Equivariant Perverse Sheaves and Quasi-Hereditary Algebras

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- Intersection cohomology: a replacement for the usual cohomology for dealing with spaces with singularities.
- However, for the main applications of this theory as discovered by K-L, one needs to consider schemes with actions by linear algebraic groups.
- A key shortcoming: it does not see the group action at all! Hence the need for a variant that does see the group action clearly.

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- However, this was not even touched upon in the work of Macpherson-Goresky or Beilinson-Bernstein-Deligne. So it was left to myself, Brylinski and Bernstein-Lunts to carry out this program.
- In summary, the idea is to combine Borel-style equivariant cohomology with intersection cohomology and also set up a framework for studying this by introducing Equivariant Derived Categories.
- A few years later, I was able to get Michel Brion also involved in this program. Together, we were able to apply this theory to spherical varieties, extending previously known results from toric varieties to spherical varieties.

Framework for the talk

- k: a fixed perfect base field, arbitrary characteristic $p \geq 0$ and of finite ℓ-cohomological dimension for any prime $\ell \neq p$. Further restrictions on k as we move along.

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- restrict to schemes of finite type over k.
- X: a quasi-projective scheme provided with the action of a *connected linear algebraic group* G, all defined over k.
- Then the derived category of sheaves on X needs to be replaced by the equivariant derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X})$ of complexes of sheaves with bounded, equivariant and constructible cohomology sheaves. The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X})$ incorporates the group action whereas the ordinary derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{X})$ does not.

Equivariant Derived Categories and Equivariant Perverse Sheaves:I

- One begins with $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$: this given by a suitable form of the Borel construction. Can view this as a simplicial scheme given in degree n by $\mathrm{G}^{\mathrm{n}} \times \mathrm{X}$, with structure maps induced by the group action, projection etc. Alternate model as an ind-scheme $\{\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}|\mathrm{m}\}$.

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- For complex varieties, one can take the derived category of complexes of sheaves of $\mathbb{C}$-vector spaces. In pos chars, one works ℓ-adically on the étale topology.
- $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X})$: the full subcategory of the derived category on $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ of complexes whose cohomology sheaves are G-equivariant, constructible and bounded.

Equivariant Derived Categories and Equivariant Perverse Sheaves:II

- Assume X provided with a G-stable filtration: $U_0 \xrightarrow{j_0} U_1 \cdots \xrightarrow{j_n} U_{n+1} = X$ with $U_i - U_{i-1}$ smooth, U_i open.

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- The Equivariant Intersection Cohomology complex: $IC^{\mathrm{G}}(\mathcal{L}) = \tau_{\leq p(n)} Rj^{\mathrm{G}}_{n*} \cdots \tau_{\leq p(0)} Rj^{\mathrm{G}}_{0*}(\mathcal{L})$.

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- $IH_{\mathrm{G}}^*(X) = \mathbb{H}^*(\mathrm{EG \times_G X, IC^G}(\mathcal{L}))$.
- p: a perversity function. We will only use the *middle perversity*, m.

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- The strata are $\mathrm{U_i} - \mathrm{U_{i-1}}$, i. These are G-stable. Given such a G-stable stratfication, $\mathcal{S}_G$, $\mathrm{D^b_{G,c}}(\mathrm{X}, \mathcal{S}_\mathrm{G})$ is the full subcategory where the cohomology sheaves are locally constant on the given strata.

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- Then the heart of this t-structure on $\mathrm{D^b_{G,c}(X, \mathcal{S}_G)}$ is the *Abelian category* of *Equivariant Perverse Sheaves*:$\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$.

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- This category is Artinian and Noetherian.
- Henceforth we will assume G acts on X with finitely many orbits and the stratum $\mathrm{U_i} - \mathrm{U_{i-1}}$ is a disjoint union of orbits of the same dimension.

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(i) *The simple objects in $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ are the equivariant intersection cohomology complexes, $IC^{\mathrm{G}}(\mathcal{L})$, which denotes the equivariant intersection cohomology complex obtained by starting with the irreducible G-equivariant local system $\mathcal{L}$ on some stratum S.*

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(ii) *Assume next that k is algebraically closed. Then the category* $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *has enough projectives and every object has a projective cover.*

Local systems and G-equivariant local systems:I

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- Over $\mathbb{C}$, the G-*equivariant local systems* on a G-scheme X correspond to representations of the fundamental group $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, x)$ in $\mathbb{C}$ or $\mathbb{Q}$-vector spaces

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- In positive characteristics, the G-*equivariant ℓ-adic local systems* on a G-scheme X correspond to ℓ-adic representations of the étale fundamental group $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, x)$.

Local systems and G-equivariant local systems:II

- If X itself is an orbit for the G-action on a larger scheme, $X \cong G/G_x$, where x denotes a fixed point on X, $\pi_1(EG \times_G X, \quad x) \cong \pi_1(BG_x)$.

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- In pos. char p, $\pi_1(EG \times_G X, \quad x)\widehat{\ }_{\ell} \cong G_x/G_x^o\widehat{\ }_{\ell}$.
- We will often make the following assumption: $\pi_1(EG \underset{G}{\times} X, x)$ acts on the stalk $\mathcal{L}_x$ through a finite quotient group F.

Proposition

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(i) Under the above assumption, every G*-equivariant local system is semi-simple.*

(ii) The above assumption holds if the base field is algebraically closed and X *has a transitive action by* G*, so that* X *identifies with an orbit for the* G*-action.*

(iii) When X *has only finitely many* G*-orbits, each orbit has enough projectives in the category of equivariant local systems.*

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- For schemes defined over fields of positive characteristics, one needs to use the full machinery of ℓ-adic sheaves worked out by Deligne, for example, or others.
- The key point: one often needs to work with sheaves, whose cohomology will be vector spaces over algebraically closed fields. In the ℓ-adic case, this means, one needs to use the algebraic closure of $\mathbb{Q}_\ell$. This is due to Deligne, and we will need to invoke at least part of this theory. We will ignore much of these technical issues in this talk.

Remarks on the proof of Theorem I

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- That there are enough projectives in the category of equivariant perverse sheaves is far from automatic, and this holds only because there are only finitely many G-orbits on the given scheme, and that the base field is algebraically closed.
- One starts with: the category of G-equivariant local systems on each orbit has enough projectives. This reduces to $\pi_1(EG \times_G (G/H)) \cong \pi_1(BH) \cong H/H^o$, which is a finite group.

Remarks on the proof of Theorem I

- Nevertheless, it needs to be pointed out that the existence of bounded projective resolutions is far from clear. To know such bounded resolutions exist, one often needs to check a condition, which amounts to knowing $H^2(BG_x, \mathbb{Q}) = 0$ for all stabilizer groups G_x and this is usually never trivial unless G_x itself is trivial.

What is known about the stabilizer groups

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- If the base field is $\mathbb{C}$, the stabilizer groups are all extensions of connected groups by finite abelian groups for all spherical varieties.
- The immersions $j : \mathcal{O} \to X$ for any G-orbit are affine for all toric varieties and also for all toroidal spherical varieties.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: I

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- The orbit stratification is rarely acyclic. However their constructions with perverse sheaves seem to extend to equivariant perverse sheaves and that leads to the next result of the talk.
- An algebra Λ over a field K is quasi-hereditary (after Cline, Parshall and Scott) if there is a chain of 2-sided ideals $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{i-1} \subseteq L_i \subseteq \cdots \subseteq L_n = L$, so that each L_i/L_{i-1} is a *heredity ideal* in Λ/L_{i-1}. The last means that $(L_i/L_{i-1})^2 = L_i/L_{i-1}$, L_i/L_{i-1} is projective as a left or right Λ/L_{i-1}-module, and that $L_i/L_{i-1}J(\Lambda/L_{i-1})L_i/L_{i-1} = 0$.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: II

- Let $\mathbf{S}_L = \{(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) | \mathcal{O} \varepsilon \mathcal{S}, \mathcal{L}_{\mathcal{O}}$ a G-equivariant irreducible local system on $\{\mathcal{O}\}$.

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- Let $\mathbf{S}_L = \{(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) | \mathcal{O} \varepsilon \mathcal{S}, \mathcal{L}_{\mathcal{O}}$ a G-equivariant irreducible local system on $\{\mathcal{O}\}$.
- $(\mathcal{O}', \mathcal{L}'_{\mathcal{O}'}) \leq (\mathcal{O}, \mathcal{L}_{\mathcal{O}})$ if
  (i) $\mathcal{O}' \subseteq \bar{\mathcal{O}}$ and
  (ii) there is a map $\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')] \rightarrow Rj^{!}_{\mathcal{O}',\mathrm{G}} j^{p}_{\mathcal{O},\mathrm{G}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})])$ in $\mathrm{D_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ that induces a monomorphism on cohomology sheaves in degree $-dim(\mathcal{O}')$.

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  $\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')] \to Rj^!_{\mathcal{O}',\mathrm{G}} j^p_{\mathcal{O},\mathrm{G}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})])$ in $\mathrm{D}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ that induces a monomorphism on cohomology sheaves in degree $-dim(\mathcal{O}')$.
- Theorem: This defines a partial order on $\mathbf{S}_L$.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: III

- When the stabilizer groups are all connected, the G-equivariant local systems on each orbit will reduce to the constant local systems, and then the above partial order on $\mathbf{S}_L$ reduces to the usual partial order on the orbits.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: III

- When the stabilizer groups are all connected, the G-equivariant local systems on each orbit will reduce to the constant local systems, and then the above partial order on $\mathbf{S}_L$ reduces to the usual partial order on the orbits.
- The *standard objects* in $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$: let $j : \mathcal{O} \to \mathrm{X}$ denote the immersion of a G-orbit into X. If $\mathcal{L}_{\mathcal{O}}$ is a G-equivariant local system on the orbit $\mathcal{O}$, $j^p_{\mathcal{O},\mathrm{G}!}(\mathcal{L}_{\mathcal{O}}) = \tau^p_{\geq 0} j_{\mathcal{O},\mathrm{G}!}(\mathcal{L}_{\mathcal{O}})$: the standard object associated to $\mathcal{L}_{\mathcal{O}}$.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: III

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- Those familar with representation theory, may identify these with the analogue of what are called *Verma modules*.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: IV

Assuming the above situation we obtain the following technical result where $\mathcal{L}_S$ ($\mathcal{L}_T$) is an irreducible G-equivariant local system on the orbit S (T, respectively).
$\mathrm{Hom}^n_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(j^p_{S,G!}(\mathcal{L}_S[d_S]), j^p_{T,G!}(\mathcal{L}'_{\mathrm{T}}[d_{\mathrm{T}}]))$ then can be identified with

- 0, if $n \leq -1$,

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Construction of Quasi-hereditary algebras from equivariant perverse sheaves: V

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- and $\mathrm{A} = Hom_{\mathrm{D_G(X)}}(\mathrm{T},\mathrm{T})$.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: VI: Theorem II

Theorem

Theorem II. *Assume the above situation. Then* A *is a quasi-hereditary algebra. The corresponding highest weight category* $\bar{\mathcal{C}}$ *is given by the category of all finitely generated modules over* A.

Proof.

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Proof.

- The technical results on the previous slide show A is a quasi-hereditary algebra by invoking [(5.9) Theorem, PS].
- By the same result of Parshall and Scott, the category of all finitely generated modules over A is then a highest weight category.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: VII

- A major difference with the non-equivariant setting (where the strata are acyclic): the category of equivariant perverse sheaves $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ is almost never a highest weight category.

Construction of Quasi-hereditary algebras from equivariant perverse sheaves: VII

- A major difference with the non-equivariant setting (where the strata are acyclic): the category of equivariant perverse sheaves $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ is almost never a highest weight category.
- This eventually reduces to the fact that $\mathrm{H}^2(B\mathrm{G}_x, \mathbb{Q})$ ($\mathrm{H}^2_{et}(B\mathrm{G}_x, \mathbb{Q}_\ell)$ in pos. char) is almost never trivial.

Historical perspective

- Observed by K-L (among possibly others) in the early 1980s, that $IH^i(X) = 0$, for all *odd* i and that $\mathcal{H}^i(IC_X) = 0$ for all *odd* i, for many algebraic varieties, such as Schubert varieties (for example).

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- The proof that K-L gave for this was not geometric. Therefore, it was posed as an open problem how to find a geometric proof of this vanishing.
- Equivariant Intersection cohomology answered this very nicely, for many of these varieties: dates from 1987.
- In a follow-up paper in 2001, Brion and myself extended this to all spherical varieties defined over $\mathbb{C}$. Presently we will extend this to all spherical varieties defined over alg closed fields of pos. chars.

Vanishing of global odd dimensional intersection cohomology:I

- Given a G-spherical variety X, there is a weak G-equivariant resolution of singularities, $\pi : \tilde{X} \to X$, so that π is birational, $\tilde{X}$ is toroidal, is *rationally smooth* and has only finitely many fixed points for the action of T.

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- Then the fixed point formula for torus action in equivariant cohomology, along with the degeneration of the equivariant cohomology spectral sequence shows:

$$H^*_{\mathrm{T}}(\tilde{X}) \cong H^*(BT) \otimes H^*(\tilde{X}),$$

$$S^{-1}H^*_{\mathrm{T}}(\tilde{X}) \cong S^{-1}H^*(BT) \otimes \oplus_{x \boldsymbol{\varepsilon} \tilde{X}^{\mathrm{T}}} H^*(x, Ri^!_x(\mathbb{Q}_\ell)).$$

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- This means $H^i(\tilde{X}, \mathbb{Q}_\ell) = 0$ for all *odd* i.

Vanishing of global odd dimensional intersection cohomology:II

- Now the decomposition theorem in intersection cohomology, shows $\oplus_{iodd} IH^i(X)$ is a summand of $\oplus_{iodd} H^i(\tilde{X}, \mathbb{Q}_\ell)$. This proves $IH^i(X) = 0$ for all *odd* i.

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- To use this to deduce the vanishing of *odd* dimensional intersection cohomology sheaves on X, one first proves $\mathcal{H}^i(IC_X)_x = 0$, for all odd i, where x is a fixed point of the maximal torus in G.

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- To use this to deduce the vanishing of *odd* dimensional intersection cohomology sheaves on X, one first proves $\mathcal{H}^i(IC_X)_x = 0$, for all odd i, where x is a fixed point of the maximal torus in G.
- This makes use of a fixed point formula for torus actions in equivariant intersection cohomology, and the degeneration of the spectral sequence in equivariant intersection cohomology: the latter is a deep result, and makes use of the fact that one has Hard-Lefschetz for intersection cohomology and also uses a nice criterion of Deligne for the degeneration of Leray spectral sequences.

Vanishing of global odd dimensional intersection cohomology:III: Theorem III

Theorem

Let G *denote a connected reductive group,* X *a* G*-spherical variety defined and of finite type over Spec* k *and* $\mathcal{L}$ *a* G*-equivariant* ℓ*-adic local system on the open dense* G*-orbit, with* $\ell \neq p$*.*
Then $\mathcal{H}^i(\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L})) = 0$ *for all odd* i*.*

Some ideas on the proof: One uses ascending induction on the dimension of the spherical variety. In dimension 1, it is trivial. The vanishing at the stalks at the fixed points of the maximal torus, follows from the torus fixed point theorem. To deduce the vanishing of the stalks at other points, one makes use of the slice structure.

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