_i}} \right)^{n_i} = u_i^{e_i < \mathbf{m}_i, \widetilde{\boldsymbol{\eta}} >} \chi^{n_i e_i (<\mathbf{m}_i, \widetilde{\boldsymbol{\eta}} > \mathbf{e} - \mathbf{m}_i)}.$$

From (4.13), we see that for a class  $\Lambda = \prod_{i=1}^{r} (b_i, t_i)_{\zeta}^{e_i}$  to lie in the kernel of the residue map  $res_{\mathcal{B},\mathcal{D}} : \mathcal{B} \to \mathcal{D}$  it is necessary and sufficient that the following condition holds:

(4.14) 
$$\prod_{i=1}^{r} \bar{u}_{i}^{e_{i} < \mathbf{m}_{i}, \widetilde{\eta}_{j} >} \cdot \chi^{\sum_{i=1}^{r} e_{i} n_{i} (<\mathbf{m}_{i}, \widetilde{\eta}_{j} > \mathbf{e} - \mathbf{m}_{i})} \in \mathrm{K}(\mathrm{V}(\tau_{j}))^{*m}$$

for all  $j = 1, ..., \mu$  where  $b_i = u_i \pi^{n_i}, u_i \in \mathbb{R}^*$  with  $\bar{u}_i$  the image of  $u_i$  in  $k^*$ , and  $n_i \in \mathbb{Z}$ . Since  $K(V(\tau_j))$  is the quotient field of  $k[\eta_i^{\perp}]$ , (4.14) holds if and only if

(4.15) 
$$\sum_{i=1}^{r} e_{i} n_{i} (\langle \mathbf{m}_{i}, \widetilde{\boldsymbol{\eta}}_{j} \rangle \mathbf{e} - \mathbf{m}_{i}) = \sum_{i=1}^{r} e_{i} n_{i} \langle \mathbf{m}_{i}, \widetilde{\boldsymbol{\eta}}_{j} \rangle \mathbf{e} - \sum_{i=1}^{r} e_{i} n_{i} \mathbf{m}_{i} \in m \widetilde{\mathbf{M}} \text{ and,}$$
$$\prod_{i=1}^{r} \overline{u}_{i}^{e_{i} \langle \mathbf{m}_{i}, \widetilde{\boldsymbol{\eta}}_{j} \rangle} \in (k^{*})^{m}.$$

Since  $\mathbf{m}_1, \ldots, \mathbf{m}_r, \mathbf{e}$  form a basis for  $\mathbf{M}$ , the first condition in (4.15) in is equivalent to requiring that

$$(4.16) m|e_i n_i \text{ for all } i = 1, \dots, r.$$

Moreover, under this assumption, we may assume that  $b_i = u_i \in \mathbb{R}^*$  for all  $i = 1, \ldots, r$ .

**Lemma 4.6.** Let  $u_i \in \mathbb{R}^*$  for  $i = 1, \dots, r$ . Then the condition that the algebra  $\Lambda = \prod_{i=1}^r (u_i, t_i)_{\zeta}^{e_i}$  is unramified along each of the rays  $\rho_i$ ,  $i = 1, \dots, \nu$  in  $\tilde{\Delta}(1)$  contained in  $\mathbb{N}_{\mathbb{R}} \times 0$  implies that  $\Lambda$  is also unramified along each of the rays  $\tau_j$ ,  $j = 1, \dots, \mu$  in  $\tilde{\Delta}(1)$  that are not contained in  $\mathbb{N}_{\mathbb{R}} \times 0$ .

*Proof.* The required argument is similar to the one used in the discussion centered around (4.10), but we provide it here for the readers' convenience. To see this, fix a ray  $\tau \in \tilde{\Delta}(1)$  that is not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$  and let  $\tilde{\boldsymbol{\eta}}$  denote the primitive vector along  $\tau$  in  $\tilde{\mathbf{N}}$ . Since  $\pi(\tilde{\boldsymbol{\eta}})$  lies in  $\mathbf{N}'$  (by our assumption in (4.9), there exist some integers  $\alpha_1, \ldots, \alpha_u$  such that  $\pi(\tilde{\boldsymbol{\eta}}) = \sum_{j=1}^u \alpha_j a_j \mathbf{n}_j$ . On the other hand, because  $\mathbf{m}_i \in \mathbf{M}$ , we have

(4.17) 
$$\langle \mathbf{m}_i, \widetilde{\boldsymbol{\eta}} \rangle = \langle \mathbf{m}_i, \pi(\widetilde{\boldsymbol{\eta}}) \rangle = \sum_{j=1}^u \langle \mathbf{m}_i, \alpha_j a_j \mathbf{n}_j \rangle = \alpha_i a_i \quad \text{for each } i = 1, \dots, r.$$

Now,  $\Lambda$  is unramified along  $\tau$  if and only if  $\prod_{i=1}^{r} \overline{u}_{i}^{e_{i} < \mathbf{m}_{i}, \widetilde{\eta} >} \in k^{*m}$ . But, using (4.17), we can write

$$\prod_{i=1}^{r} \overline{u}_{i}^{e_{i} < \mathbf{m}_{i}, \widetilde{\boldsymbol{\eta}} >} = \prod_{i=1}^{r} \overline{u}_{i}^{\alpha_{i}a_{i}e_{i}} = \prod_{i=1}^{r} \left(\overline{u}_{i}^{a_{i}e_{i}}\right)^{\alpha_{i}}.$$

Now, if  $u_i^{a_i e_i} \in \mathbf{K}^{*m}$ , then  $\overline{u}_i^{a_i e_i} \in k^{*m}$  and, thus,  $\prod_{i=1}^r \overline{u}_i^{e_i < \mathbf{m}_i, \widetilde{\eta} >} \in k^{*m}$  as well.

Let  $\mathcal{B}'$  denote the subgroup of  $\mathcal{B}$  generated by the algebras of the form  $\Lambda = \prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$ , as  $u_i$  varies among the units in R. The above discussion shows that for such an algebra  $\Lambda \in \mathcal{B}'$  to lie in the kernel of  $res_{\mathcal{B},\mathcal{D}}: \mathcal{B} \to \mathcal{D}$ , it is sufficient that it be unramified along the rays of  $\tilde{\Delta}$  that are contained in  $\mathbf{N}_{\mathbb{R}} \times 0$ .

**Proposition 4.7.** The subgroup  ${}_{m}Br(X) \cap \mathcal{B}$  is generated by the classes  $\Lambda = \prod_{i=1}^{r} (b_i, t_i)^{e_i}_{\zeta}$  satisfying the following conditions:

- (i)  $m|e_in_i$ , for each  $i = 1, \dots, r$ , where  $b_i = u_i \pi^{n_i}$ ,  $u_i \in \mathbb{R}^*$ , and
- (ii)  $b_i \in \mathbf{K}^*$ , i = 1, ..., r, and  $0 \le e_i < m$ , so that for each s = 1, ..., u, if  $m_s = hcf\{m, e_s, ord_m(u_s)\}$ , then  $(\frac{ord_m(u_s)}{m_s})|a_s$ , and where  $ord_m(u)$  denotes the order of the image of u in  $\mathbf{K}^*/(\mathbf{K}^*)^m$ .

Proof. Clearly  $\ell^n \operatorname{Br}(X) \cap \mathcal{B} = \operatorname{kernel}(\operatorname{res}_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}) \cap \operatorname{kernel}(\operatorname{res}_{\mathcal{B},\mathcal{D}} : \mathcal{B} \to \mathcal{D})$ . Next observe that the condition in (4.16) (which is the condition (i)) implies the condition in (4.12). Observe that (4.13), (4.14) and (4.15) show that the condition (4.16) is necessary for  $\Lambda$  to be in the kernel of the residue map  $\operatorname{res}_{\mathcal{B},\mathcal{D}} : \mathcal{B} \to \mathcal{D}$ . Therefore, the discussion centered around (4.11) and (4.12) shows one may reduce to considering algebras of the form  $\Lambda = \prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$ , with  $u_i \in \mathbb{R}^*$ .

Moreover, we just showed that an algebra of the form  $\Lambda = \prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$  is unramified along the rays  $\tau_j$ ,  $j = 1, \dots, m$ , if it is unramified along the rays  $\rho_i$ ,  $i = 1, \dots, \nu$ : clearly the latter condition is equivalent to  $\Lambda = \prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$  being in the kernel of the residue map  $res_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}$ . Finally an algebra of the form  $\Lambda = \prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$  is in the kernel of the residue map  $res_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}$ . Finally an algebra of the form that for each  $0 \leq e_i < m$ , so that for each  $s = 1, \dots, u$ , if  $m_s = hcf\{m, e_s, ord_m(u_s)\}$ , then  $(\frac{ord_m(u_s)}{m_s})|a_s$ , and where  $ord_m(u)$  denotes the order of the image of u in  $\mathbb{R}^*/(\mathbb{R}^*)^m$ . Observe that for an element  $c \in \mathbb{K}^*$  ( $\mathbb{R}^*$ ),  $ord_m(c)$  denotes the order of its in image in  $\mathbb{K}^*/(\mathbb{K}^*)^m$  (in  $\mathbb{R}^*/(\mathbb{R}^*)^m$ , respectively). This completes the proof of the proposition.

Computation of  $kernel(res|_{mBr(Spec K)})$ . Finally, we need to determine the kernel of the map *res* restricted to  ${}_{m}Br(Spec K) \subseteq {}_{m}Br(T_{K})$ . Again we break this into two steps: (i)  $kernel(res : {}_{m}Br(Spec K) \rightarrow C)$  and (ii)  $kernel(res : {}_{m}Br(Spec K) \rightarrow D)$ .

Since the rays  $\rho_i$ ,  $i = 1, \dots, \nu$  are all contained in  $\mathbf{N}_{\mathbb{R}} \times \mathbf{0}$ , these rays correspond to the rays in the fan of the toric variety  $X_K$ . Therefore, it is clear that the residue map  $res : {}_m Br(Spec K) \to \mathcal{C}$  is trivial proving  ${}_m Br(Spec K) = kernel(res : {}_m Br(Spec K) \to \mathcal{C}).$ 

Next we determine the kernel of the residue map  $res : {}_{m}Br(Spec K) \to \mathcal{D}$ . For that, we begin with the commutative diagram:

The left-most vertical map is induced by the structure map  $X \to \operatorname{Spec} R$  and the next vertical map is induced by the structure map  $\pi : T_K \to \operatorname{Spec} K$ . The exactness of the top row is shown in [CTS, Theorem 3.6.1]. That the diagram commutes follows from Proposition 2.9(iii). One may now observe from [Qu, Proposition 2.1.3] that the cones of the fan not contained in  $\mathbf{N}_{\mathbb{R}} \times 0$  correspond to the orbits of the torus  $T_k$ : since  $T_k$  is abelian, these correspond to (possibly) smaller dimensional tori over k. Hence it follows that the special fiber  $X_k$  has a k-rational point, so that the last vertical map in (4.18) is injective. Observe that this map, maps into the summand  $\mathcal{D}$ . Therefore, the kernel of the composition of the the middle vertical map followed by the residue map  $res : {}_m \operatorname{Br}(T_K) \to \mathcal{D}$  equals the kernel of the top map  $res : {}_m \operatorname{Br}(\operatorname{Spec} K) \to \operatorname{H}^1_{\operatorname{et}}(\operatorname{Spec} k, \mu_m(0))$ : clearly this is  ${}_m \operatorname{Br}(\operatorname{Spec} R)$  by the exactness of the top row. These show that  ${}_m \operatorname{Br}(X) \cap {}_m \operatorname{Br}(\operatorname{Spec} K) = {}_m \operatorname{Br}(\operatorname{Spec} R)$ .

Therefore, summarizing the above discussion, we obtain the following theorem:

**Theorem 4.8.** Assume that R is a discrete valuation ring with field of fractions K and residue field k. Assume m is a positive integer invertible in R and that R contains a primitive m-th root of unity  $\zeta$ . Let X denote a regular toric scheme over Spec R in the sense of [KKMSD, Chapter 4, section 3] and satisfying the assumption (4.9). Let  $a_1|a_2|\cdots|a_u$  denote the invariant factors of the given fan as in (4.5). Then

$${}_{m}\mathrm{Br}(\mathrm{X}) \cong {}_{m}\mathrm{Br}(\mathrm{Spec}\,\mathrm{R}) \oplus ({}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A}) \oplus ({}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B})$$

where  $(_m Br(X) \cap \mathcal{A})$  is the subgroup generated by  $\{\Lambda = \prod_{i < j} (t_i, t_j)_{\zeta}^{e_{i,j}} | m > e_{i,j} \ge 0\}$  satisfying the following conditions: for each  $s = 1, \ldots, min\{u, r-1\}$ , if  $m_s = hcf\{m, e_{1,s}, e_{2,s}, \cdots, e_{s-1,s}, e_{s,s+1}, \cdots, e_{s,r}\}$ , then  $(\frac{m}{m_s})|a_s$ .

(ii)  ${}_{m}Br(X) \cap \mathcal{B}$  is the subgroup generated by

 $\{\Lambda = \Pi_{i=1}^{r}(b_{i}, t_{i})_{\zeta}^{e_{i}} | b_{i} \in \mathbf{K}^{*}, i =, \cdots, \mathbf{r}, m > \mathbf{e}_{i} \ge 0, m | e_{i}n_{i}, i = 1, \cdots, r, b_{i} = u_{i}\pi^{n_{i}}, u_{i} \in \mathbf{R}^{*}\}$ 

so that for each  $s = 1, \dots, u$ , if  $m_s = hcf\{m, e_s, ord_m(b_s)\}$ , then  $(\frac{ord_m(b_s)}{m_s})|a_s|$  where  $ord_m(b)$  denotes the order of the image of b in  $K^*/(K^*)^m$ .

- (iii) Moreover,  ${}_{m}Br(X) \cap \mathcal{A} \cong Hom_{(Ab.groups)}(\wedge^{2}(N/N'), \mathbb{Z}/m\mathbb{Z}).$
- (iv) In view of the condition  $m|e_in_i$  in (ii), the class of the cyclic algebra  $(b_i, t_i)^{e_i}_{\zeta}$  identifies with the class of the cylic algebra  $(u_i, t_i)^{e_i}_{\zeta}$ , with  $b_i = u_i \pi^{n_i}$ . Therefore,  ${}_m Br(X) \cap \mathcal{B}$  also identifies with  $Hom_{(Ab,groups)}(\mathbf{N}/\mathbf{N}', \mathbb{R}^*/(\mathbb{R}^*)^m)$ .

## Corollary 4.9. Assume the hypotheses of Theorem 4.8.

In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension $(\sigma) \geq r-1$ , or if u = r-1, and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, u$ , then  ${}_m Br(X) \cap \mathcal{A}$  is trivial.

In case there is a cone  $\sigma$  in the fan  $\Delta$  with dimension $(\sigma) \geq r$ , or if u = r and the invariant factors  $a_i = 1$ , for all  $i = 1, \dots, u$ , then both  ${}_m Br(X) \cap \mathcal{B}$  and  ${}_m Br(X) \cap \mathcal{A}$  are trivial, so that  ${}_m Br(X) \cong {}_m Br(Spec R)$ .

Proof. The proof of the first statement is identical to the proof of the first statement in Corollary 4.2 or Corollary 3.3. Therefore, we proceed to the proof of the second statement. In view of the condition in Theorem 4.8(ii) that  $m|e_in_i$ , for all  $i = 1, \dots, r$ , the arguments as in (4.11) and (4.12) show we reduce to considering cyclic algebras of the form  $\prod_{i=1}^{r} (u_i, t_i)_{\zeta}^{e_i}$ , where each  $u_i \in \mathbb{R}^*$ . Since each  $a_s = 1$ , for all  $s = 1, \dots, u = r$ , it follows from condition (ii) in Theorem 4.8 that  $\frac{ord_m(u_s)}{m_s}|a_s = 1$ , for each  $s = 1, \dots, r$ , so that  $m_s = ord_m(u_s)$ . Since  $m_s|e_s$ , for each  $s = 1, \dots, r$ , it follows readily that the class of each  $(u_s, t_s)_{\zeta}^{e_s}$  is trivial. This proves the second statement.

**Remark 4.10.** The following facts may be worth pointing out: the detailed analysis above shows that the kernels of the residue maps  $res : \mathcal{A} \to \mathcal{C}$  and  $res : \mathcal{B} \to \mathcal{C}$  are always the same in Theorems 3.2, 4.1 and 4.8. Thus the main possible difference in Theorem 4.8 is in the computations of the kernels of the residue maps  $res : \mathcal{A} \to \mathcal{D}$  and  $res : \mathcal{B} \to \mathcal{D}$ .

The simplifying assumption (4.9), then ensures that the kernel of  $res : \mathcal{A} \to \mathcal{D}$  is contained in the kernel of  $res : \mathcal{A} \to \mathcal{C}$  and similarly that the kernel of  $res : \mathcal{B} \to \mathcal{D}$  is contained in the kernel of  $res : \mathcal{B} \to \mathcal{C}$ . However, considerable efforts are needed to show this as the detailed analysis in the proof of Theorem 4.8 shows. Thus the computation of the *m*-torsion part of the Brauer group of the toric scheme X over the DVR R roughly proceeds as in Theorems 3.2 and 4.1, under the simplifying hypothesis (4.9).

## 5. The period-index problem for toric varieties

This section is devoted to a discussion of the period-index problem for smooth toric varieties, beginning with tori. Our interest in this problem started with the paper of Hotchkiss (see [Hotch]) where he disproved the period-index conjecture for complex analytic tori. See [dJ04] and [Lie08] for basic background on this problem.

The starting point of the discussion in this section is the calculation of the Brauer group of a split torus as in Corollary 2.4. We will assume throughout the following discussion that the base field K has a primitive m-th root of unity, for a fixed positive integer m. (Clearly such an assumption is satisfied when the field is algebraically closed, but in the second half of this section, we will also consider base fields that are not necessarily algebraically closed.) We begin with the following preliminary results.

**Proposition 5.1.** Let T denote a torus of dimension n over the algebraically closed field K. Let  $\zeta$  denote a primitive m-root of unity in K, where m is prime to the characteristic of K. Let  $\beta_i = \prod_{j=i+1}^n (t_i, t_j)_{\zeta}^{a_{i,j}}$ . Then the smallest degree of the extension field over K in which  $\beta_i$  splits is  $m/u_i$ , where  $u_i = hcf\{a_{i,i+1}, \dots, a_{i,n}, m\}$ .

*Proof.* Let  $b_{i,j}$  denote positive integers chosen so that  $a_{i,j} = u_i b_{i,j}$ , for  $j = i + 1, \dots, n$ . Therefore,

$$a_{i,j}(m/u_i) = (u_i b_{i,j})(m/u_i) = b_{i,j}m,$$

so that

$$\beta_i^{m/u_i} = \prod_{j=i+1}^n (t_i, t_j)_{\zeta}^{b_{i,j}m}.$$

This proves that the cyclic algebra  $\beta_i$  splits in a field extension of degree  $m/u_i$ .

Suppose  $\beta_i$  splits in a field extension of degree q. Then  $m|a_{i,j}q, j = i + 1, \dots, n$ . Let  $m = m'u_i$ , and  $a_{ij} = u_i b'_{i,j} v'_i$  where  $v'_i$  is chosen as in the following lemma. Therefore, the condition  $m|a_{i,j}q, j = i + 1, \dots, n$  implies  $m'|b_{i,j}v'_iq, j = i + 1, \dots, n$ . Now the following lemma shows that  $hcf\{m', b'_{i,i+1}v'_i, \dots, b'_{i,n}v'_i\} = 1$ . Therefore, it follows that m'|q. This proves that  $m' = m/u_i$  is the smallest degree of the extension field in which  $\beta_i$  splits.

**Lemma 5.2.** Let  $u_i$  be as in Proposition 5.1,  $v_i = hcf\{a_{i,i+1}, \dots, a_{i,n}\} = u_iv'_i$ ,  $m' = \frac{m}{u_i}$  and  $b'_{i,j} = \frac{a_{i,j}}{v_i}$ ,  $j = i + 1, \dots, n$ . Then the following hold:

- (i)  $hcf\{b'_{i,i+1}, \cdots, b'_{i,n}, m'\} = 1$ , and,
- (ii)  $hcf\{m', v'_i\} = 1.$

*Proof.* Let  $w = hcf\{b'_{i,i+1}, \cdots, b'_{i,n}, m'\}$ . Then  $b'_{i,j} = c_{i,j}w$  and

$$a_{i,j} = b'_{i,j} \cdot v_i = c_{i,j} w v_i, v_i = u_i v'_i, j = i + 1, \cdots, n.$$

Therefore,  $wu_i|a_{i,j}$ ,  $j = i + 1, \dots, n$ . Since w|m', it follows  $wu_i|m'u_i = m$ . Thus  $wu_i|u_i$ , where  $u_i$  in fact is the  $hcf\{a_{i,i+1}, \dots, a_{i,n}, m\}$ . It follows that w = 1, proving the first statement.

Let  $w' = hcf\{m', v'_i\}$ . Then m' = w'm'' and  $v'_i = w'v''_i$ . The last equality shows  $w'|v'_i$  and hence  $w'|v_i$ . Recall  $a_{i,j} = b'_{i,j}v_i = b'_{i,j}v'_iu_i$ . Since  $w'|v'_i$ , it follows that  $u_iw'|a_{i,j}$ , for all  $j = i + 1, \dots, n$ . Since  $m = m'u_i = w'm''u_i$ , it follows that  $u_iw'|m$  as well. Therefore,  $u_iw'|u_i$ , and hence w' = 1, thereby proving the second statement.

The following is our main result on the period-index problem for tori over algebraically closed fields.

**Theorem 5.3.** Let K denote an algebraically closed field and let T denote an n-dimensional torus over K.

(i) Assume char(K) = 0. For each positive integer m, the m-torsion part of the Brauer group of T, namely  ${}_{m}Br(T)$  is the free  $\mathbb{Z}/m\mathbb{Z}$ -module with basis given by the  $\binom{n}{2}$  cyclic algebras  $(t_{i}, t_{j})_{\zeta}$ ,  $1 \leq i < j \leq n$ , where  $\zeta$  is an m-th root of unity. Therefore, if  $\alpha = \prod_{i < j} (t_{i}, t_{j})_{\zeta}^{a_{i,j}}$ ,  $a_{i,j} \geq 0$ , and with  $u = hcf\{a_{i,j}, m | i, j\}$ , the order of  $\alpha$  in  ${}_{m}Br(T)$  is  $m' = \frac{m}{u}$ .

- (ii) Index( $\alpha$ ) may be calculated as follows. For  $i = 1, \dots, n-1$ , let  $u_i = hcf\{a_{i,i+1}, \dots, a_{i,n}, m\}$ . Then the index of  $\prod_{j=i+1}^{n} (t_i, t_j)_{\zeta}^{a_{ij}}$  is given by  $\frac{m}{u_i}$ . Therefore, the index of  $\alpha = \prod_{i=1}^{n-1} (\frac{m}{u_i}) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}}$ . Consequently Index( $\alpha$ )|Period( $\alpha$ )<sup>n-1</sup>.
- (iii) Assume char(K) = p > 0. For each positive integer m relatively prime to p, the same conclusions as in (i) hold.

Proof. We first prove that the period of  $\alpha$  is m' = m/u. To see this first observe that  $a_{i,j} = c_{i,j}u$  for some sequence of integers  $c_{i,j}$ . Therefore,  $a_{i,j}m' = c_{i,j}um' = c_{i,j}u(m/u) = c_{ij}m$ . Therefore,  $\alpha^m = *$ . Next we proceed to show that if p is an integer so that  $\alpha^p = *$ , then m'|p, which will complete the proof that the period of  $\alpha$  is m'. Therefore, let p denote an integer so that  $\alpha^p = *$ . Then  $m|a_{i,j}p$ , for all i, j. Writing  $a_{i,j} = c_{i,j}u$ , this means

$$m = m'u|a_{i,j}p = c_{ij}up$$
, for all  $i < j$ .

Therefore,  $m'|c_{i,j}p$ , for all i < j. Once we show that  $1 = hcf\{m', c_{i,j}|i < j\}$ , it will follow that m'|p proving the period of  $\alpha$  is m'.

Therefore, let  $hcf\{m', c_{i,j} | i < j\} = w$ . Then m' = wm'', for some m'' and  $c_{i,j} = d_{ij}w$ . Therefore,

$$m = m'u = wm''u$$
, and  $c_{i,j} = d_{i,j}u$ 

for some integers  $d_{i,j}$ . Therefore,  $a_{i,j} = c_{i,j}u = d_{i,j}wu$ . Thus  $wu|a_{i,j}$  for all i < j and wu|m. Therefore, wu|u, since  $u = hcf\{m, a_{i,j}|i < j\}$ . This readily implies w = 1. These complete the proof of (i) in view of Corollary 2.4.

We list the 
$$\binom{n}{2}$$
-factors  $(t_i, t_j)_{\zeta}^{a_{i,j}}$ ,  $i < j$ , in matrix form  

$$\begin{pmatrix} (t_1, t_2)_{\zeta}^{a_{1,2}} & (t_1, t_3)_{\zeta}^{a_{2,3}} & \cdots & (t_1, t_n)_{\zeta}^{a_{1,n}} \\ 1 & (t_2, t_3)_{\zeta}^{a_{2,3}} & \cdots & (t_2, t_n)_{\zeta}^{a_{2,n}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & & (t_{n-1}, t_n)_{\zeta}^{a_{n-1,j}} \end{pmatrix}$$

Now we invoke Proposition 5.1 to the product of the cyclic algebras in each row. We observe then that product of the (n-1)-factors in the first row is split by adding an  $m/u_1$ -th root of  $t_1$ , the product of the (n-2) factors in the second row is split by adding an  $m/u_2$ -th root of  $t_2$ , and so on, with just the one factor  $(t_{n-1}, t_n)_{\zeta}^{a_{n-1,n}}$  in the n-1-th row which is split by adding an  $m/u_{n-1}$ -th root of  $t_{n-1}$ , so that all these terms together account for the  $\binom{n}{2}$ -factors  $(t_i, t_j)_{\zeta}^{a_{i,j}}$  in  $\alpha$ . Therefore,

$$Index(\alpha) = \prod_{i=1}^{n-1} (\frac{m}{u_i}) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}}$$

Since  $u|u_i$ , for all  $i = 1, \dots, n-1$ , it follows that  $u^{n-1}|\prod_{i=1}^{n-1} u_i$  and therefore

$$Index(\alpha) = \frac{m^{n-1}}{u_1 \cdots u_{n-1}} |\frac{m^{n-1}}{u^{n-1}} = Period(\alpha)^{n-1},$$

thereby completing the proof of (ii). The proof of (iii) is similar and is therefore skipped.

**Corollary 5.4.** Let X denote a smooth toric variety of dimension n over an algebraically closed field K. Then for each integer m relatively prime to char(K), and any class  $\alpha \in {}_{m}Br(X)$ ,

$$Index(\alpha)|Period(\alpha)^{n-1}.$$

*Proof.* This is clear from Theorem 5.3 in view of corollary 2.4.

**Remark 5.5.** One may observe that the recent result of Hotchkiss (see [Hotch]) shows that the period-index conjecture is false for complex analytic tori, by showing the existence of certain classes in the Brauer group of a complex analytic torus of rank r at least 3, for which the index does not divide period<sup>r-1</sup>. Therefore, the results of Theorem 5.3 show that such counter-examples do not show up for algebraic tori.

Next we proceed to extend our results to toric varieties defined over fields of cohomological dimension 1, which includes fields that are quasi-algebraically closed and finite fields. Let K denote a field of cohomological dimension 1 and let T denote an *n*-dimensional split torus over K. We will assume K contains a primitive *m*-th root of unity. Corollary 2.4 shows that the *m*-torsion part of the Brauer group  $Br(T)_m$  has two main components: (i) the part coming from cyclic algebras of the form  $(t_i, t_j)^{a_{i,j}}_{\zeta}$  as  $t_i, t_j$  run over characters of the given split torus T and (ii) the part coming from cyclic algebras of the form  $(b, t_i)^{a_i}_{\zeta}$ , as  $b \in K^*$  and  $t_i$  denotes a character of the torus T.

## **Theorem 5.6.** (i) Assume char(K) = 0. Let S denote a finite subset of $K^*$ so that the map

 $coker(\Gamma(\operatorname{Spec} \mathrm{K}, \mathbb{G}_{\mathrm{m}}) \xrightarrow{m} \Gamma(\operatorname{Spec} \mathrm{K}, \mathbb{G}_{\mathrm{m}})) \xrightarrow{\delta} \mathrm{H}^{1}_{\mathrm{et}}(\operatorname{Spec} \mathrm{K}, \mu_{m})$ 

sends the set S injectively into the nontrivial elements in  $\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K},\mu_{m})$ . If  $\alpha = \Pi_{b\epsilon\mathrm{S}}\Pi^{n}_{i=1}(b,t_{i})^{a_{i}}_{\zeta}$ ,  $a_{i} \geq 0$ , and with  $u = hcf\{a_{i}, m | i = 1, \cdots, n\}$ , the order of  $\alpha$  in  ${}_{m}\mathrm{Br}(\mathrm{T})$  is  $m' = \frac{m}{u}$ .

- (ii) Let S denote a finite subset of K<sup>\*</sup> and let  $\alpha$  denote a class as in (i). Assume that  $|S| \ge n$ . For  $i = 1, \dots, n-1$ , let  $u_i = hcf\{a_i, m\}$ . Then the index of  $\alpha_i = \prod_{b \in S} (b, t_i)_{\zeta}^{a_i}$  is given by  $\frac{m}{u_i}$ . Let  $\alpha = \prod_{i=1}^n \alpha_i = \prod_{i=1}^n \prod_{b \in S} (b, t_i)_{\zeta}^{a_i}$ . Then  $Index(\alpha) = \prod_{i=1}^n (\frac{m}{u_i}) = \frac{m^n}{u_1 \cdots u_n}$ . Consequently  $Index(\alpha) | Period(\alpha)^n$ .
- (iii) Assume char(K) = p > 0, let m be prime to char(K) and that K contains a primitive m-th root of unity. Then the same conclusions as in (i) and (ii) hold for classes in mBr(T).

Proof. We first observe that the period of the cyclic algebra  $(b, t_i)_{\zeta}$  is m: this may be proven exactly as in [Mag, Proof of Lemma 5]. We will next prove that the period of  $\alpha$  is m/u. This will be a variant of the proof of the first statement in Theorem 5.3. To see this first observe that  $a_i = c_i u$  for some sequence of integers  $c_i$ . Therefore,  $a_i m' = c_i u m' = c_i u (m/u) = c_i m$ . Therefore,  $\alpha^m = *$ . Next we proceed to show that if p is an integer so that  $\alpha^p = *$ , then m'|p, which will complete the proof that the period of  $\alpha$  is m'. Therefore, let p denote an integer so that  $\alpha^p = *$ . Then  $m|a_ip$ , for all i, since the period of the cyclic algebra  $(b, t_i)_{\zeta}$  is m. Writing  $a_i = c_i u$ , this means

$$m = m'u|a_ip = c_iup$$
, for all  $i = 1, \cdots, n$ .

Therefore,  $m'|c_ip$ , for all  $i = 1, \dots, n$ . Once we show that  $1 = hcf\{m', c_i | i = 1, \dots, n\}$ , it will follow that m'|p proving the period of  $\alpha$  is m'.

Therefore, let  $hcf\{m', c_i | i = 1, \dots, n\} = w$ . Then m' = wm'', for some m'' and  $c_i = d_i w$ . Therefore,

$$m = m'u = wm''u$$
, and  $c_i = d_iw$ 

for some integers  $d_i$ . Therefore,  $a_i = c_i u = d_i w u$ . Thus  $wu|a_i$  for all  $i = 1, \dots, n$  and wu|m. Therefore, wu|u, since  $u = hcf\{m, a_i|i = 1, \dots, n\}$ . This readily implies w = 1. These complete the proof of (i) in view of Corollary 4.2.

It should be clear from the above discussion that the index of  $\alpha_i = m/u_i$ , for each  $i = 1, \dots, n$ . Therefore, the index of  $\alpha$  is  $\prod_{i=1}^n \frac{m}{u_i} = \frac{m^n}{u_1 \cdots u_n}$ . Since  $u|u_i$ , for each  $i = 1, \dots, n$ , it follows that  $u^n|u_1 \cdots u_n$ , proving that  $Index(\alpha)|Perod(\alpha)^n = (\frac{m}{u})^n = \frac{m^n}{u^n}$ . We skip the proof of the last statement which should be clear.  $\Box$ 

**Corollary 5.7.** Assume the base field K is the function field of a curve over an algebraically closed field. Let X denote a smooth split toric variety over K of dimension n. Then for each integer m relatively prime to char(K), and any class  $\alpha \in {}_{m}Br(X)$ ,

$$Index(\alpha)|Period(\alpha)^n$$
.

*Proof.* First one reduces to the case of a split torus over K. Let C denote the given affine curve over the algebraically closed field k. Let  $U = C - \{x_i | i = 1, \dots, n\}$  and  $X = \{x_i | i = 1, \dots, n\}$ . Then the localization sequence

$$0 \to \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{C},\mu_{m}) \to \mathrm{H}^{1}_{\mathrm{et}}(\mathrm{U},\mu_{m}) \to \mathrm{H}^{2}_{\mathrm{et},\mathrm{X}}(\mathrm{C},\mu_{m}) \to \mathrm{H}^{2}_{\mathrm{et}}(\mathrm{C},\mu_{m}) \cong 0$$

and the isomorphism  $\mathrm{H}^{2}_{\mathrm{et},\mathrm{X}}(\mathrm{C},\mu_{m}) \cong \bigoplus_{i=1}^{n} \mathrm{H}^{0}_{\mathrm{et}}(\mathrm{x}_{i},\mu_{m}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/\mathrm{m}$  show that on taking  $n \to \infty$ ,

 $\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Spec}\,\mathrm{K}(\mathrm{C}),\mu_{m})$  is of infinite rank as a free  $\mathbb{Z}/m$ -module. Now the statement follows from corollary 2.4, Theorem 5.6 and the observation that any algebraically closed field of characteristic p contains primitive m-th roots of unity for all m prime to p.

**Corollary 5.8.** Assume the base field is the finite field  $\mathbb{F}_q$  of characteristic p and let X denote a split smooth toric variety over  $\mathbb{F}_q$  of dimension n. Let m|q-1. If  $\alpha \in {}_m Br(X)$ ,

$$Index(\alpha)|Period(\alpha)^{n-1}.$$

*Proof.* Here the key observation is that  $\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{F}_{q},\mu_{m}) = \mathbb{Z}/\mathrm{m}$ . Therefore, the period and index of the class  $\alpha = \prod_{i=1}^{n} (b,t_{i})_{\zeta}$  is m, where  $b \in \mathbb{F}_{q}^{*}$  corresponds to the generator of  $\mathrm{H}^{1}_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{F}_{q},\mu_{m})$ . The conclusion now follows from Theorems 5.3 and 5.6 once one observes that  $\mathbb{F}_{q}$  contains a primitive m-th root of unity.  $\Box$ 

## 6. Examples

We will assume throughout this section that  $H^1_{et}(\operatorname{Spec} K, \mu_m(0))$  is not trivial.

We will consider several examples in detail, all of which will be given by non-projective toric schemes. The reason for restricting to non-projective schemes is so that there is no likely shortcut to obtaining the results we obtain, and one is forced to invoke the results in Theorems 3.2, 4.1 and 4.8 to draw the conclusions we arrive at in these examples.<sup>7</sup>

We will start with an affine toric variety or scheme, often defined by a single cone, which has singularities, and then produce a regular toric variety or scheme by suitably subdividing the given fan (or cone). We will view the resulting fan as sitting in different lattices and proceed to interpret the results on the Brauer groups of the resulting toric varieties and schemes in the context of our main theorems, Theorems 3.2, 4.1 and 4.8.

**Example 6.1.** (i) Let R denote a commutative Noetherian ring with 1 as in 1.1 and let  $\{\mathbf{e}_1, \mathbf{e}_2\}$  form the standard basis for the free R-module R<sup>2</sup>. Consider the affine toric scheme over R defined by the cone  $\sigma = cone(\mathbf{e}_2, 3\mathbf{e}_1 - 2\mathbf{e}_2)$ . One can the subdivide the fan by introducing two additional cones  $\sigma_1 = cone(\mathbf{e}_2, \mathbf{e}_1)$ ,

<sup>&</sup>lt;sup>7</sup>For projective schemes over a field that are regular, one could use the fact that the Brauer group is a birational invariant (see [CTS, Theorem 6.2.7]) to short-circuit some of these calculations.

 $\sigma_2 = cone(\mathbf{e}_1, 2\mathbf{e}_1 - \mathbf{e}_2)$ , and  $\sigma_3 = cone(2\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 2\mathbf{e}_2)$ . Let X denote the resulting toric scheme. One may readily see that all the cones  $\sigma_i$ , i = 1, 2, 3 are smooth cones, and therefore the toric schemes corresponding to each  $\sigma_i$  is the corresponding affine space  $\mathbb{A}^2_{\mathbb{R}}$ . The above example may be found in [Ful, 2.6], when the base is Spec  $\mathbb{C}$ .

Now Corollary 3.3 (where R = K is the base field) will show that  ${}_{m}Br(X) \cong {}_{m}Br(Spec K)$  since all the cones in the fan are of rank 2, which is also the rank of the lattice **N**.

(ii) Next we will view the same fan as sitting in a lattice **N** of rank r >> 0. This corresponds to assuming that the toric scheme is of dimension r, while the fan is still of dimension 2.

In this case, if we assume  $r \geq 4$ ,

 ${}_{m}Br(X) = {}_{m}Br(Spec K) \oplus {}_{m}Br(X) \cap \mathcal{A} \oplus {}_{m}Br(X) \cap \mathcal{B}$  with

 ${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} \cong \mathrm{Hom}_{\mathrm{Ab},\mathrm{groups}}(\Lambda^{2}(\mathbf{N}/\mathbf{N}^{2}), \mathbb{Z}/\mathrm{m}\mathbb{Z}) \text{ and } {}_{\mathrm{m}}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} \cong \mathrm{Hom}_{\mathrm{Ab},\mathrm{groups}}(\mathbf{N}/\mathbf{N}', \mathrm{K}^{*}/\mathrm{K}^{*\,m}).$ 

In particular, by the explicit computation in Theorem 3.2(i) and (ii) in terms of cyclic algebras, this means:

(6.1) 
$${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} = \text{ the subgroup generated by } \{\Lambda = \Pi_{i < j}(t_i, t_j)_{\zeta}^{e_{i,j}} | |i = 3, \cdots r\}, \text{ and}$$
  
 ${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} = \text{ is generated by } \{\Lambda(b) = \Pi_{i=1}^{r}(b_i, t_i)_{\zeta}^{e_i} | |i = 3, \cdots, r\}$ 

as  $b_i \in \mathbf{K}^*$  varies among the corresponding classes in  $\mathbf{H}^1_{\text{et}}(\text{SpecK}, \mu_m(0))$ .

In case r = 3, the term  $Br(X) \cap \mathcal{A} = \{0\}$ .

(iii) We will now reconsider the same example in the context of Theorem 4.1. In the context of Theorem 4.1, we will assume that the base scheme is an excellent Dedekind domain R and that the simplifying assumption (4.1) holds. Recall that this implies all the toric orbits on the toric scheme X and the toric scheme X are flat over Spec R. Then Theorem 4.1 shows that the same conclusions as above hold with the base field K replaced by the ring R. For example, if the Dedekind domain  $R = \mathcal{O}_K$  is the ring of integers in a number field K, then one knows  $Br(Spec R) \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$  if K has r real places and  $Br(R) \cong 0$  otherwise. (See, for example, [Y, p. 6].) Therefore, in particular,  ${}_mBr(Spec R) \cong 0$ , if  $m \neq 2$ , or if K has no real places, and  ${}_mBr(Spec R) \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$  if K has r real places and m = 2. Now Theorem 4.1 and Corollary 4.2 provide a computation of any toric scheme that is smooth over such a ring of integers and satisfying the simplifying assumption 4.1 and where m is invertible in  $\mathcal{O}_K$  and  $\mathcal{O}_K$  has a primitive m-root of unity.

(iv) Finally, we will also consider the same example in the context of Theorem 4.8. Here the base scheme is the spectrum of an excellent DVR, with residue field k and the toric scheme X is defined as in [KKMSD] or [GPS]. Now Theorem 4.8 and Corollary 4.9 show the corresponding conclusions hold under the simplifying assumption (4.9).

**Example 6.2.** Let R denote a field, an excellent Dedekind domain or a DVR as in 1.1 and let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  form the standard basis for the free R-module R<sup>3</sup>. Let  $\sigma = cone(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3)$ . This is an affine toric 3-fold and has a resolution of singularities by subdividing the cone as indicated in *op. cit.* The resulting fan has several cones, all of which are smooth cones: one knows for example, that subdividing the given cone into the two cones  $\sigma_1 = cone(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3)$  and  $\sigma_2 = cone(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3)$  will resolve the singularity.

(See [Pop, p. 62]. The above example may be also found in [CLS, 11.1.12], again when the base is Spec  $\mathbb{C}$ . We will consider the resulting regular toric scheme henceforth and denote it by X.

(i) Observe that this toric scheme is of dimension 3 over Spec R and we view its fan in a 3-dimensional lattice.

Now Theorems 3.2, 4.1 (with the simplifying assumption (4.1)) and 4.8 (with the simplifying assumption (4.9)) along with Corollaries 3.3, 4.2 and 4.9 will show that  ${}_{m}Br(X) \cong {}_{m}Br(Spec R)$ , since all the cones in the fan are of rank 3, which is also the rank of the lattice **N**.

(ii) Next we will view the same fan as sitting in a lattice **N** of rank r >> 0 under the simplifying assumption (4.1) or (4.9).

In this case, if we assume  $r \geq 5$ ,

$$_{m}Br(X) = {}_{m}Br(Spec R) \oplus {}_{m}Br(X) \cap \mathcal{A} \oplus {}_{m}Br(X) \cap \mathcal{B}$$
 with

 ${}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{A} \cong \mathrm{Hom}_{\mathrm{Ab},\mathrm{groups}}(\Lambda^{2}(\mathbf{N}/\mathbf{N}^{2}), \mathbb{Z}/\mathrm{m}\mathbb{Z}) \text{ and } {}_{m}\mathrm{Br}(\mathrm{X}) \cap \mathcal{B} \cong \mathrm{Hom}_{\mathrm{Ab},\mathrm{groups}}(\mathbf{N}/\mathbf{N}', \mathrm{R}^{*}/\mathrm{R}^{*m}).$ 

In particular, by the explicit computation in Theorems 3.2, 4.1 and 4.8(i) and (ii), this means:

- (i)  ${}_{m}Br(X) \cap \mathcal{A} = \text{the subgroup generated by } \{\Lambda = \prod_{i < j} (t_i, t_j)_{\zeta}^{e_{i,j}} | i = 4, \cdots r\}, \text{ and }$
- (ii)  ${}_{m}\operatorname{Br}(\mathbf{X}) \cap \mathcal{B} = \text{is generated by } \{\Lambda(b) = \prod_{i=1}^{r} (b_i, t_i)_{\zeta}^{e_i} | |i = 4, \cdots, r\} \text{ as } b_i \in \mathbf{R}^* \text{ varies among the corresponding classes in } \mathbf{H}^1_{\operatorname{et}}(\operatorname{Spec} \mathbf{R}, \mu_m(0)).$

In case r = 4, the term  $Br(X) \cap \mathcal{A} = \{0\}$ .

**Example 6.3.** We will next discuss the simplifying assumption (4.9).

We will view the fan for the toric scheme we are considering as sitting in the lattice  $\mathbf{N} = \mathbb{Z}^4$ , with basis  $\mathbf{e}_i$ ,  $i = 1, \dots, 4$ . We will further assume that the sublattice  $\mathbf{N}'$  generated by the primitive vectors along the rays of the fan of the given toric scheme is spanned by  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . In fact, one may take the toric scheme considered in Example 6.1, with the ambient lattice containing the fan of rank 4: clearly it meets all the above conditions.

Let **e** denote the extra basis vector for  $\mathbf{N}$ , in addition to the basis for **N**. Now consider the situation where the special fiber is a toric scheme with fan given by a cone spanned by  $\mathbf{v}_1 = (\mathbf{e}_3, 1)$  (=  $\mathbf{e}_3 + \mathbf{e}$ ) and  $\mathbf{v}_2 = (\mathbf{e}_2 + \mathbf{e}_3, 1)$  (=  $\mathbf{e}_2 + \mathbf{e}_3 \oplus \mathbf{e}$ ). Now the projections  $\pi(\mathbf{v}_1) = \mathbf{e}_3$ , and  $\pi(\mathbf{v}_4) = \mathbf{e}_2 + \mathbf{e}_3$ , so that the projection of the rays in the fan for the special fiber to the lattice **N** is not contained in **N**'. This is an example of cases, where the simplifying assumption (4.9) is not satisfied.

The resulting Brauer group will be strictly smaller than the one described in (6.1): for instance,  ${}_{m}Br(X)\cap \mathcal{A}$ will consist of only those classes which are also unramified with respect to  $\mathbf{e}_{3}$ . Without the condition coming in from being unramified with respect to  $\mathbf{e}_{3}$ ,  ${}_{m}Br(X)\cap \mathcal{A}$  is generated by the cyclic algebra  $(t_{3},t_{4})^{e_{3,4}}_{\zeta}$ . The condition of this class being unramified with respect to  $\mathbf{e}_{3}$  is effectively adding  $\mathbf{e}_{3}$  to the sublattice spanned by the fan. Therefore, it will follow that one will need  $m|e_{3,4}$ , implying that  ${}_{m}Br(X)\cap \mathcal{A}$  will now be trivial in this example. If the simplifying assumption (4.9) is not put in, the resulting Brauer groups will be smaller in general, as one will need to consider additional conditions for a cyclic algebra to be in the corresponding Brauer group, as the above example shows. The part  ${}_{m}Br(X) \cap \mathcal{B}$  will also be smaller for similar reasons.

If, on the other hand, we define the rays in the special fiber to be as follows, we obtain a situation where the simplifying assumption (4.9) is satisfied. We will continue to assume that  $\{\mathbf{e}_1, \mathbf{e}_2\}$  is a basis for N'. Now let  $\mathbf{u}_i, i = 1, 2$  denote two vectors in  $\widetilde{\mathbf{N}}$ , which generate the fan of the toric scheme corresponding to the special fiber. Then we require that  $\pi(\mathbf{u}_i)$  is a linear combination of  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . Observe that this ensures  $\pi(\mathbf{u}_i)$  lies in the sublattice N'. Then the simplifying assumption (4.9) is clearly satisfied. This example also shows that the simplifying assumption (4.9) allows considerable freedom in the choice of the toric scheme forming the special fiber.

**Example 6.4.** We consider the following 4-dimensional toric variety (scheme) defined by a fan which consists of 4-cones, defined as follows:

(6.2) 
$$\sigma_1 = cone(\mathbf{e}_2, \mathbf{e}_1), \sigma_2 = cone(\mathbf{e}_1, 2\mathbf{e}_1 - \mathbf{e}_3), \sigma_3 = cone(2\mathbf{e}_1 - \mathbf{e}_3, 3\mathbf{e}_1 - 2\mathbf{e}_3), \text{ and } \sigma_4 = cone(\mathbf{e}_1, 2\mathbf{e}_1 - \mathbf{e}_4)$$

One can see that the fan imbeds only into a lattice of dimension 4 or higher. This example is also given by a non-projective and regular scheme. One may readily see that all the cones  $\sigma_i$ , i = 1, 2, 3, 4 are smooth cones, and therefore the toric schemes corresponding to each  $\sigma_i$  is the corresponding affine space  $\mathbb{A}^2_{\mathbb{R}}$ .

Next, writing down the coordinate vectors of the vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $2\mathbf{e}_1 - \mathbf{e}_3$  and  $2\mathbf{e}_1 - \mathbf{e}_4$  in terms of the standard basis  $\{\mathbf{e}_i | i = 1, \cdots, 4\}$  shows that

$$det(\mathbf{A}) = 1$$
, where  $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ .

This shows (using Cramer's rule integrally) that one has solutions for the matrix equations

$$A.\mathbf{x} = \mathbf{e}_i, i = 1, \cdots, 4$$

as vectors in the lattice **N**. It follows that the vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $2\mathbf{e}_1 - \mathbf{e}_3$  and  $2\mathbf{e}_1 - \mathbf{e}_4$  in fact form a basis for the 4-dimensional lattice **N** spanned by  $\mathbf{e}_i$ ,  $i = 1, \dots, 4$ . Therefore, one may see that u = r = 4. It also follows that the invariant factors  $a_i = 1$ ,  $i = 1, \dots, 4$ , despite the fact that the cones in the fan are all two dimensional.

Observe also that since each of the cones is 2-dimensional and smooth, the toric scheme is regular. Now it follows from Theorem 3.2, when the lattice containing the above fan is 4-dimensional (under the assumption that the base ring R is a field K), that  ${}_{m}Br(X) = {}_{m}Br(Spec K)$ . Similarly under the assumption that the base ring R is a Dedekind domain and the simplifying assumption (4.1), Theorem 4.1 shows that  ${}_{m}Br(X) = {}_{m}Br(Spec R)$ . Under the assumption that the base ring R is a DVR and the simplifying assumption (4.9), Theorem 4.8 shows that  ${}_{m}Br(X) = {}_{m}Br(Spec R)$ .

On increasing the lattice dimension to be 5, Theorems 3.2, 4.1 and 4.8 will show that  ${}_{m}Br(X) \cap \mathcal{B}$  will also be nontrivial. If the lattice dimension is 6 or higher, then the same theorems also will show that  ${}_{m}Br(X) \cap \mathcal{A}$ will also be nontrivial. **Example 6.5.** This final example is to show that when the residue field k imbeds into the DVR R, it is possible to shed further light on the construction of toric schemes. (See [Qu, Remark 2.1.10].) Here one may take the ring of p-adic integers for a fixed prime p. This is a complete DVR with residue field  $\mathbb{Z}/p\mathbb{Z}$ . As another example, one may take the formal power series ring  $R = \mathbb{Z}/p\mathbb{Z}[[x]]$  for a fixed prime p. This is also a DVR and has the special property that the residue field  $k = \mathbb{Z}/p\mathbb{Z}$  is a subring of R. Therefore, the discussion in [Qu, Remark 2.1.10] applies in this case. This means that if  $\tilde{\Delta}$  is a fan in  $\tilde{N}$ , the map of fans  $\tilde{\Delta} \to \mathbb{Q}_{\geq 0}$  induces a map

$$X(\tilde{\Delta})_k \to \mathbb{A}^1_k$$

of toric varieties over the field  $k = \mathbb{Z}/p\mathbb{Z}$ , defined by the homomorphism of k-algebras  $k[x] \to \mathbb{R}, x \mapsto \pi$ . In fact in this case if  $X_{\mathbb{R}}$  denotes the corresponding toric scheme over  $\mathbb{R}$ ,

$$X_{R} \cong X(\hat{\Delta})_{k} \times_{\mathbb{A}^{1}_{k}} (\operatorname{Spec} R).$$

### 7. Appendix: Motivic cohomology over regular Noetherian base schemes

First one may observe that the higher cycle complex may be defined over any base scheme B: if X is a scheme of finite type over B, and  $c \ge 0$  is a fixed integer, one defines  $Z^{c}(X, .)$  to be the chain complex defined in degree n, by

(7.1)  $\{Z = a \text{ pure codim } c \text{ cycle on } X \times_B \Delta_B[n] | Z \text{ intersects the faces of } X \times \Delta_B[n] \text{ properly} \}.$ 

**Definition 7.1.** We let  $\mathbb{Z}(c)$  denote the co-chain complex  $Z^{c}(X, .)[-2c]$  in cohomological degree m, that is,  $\mathbb{Z}(c) = Z^{c}(X, 2c - m)$ . (Observe that  $\mathbb{Z}(c)$  is contravariantly functorial for flat maps.) If X denotes a smooth scheme of finite type over B, we will let  $\mathbb{Z}^{X}(c)$  denote the restriction of the complex  $\mathbb{Z}(c)$  to the small Zariski or Nisnevich site of X. We call the complex  $\mathbb{Z}(c)$  the motivic complex of weight c. If c < 0 is an integer, we define  $\mathbb{Z}(c)$  to be  $\{0\}$ .

Next we will assume that  $B = \operatorname{Spec} R$ , where R is a Dedekind domain.

**Proposition 7.2.** Assume in addition that X denotes a smooth scheme of finite type over B. Then  $\mathbb{Z}^{X}(1)[1]$  identifies with  $\mathbb{G}_{m}^{X}$ , which denotes the restriction of the sheaf  $\mathbb{G}_{m}$  to the small Nisnevich site of X.

*Proof.* This is discussed in [Bl, section 6], where the discussion does not assume the base scheme is a field.  $\Box$ 

**Definition 7.3.** (Motivic cohomology) Let X denote a scheme of finite type over B. We let  $H_M^{i,j}(X) = H_{Zar}^i(X, \mathbb{Z}(j))$ , where  $H_{Zar}$  denotes the hypercohomology on the Zariski site.

It is observed in [Geis, Corollary 3.3], that one obtains the identification  $H^i_{Zar}(X, \mathbb{Z}(j)) \cong H_{Zar}(B, \pi_*(\mathbb{Z}(j)))$ , where  $\pi : X \to B$  denotes the structure map.

Let X denote a scheme of finite type over B and let Z denote a closed subscheme of X of pure codimension c with open complement U. Let  $i : Z \to X$  and  $j : U \to X$  denote the corresponding immersions. Then it is shown in [Geis, Corollary 3.3] that one obtains the distinguished triangle

(7.2) 
$$0 \to i_* \mathbb{Z}^{\mathbb{Z}}(n-c)[-2c] \to \mathbb{Z}^{\mathbb{X}}(n) \to j_* \mathbb{Z}^{\mathbb{U}}(n)$$

in the derived category of Zariski sheaves on X. In particular, this provides the identification of the terms in the long-exact sequence forming the top row in the diagram:

**Proposition 7.4.** Assume the above situation. If  $n < c = codim_X(Z)$ , where  $codim_X(Z)$  denotes the codimension of Z in X, then  $H_Z^i(X, \mathbb{Z}(n)) = 0$ . A corresponding result also holds when  $\mathbb{Z}(n)$  is replaced by  $\mathbb{Z}/m(n)$ , when m is invertible in R.

*Proof.* The first statement is clear from the identification of the first terms in the commutative diagram (7.3), since when n < c, the complex  $\mathbb{Z}^{\mathbb{Z}}(n-c)$  is trivial. This proves the first statement.

To obtain the second statement, one first tensors the localization sequence in (7.2) with  $\mathbb{Z}/m$  (i.e., the ring of integers modulo m) to obtain the corresponding localization sequence involving the motivic complexes  $\mathbb{Z}/m(n)$ . This then provides the commutative diagram corresponding to the one in (7.3) where the integral motivic complex  $\mathbb{Z}(n)$  is replaced by  $\mathbb{Z}/m(n)$ .

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