Equivariant Intersection Cohomology

Recall the definition of the intersection cohomology complex

Throughout the talk we will restrict to algebraic varieties defined over a field, which we may assume (for the sake of simplicity) is $\mathbb{C}$; much of our work will also hold in positive characteristic.

$X$: a possibly singular variety provided with a filtration:

$$U_1 \xrightarrow{j_1} U_2 \ldots U_n \xrightarrow{j_n} U_{n+1} = X, \dim_k X = n$$

so that each $j_i$ is an open immersion and each $U_{i+1} - U_i$ is smooth and of codimension $= i$ in $X$.

$\mathcal{L}$: a local system on $U_1$
\( p \): (non-negative even integers) \( \rightarrow \) (integers) is a non-decreasing function so that

\[
p(2k + 2) - p(2k) = 0, 1 \text{ or } 2, \quad p(2) = -n + 1.
\]

Now \( IC_p(\mathcal{L}) \) is characterized in the derived category by:

i). \( IC_p(\mathcal{L})|_{U_0} \cong \mathcal{L}[n] \)

ii). \( \mathcal{H}^i(\mathcal{L})|_{U_{k+1} - U_k} \cong 0 \) if \( i \geq p(2k) \)

iii). \( \mathcal{H}^i_{U_{k+1} - U_k}(IC_p(\mathcal{L})) \cong 0 \) if \( i \leq p(2k) \)

iv). \( \mathcal{H}^i(\mathcal{L}) = 0 \) if \( i > n \) or if \( i < -n \)

Now Deligne’s construction:

\[
IC_p(\mathcal{L}) = \sigma_{<p(2n)}Rj_\ast\sigma_{<p(2)}Rj_1\ast(\mathcal{L}[n])
\]

\[
IH^*_p(X; \mathcal{L}) = \mathbb{H}^*(X; IC_p(\mathcal{L})){/eq}
Incorporate a group action

Let $G$ act on $X$. (Now $\mu, pr_2 : G \times X \to X$, $\sigma : X \to G \times X$.) Now we obtain the simplicial space $EG \times X$ given by: $(EG \times X)_n = G^n \times X$,

$$d_i (g_0, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n, x) = (g_1, \ldots, g_{n-1}, x) \text{ if } i = 0$$

$$= (g_0, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n, x) \text{ if } 0 < i < n$$

$$= (g_0, \ldots, g_{n-2}, g_{n-1}, x) \text{ if } i = n$$

$$s_i (g_0, \ldots, g_{i-1}, g_i, \ldots, g_n, x) = (g_0, \ldots, g_{i-1}, e, g_i, \ldots, g_{n-2}, x), 0 \leq i \leq n - 1$$

$Top(EG \times X)_n = \text{the Grothendieck topology with}$

$objects: \ U_n \text{ in } Top((EG \times X)_n)$

$morphisms: \text{ given } U_n \text{ in } Top((EG \times X)_n) \text{ and } U_m \text{ in } Top((EG \times X)_m) \text{ a map } U_n \to U_m \text{ is a map lying}$

$\text{over some structure map } (EG \times X)_n \to (EG \times X)_m.$
A sheaf $F$ on $\text{Top}(EG \times X)_G$ is a collection of sheaves $\{F_n|n\}$ so that each $F_n$ is a sheaf on $\text{Top}((EG \times X)_n)_G$ and provided with maps $\phi_\alpha : \alpha^*(F_m) \to F_n$ for each structure map $\alpha$. (These are required to satisfy a compatibility condition.)

Such a sheaf is $G$-equivariant if each of the maps $\phi_\alpha$ is an isomorphism. The category of $G$-equivariant sheaves is an abelian sub-category closed under extensions: therefore one defines $D^G_b(X; \mathbb{Q})$ (= the equivariant derived category) to be the full sub-category of $D_b(EG \times X; \mathbb{Q})$ consisting of complexes whose cohomology sheaves are $G$-equivariant.

Note: if $f : X \to Y$ is a $G$-equivariant map, the induced map $EG \times X_G \to EG \times Y_G$ is denoted $f^G$. This is given by $(f^G)_n = f \times id^n$. Now one may de-
fine the derived functors $Rf^*_G$: observe that $Rf^*_G = \{R(f^G)_{n*}|n\}$.

**Equivariant intersection cohomology:**

$$IC^G_p(\mathcal{L}) = \sigma_{<p(2)}Rj_{n*}^G\ldots\sigma_{<p(2)}Rj_{1*}^G(\mathcal{L}[n]), \quad \mathcal{L} - a$$

$G$-equivariant local system on $EG \times U_1$

$$IH^*_{G,p}(X; \mathcal{L}) = \mathbb{H}^*(EG \times X; IC^G_p(\mathcal{L}))$$

**Main results on equivariant intersecton cohomology**

**Proposition 1.** $IH^*_{G,p}(X; \mathcal{L})$ is a module over $H^*(BG; \mathbb{Q})$

**Theorem 1.** There exists a Leray-spectral sequence:

$$E_2^{s,t} = H^s(BG; R^t\pi_*(IC^G_p(\mathcal{L}))) \Rightarrow IH^{s+t}_{G,p}(X; \mathcal{L})$$
where \( \pi : EG \times X \rightarrow BG \) is the obvious map. Moreover if \( \tilde{x} \in (BG)_n \),

\[
(R^t \pi_* IC^G_p(\mathcal{L})) \cong IH^t_p(X; \mathcal{L})
\]

**Theorem 2.** The above spectral sequence with \( \mathcal{L} = \mathbb{Q} \) degenerates in the following cases and provides the isomorphism:

\[
IH^*_{G,p}(X; \mathbb{Q}) \cong H^*(BG; \mathbb{Q}) \otimes IH^*_p(X; \mathbb{Q})
\]

(a) \( G \) acts trivially on \( X \) and \( p \) is arbitrary

(b) \( G \) is connected, \( p = m \) and \( X \) is projective

Next we consider torus actions. Let \( G = T, i : X^T \rightarrow X \). \( S = H^*(BT; \mathbb{Q}) \rightarrow 0 \).

**Theorem 3.** \( S^{-1} IH^*_{T,p}(X; \mathbb{Q}) \cong S^{-1} H^*(BT; \mathbb{Q}) \otimes \mathbb{H}^*(X^T; Ri^! IC^T_p(\mathbb{Q})) \)
Next we consider the action of a complex reductive group $G$.

**Theorem 4.** Assume that $G$ acts on $X$ with finite stabilizers and so that a geometric quotient $X/G$ exits as a scheme. Then

$$IH^*_G(X; \mathbb{Q}) \cong IH^*_p(X/G; \mathbb{Q})$$

Next we consider equivariant derived categories in more detail. Let $D^G_{b,c}(X; \mathbb{Q})$ denote the full subcategory of $D^G_b(X; \mathbb{Q})$ with constructible cohomology sheaves.

**Theorem 5.** For each fixed perversity $p$, there exists a non-standard $t$-structure on $D^G_{b,c}(X; \mathbb{Q})$ whose heart will be called the category of $G$-equivariant perverse sheaves on $X$ and denoted $C^G_p(X)$. 

7
The simple objects in the above abelian category $C^G_p(X)$ are given by the complexes $IC^G_p(\mathcal{L}_C)[d_C]$, where $C$ is a $G$-stable locally closed smooth sub-variety of $X$ of dimension $d_C$ and $\mathcal{L}$ is an irreducible $G$-equivariant local system on $EG \times C$.

**Theorem 6.** The category $C^G_m(X)$ is equivalent to the subcategory of $C_m(X)$ consisting of perverse sheaves $F$ on $X$ provided with the following data:

there exists an isomorphism $\phi : \mu^*(F) \to \text{pr}_2^*(F)$ (of perverse sheaves on $G \times X$) so that $\sigma^*(\phi) = id$ and there exists a cocycle condition between the pull-backs $d_0^*(\phi), d_1^*(\phi)$ and $d_2^*(\phi)$ as perverse sheaves on $G \times G \times X$. Morphisms are maps of perverse sheaves on $X$ that preserve the above structure.
Connections with the theory of $D$-modules

Assume $X$ is a smooth algebraic variety and $X_{an} = \text{the associated analytic space}$. An algebraic (left) $D_{X}$-module $M$ is strongly $G$-equivariant if there exists an isomorphism $\phi : \mu^{*}(M) \rightarrow pr_{2}^{*}(M)$ (as $D_{G \times X} \cong D_{G} \boxtimes D_{X}$-modules) so that $\sigma^{*}(\phi) = id$ and there exists a co-cycle condition between the pull-backs $d_{0}^{*}(\phi)$, $d_{1}^{*}(\phi)$ and $d_{2}^{*}(\phi)$ as $D_{G \times G \times X}$-modules.

Now the De-Rham functor $DR$ induces an equivalence of categories:

$$Mod_{r.h}^{s-G}(D_{X}) \xrightarrow{\sim} C_{m}^{G}(X_{an})$$

where the left hand side = the category of strongly $G$-equivariant regular holonomic $D_{X}$-modules. Under this correspondence the equivariant intersection cohomology complexes associated to $G$-stable locally closed
smooth sub-varieties \(C\) of \(X\) and \(G\)-equivariant irreducible local systems on \(EG \times C\) correspond to simple objects in the category on the left.

**Sample of some of the applications**

1. *Vanishing of odd dimensional intersection cohomology*

**Theorem 7.** Assume \(X\) is a projective variety provided with the action of an algebraic torus \(T\). Assume further that \(X\) is provided with a \(T\)-stable stratification \(\{S\}\) so that

a) each \(S\) is connected

b) each \(H^i(IC_m(\mathbb{Q}))\) is locally constant on each stratum \(S\)
c) each stratum $S$ has at least one fixed point for the $T$-action and $X^T$ is discrete

Now:

(i) $IH^i_m (X; \mathbb{Q}) = 0$ for all odd $i$ if and only if $H^i (IC_m (\mathbb{Q})) = 0$ for all odd $i$

(ii) Moreover $IH^i_m (X; \mathbb{Q}) = 0$ for all odd $i$, if there exists a $T$-equivariant resolution of singularities $\tilde{X} \rightarrow X$ so that $(\tilde{X})^T$ is also discrete.

**Corollary.** The conclusions of the above theorem hold for Schubert varieties and the varieties $\tilde{O}(w)$ (= the closure of the orbit $O(w)$ for the diagonal action of a reductive group $G$ on the flag-manifold $G/B \times G/B$.

2. Applications to geometric invariant theory.

Let $X$ be reductive acting on the projective variety $X$. $X^{ss} (X^s) =$ the set of semi-stable (stable, re-
respectively) points with respect to a $G$-linearized ample line bundle.

**Theorem** (Kirwan)

(i) There exists a $G$-stable stratification of $X$ indexed by a partially ordered set $\mathcal{B}$ where the open stratum is $X^{ss}$. The closure of a stratum $S_\beta$ is contained in the union of $\{S_\gamma | \gamma \geq \beta\}$. One can write $\mathcal{B} = \{\beta_0, \beta_1, \ldots, \beta_s\}$ so that, for $0 \leq j \leq s$, $U_j = S_{\beta_0} \cup \ldots \cup S_{\beta_j}$ is open in $X$ and $S_{\beta_j}$ is closed in $U_j$.

(ii) If $X$ is also smooth, one obtains:

$$0 \to H^n_G(U_j, U_{j-1}; \mathbb{Q}) \to H^n_G(U_j; \mathbb{Q}) \to H^n(U_{j-1}; \mathbb{Q}) \to 0$$

(iii) In general:

$$0 \to IH^n_{G,m}(U_j, U_{j-1}; \mathbb{Q}) \to IH^n_{G,m}(U_j; \mathbb{Q}) \to$$

$$IH^n_{G,m}(U_{j-1}; \mathbb{Q}) \to 0$$
(iv) Moreover $H^*_G(X; \mathbb{Q})(in\,(ii)) \cong H^*(BG; \mathbb{Q}) \otimes H^*(X; \mathbb{Q})$ and $IH^*_{G,m}(X; \mathbb{Q})(in\,(iii)) \cong H^*(BG; \mathbb{Q}) \otimes IH^*_m(X; \mathbb{Q})$. If $X^{ss} = X^s$ also, (see Theorem 4), $IH^*_{G,m}(X^{ss}; \mathbb{Q}) \cong IH^*_m(X//G; \mathbb{Q})$.

Therefore one may compute the Poincaré series for $IH^*(X//G; \mathbb{Q})$ using the Poincaré-series in equivariant intersection cohomology of the various pairs $(U_j, U_{j-1})$. (The Poincaré series in equivariant intersection cohomology of a pair $(U_j, U_{j-1})$ with a $G$-action is:

$$IP_t^G(U_j, U_{j-1}) = \Sigma_i dim IH^i_{G,m}(U_j, U_{j-1}; \mathbb{Q}) t^i$$

Let $T = a$ maximal torus in $G$. Now the Hilbert-Mumford criterion for semi-stability implies:

$$X^{ss} = \bigcap_{g \in G} gX^{ss}_T$$

$X^{ss}_T$ = the semi-stable points for the $T$-action.
**Theorem (B-J)**

\[ IH_{G,m}^*(X^{ss}; \mathbb{Q}) \cong (IH_{T,m}^*(X_T^{ss}; \mathbb{Q}))^a \]

where \( a \) denotes the anti-invariant part = the part corresponding to the sign-representation of \( W \). To see the \( W \)-action on \( IH_{T,m}^*(X_T^{ss}; \mathbb{Q}) \) observe:

\[ H^*(BT; \mathbb{Q}) \cong H^*(G/T; \mathbb{Q}) \otimes H^*(BG; \mathbb{Q}). \]

Now \( IH_{T,m}^*(X_T^{ss}; \mathbb{Q}) \) is a module over \( H^*(BT; \mathbb{Q}) \).

In particular if \( X^{ss} = X^s \), then:

\[ IH_m^*(X//G; \mathbb{Q}) \cong (IH_{T,m}^*(X_T^{ss}; \mathbb{Q}))^a \]

*Home-page:* http://www.math.ohio-state.edu/~joshua