

Equivariant Derived Categories for Toroidal Group Imbeddings

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project is joint work with Michel Brion.

Goals of the talk

Let X denote a projective variety over an algebraically closed field on which a linear algebraic group acts with finitely many orbits. Then, a conjecture of Soergel and Lunts in the setting of Koszul duality and Langlands' philosophy, postulates that the equivariant derived category of bounded complexes with constructible equivariant cohomology sheaves on X is equivalent to a full subcategory of the derived category of modules over a graded ring defined as a suitable graded *Ext*.

Only special cases of this conjecture have been proven so far. *The purpose of this talk is to outline a proof of this conjecture for toroidal imbeddings of complex reductive groups.*

Goals of the talk: contd

Since every equivariant imbedding of such a group is dominated by a toroidal imbedding, the class of varieties for which our proof applies is quite large.

We also show that, in general, there exist a countable number of obstructions for this conjecture to be true and that half of these vanish when the odd dimensional equivariant intersection cohomology sheaves on the orbit closures vanish. This last vanishing condition had been proven to be true in many cases of spherical varieties by [BJ] in prior work.

Outline

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- 6 Comparison of equivariant Derived categories

Equivariant Derived categories and the formulation of the conjecture:I

- (i) Let X be an algebraic variety provided with the action of a linear algebraic group G . Then $D_{G,c}^b(X)$ denotes the equivariant derived category of bounded complexes of sheaves with G -equivariant cohomology sheaves. Often the variety is singular: for example, if X has only finitely many G -orbits. Then X is stratified by the strata, that are locally closed smooth subvarieties.

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- (ii) Then $D_{G,c}^b(X)$ has two natural t -structures: (i) the usual one, whose heart is the category of equivariant sheaves and (ii) a non-standard one, whose heart is the category of equivariant perverse sheaves. Both hearts are abelian categories, and the simple objects in the latter are the *equivariant intersection cohomology complexes on the closures of the strata*.

Equivariant Derived categories and the formulation of the conjecture:II

- (iii) Examples: Toric varieties, where the strata are the T -orbits, spherical varieties for the action of a reductive group G , where the strata are either the B -orbits or the G -orbits.

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- (iv) Assume that G acts on X with finitely many orbits. Then the conjecture states the following:

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- (iii) Examples: Toric varieties, where the strata are the T -orbits, spherical varieties for the action of a reductive group G , where the strata are either the B -orbits or the G -orbits.
- (iv) Assume that G acts on X with finitely many orbits. Then the conjecture states the following:
 - (v) that $D_{G,c}^b(X)$ is equivalent to a full subcategory of the category of graded dg-modules over the dg-algebra $RHom(L, L)$, where $L = \bigoplus_i L_i$ is the sum of the equivariant intersection cohomology complexes on the orbits, and that the last dg-algebra is *formal*, i.e. is quasi-isomorphic to its cohomology algebra.

Status of the conjecture

- (i) Lunts proved this for projective complex toric varieties: (1995). Guillermou proved it for smooth complete symmetric varieties for the action of a semi-simple adjoint group (2005) and Schnurer proved it for flag varieties (2011). Every other case was still open.

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- (i) Lunts proved this for projective complex toric varieties: (1995). Guillermou proved it for smooth complete symmetric varieties for the action of a semi-simple adjoint group (2005) and Schnurer proved it for flag varieties (2011). Every other case was still open.
- (ii) We will outline a proof for the large sub-class of spherical varieties called complex projective toroidal group imbeddings. The machinery developed for this proof applies more generally to prove at least part of the conjecture for other important subclasses of spherical varieties, such as horospherical varieties, projective toric varieties in pos. chars. etc.

Why consider toroidal group imbeddings?

- (i) Toric varieties and Flag varieties form two extreme ends of the large class of varieties called spherical varieties and spherical varieties are classified by (colored) fans just as toric varieties are classified by fans. The conjecture has been already verified for toric and flag varieties.

Why consider toroidal group imbeddings?

- (i) Toric varieties and Flag varieties form two extreme ends of the large class of varieties called spherical varieties and spherical varieties are classified by (colored) fans just as toric varieties are classified by fans. The conjecture has been already verified for toric and flag varieties.
- (ii) The class of projective toroidal imbeddings is, in fact, an important subclass of spherical varieties, as the following observation will show: recall that given a complex connected reductive group G and any projective $G \times G$ -equivariant embedding X of G , there exists a projective toroidal imbedding \tilde{X} of G together with a $G \times G$ -equivariant birational map $\tilde{X} \rightarrow X$. (See [Prop. 6.2.5, BK05].)

Some examples of projective toroidal group imbeddings:I

- (i) The simplest example of group imbeddings are that of GL_2 . In this case there are at least 4 distinct $GL_2 \times GL_2$ -equivariant imbeddings of GL_2 , namely, \mathbb{A}^4 , \mathbb{P}^4 , and $\widetilde{\mathbb{A}}^4$, $\widetilde{\mathbb{P}}^4$, where the last two are blow-ups of the first two at the origin in \mathbb{A}^4 . Only the latter two are toroidal imbeddings, and clearly $\widetilde{\mathbb{P}}^4$ is the only toroidal imbedding that is projective.

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- (ii) Let G be semi-simple of adjoint type. Then $X_*(T)$ has a basis consisting of the fundamental weights. Therefore, the fan consisting of the Weyl chambers and their faces is smooth. The corresponding (smooth) toroidal embedding is the *wonderful compactification* of G . (See [section 6.1, BK05].)

Some examples of projective toroidal group imbeddings:II

- (iii) If G is semi-simple but no longer of adjoint type, then we may consider the same fan, but now it is almost never smooth. For example, if $G = \mathrm{SL}_3$ then we get a singular toroidal compactification of G . It can be constructed geometrically as the normalization of the wonderful compactification of PGL_3 in the function field $k(\mathrm{SL}_3)$. This construction works more generally for any semi-simple G : the normalization in $k(G)$ of the wonderful compactification of the adjoint group yields a canonical embedding of G , which is toroidal and projective but (again) almost never smooth. (See [section 6.2.A, BK05] for additional details.)

Main Theorem I

Let X denote a variety on which a linear algebraic group acts with finitely many orbits. Let $\mathcal{B}_G(X) = \mathrm{RHom}(L, L)$. Then, sending $K \in D_{G,c}^b(X) \mapsto \mathrm{RHom}(L, K)$ defines a fully-faithful imbedding of the equivariant derived category $D_{G,c}^b(X)$ into the derived category $D(\mathrm{Mod}(\mathcal{B}_G(X)))$. If E_i denotes the image of L_i under this imbedding and if $D^f(\mathrm{Mod}(\mathcal{B}_G(X)))$ denotes the triangulated full sub-category of $D(\mathrm{Mod}(\mathcal{B}_G(X)))$ generated by the E_i , $i = 1, \dots, n$, then $D_{G,c}^b(X)$ is equivalent to $D^f(\mathrm{Mod}(\mathcal{B}_G(X)))$.

Main Theorem II

Let \bar{G} denote a projective toroidal imbedding of the complex connected reductive group G and let $\mathcal{B}_{G \times G}(\bar{G})$ denote the differential graded algebra $RHom(L, L)$ considered above. Then $\mathcal{B}_{G \times G}(\bar{G})$ is formal as a differential graded algebra.

Example of a dg algebra that is formal. Suppose \mathcal{A} is a dg-algebra so that $\mathcal{H}^i(\mathcal{A}) = 0$ for all $i \neq 0$. Then \mathcal{A} is formal as a dga. The proof is just the diagram: $\mathcal{H}^0(\mathcal{A})[0] \leftarrow \sigma_{\leq 0}(\mathcal{A}) \rightarrow \mathcal{A}$ in which both arrows are quasi-isomorphisms. It is important to observe that both these maps preserve the multiplicative structure.

The basic general strategy of the proof

- (i) Find a sheaf of dg-algebras \mathcal{A} on a related *site*, so that (a) it is not that difficult to prove \mathcal{A} is formal and (b) then show there is a functor
- $$\psi : D_{G \times G, c}^b(\mathbf{X}) \rightarrow D(\text{dg-modules on } \mathcal{A})$$
- that preserves quasi-isomorphisms and the functor ψ preserves multiplicative structures.

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- (ii) Then show $\text{RHom}(L, L)$ is quasi-isomorphic as a dga to $\text{RHom}_{\mathcal{A}}(\psi(L), \psi(L))$.

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- (ii) Then show $\text{RHom}(L, L)$ is quasi-isomorphic as a dga to $\text{RHom}_{\mathcal{A}}(\psi(L), \psi(L))$.
- (iii) Finally show the dga $\text{RHom}_{\mathcal{A}}(\psi(L), \psi(L))$ is formal as a dga by showing it satisfies the properties as on the last slide.

Projective toroidal group imbeddings: Reduction to \bar{T}

- (i) Choose a maximal torus T of G and denote by N its normalizer in G ; the quotient N/T is the Weyl group W . Let B denote a fixed Borel subgroup containing T .

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- (i) Choose a maximal torus T of G and denote by N its normalizer in G ; the quotient N/T is the Weyl group W . Let B denote a fixed Borel subgroup containing T .
- (ii) Let \bar{T} denote the closure of T in \bar{G} ; then $(T \times T) \text{diag}(N)$ acts on \bar{T} via its quotient WT , where W acts by conjugation and T by left multiplication. This yields a (restriction) functor

$$res : D_{G \times G, c}^b(\bar{G}) \longrightarrow D_{(T \times T) \text{diag}(N), c}^b(\bar{T}).$$

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- (ii) Moreover, if $D_{(T \times T) \text{ diag}(N), c}^{b, o}(\bar{T})$ denotes the full subcategory of $D_{(T \times T) \text{ diag}(N), c}^b(\bar{T})$ generated by the $(T \times T) \text{ diag}(N)$ -equivariant sheaves that are constant along the orbits of $(T \times T) \text{ diag}(N)$ on \bar{T} , then the above functor induces an equivalence

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$$D_{G \times G, c}^b(\bar{G}) \xrightarrow{\text{res}} D_{(T \times T) \text{diag}(N), c}^{b, o}(\bar{T}).$$

- (iii) It sends the $G \times G$ -equivariant intersection cohomology complex $IC^{G \times G}(\mathbb{Q}_{\mathcal{O}})$ on a $G \times G$ -orbit \mathcal{O} to the corresponding $(T \times T) \text{diag}(N)$ -equivariant intersection cohomology complex $IC^{(T \times T) \text{diag}(N)}(\mathbb{Q}_{\mathcal{O}'})$ where \mathcal{O}' is the $(T \times T) \text{diag}(N)$ -orbit corresponding to the $G \times G$ -orbit \mathcal{O} .

Remarks on the proof

This is mostly the only place where we need to restrict to char 0 for the proof we have to work, as it involves looking at maximal compact subgroups.

We will henceforth denote by $\tilde{N} = (T \times T)\text{diag}N(T)$. Observe that $\tilde{N}/\text{diag}(T) = WT$.

The first dg-category:I

Let

$$\psi : E\tilde{N} \times_{\tilde{N}} \bar{T} \rightarrow EWT \times_{WT} \bar{T}$$

denote the map induced by the identity on \bar{T} and the quotient map $\tilde{N} \rightarrow WT$. The fibers of the simplicial map ψ at every point can be identified with $B\text{diag}(T)$ since $\text{diag}(T)$ acts trivially on \bar{T} . Recall $D_{\tilde{N},c}^{b,o}(\bar{T})$ denotes the full subcategory of the derived category $D_{\tilde{N},c}^b(\bar{T})$ generated by the constant sheaves on each orbit of \tilde{N} on \bar{T} .

Clearly the functor $R\psi_*$ sends complexes in $D_{\tilde{N},c}^{b,o}(\bar{T})$ to complexes of dg-modules over the sheaf of dg-algebras $R\psi_*(\mathbb{Q})$, i.e. to objects in the derived category $D_{WT}(\bar{T}, R\psi_*(\mathbb{Q}))$.

The first dg-category:II

We let $D_{\mathrm{WT},c}^{+,o}(\bar{T}, R\psi_*(Q))$ denote the full subcategory generated by the objects $R\psi_*(j_!\mathbb{Q})$ where $j : \mathcal{O} \rightarrow \bar{T}$ denotes the immersion associated with an \tilde{N} -orbit and we vary over such \tilde{N} -orbits. Let

$$L\psi^* : D_{\mathrm{WT},c}^{+,o}(\bar{T}, R\psi_*(\mathbb{Q})) \rightarrow D_{\tilde{N},c}^{b,o}(\bar{T})$$

denote the functor defined by sending a dg-module M to

$$\mathbb{Q} \otimes_{\psi^{-1}R\psi_*(\mathbb{Q})}^L \psi^{-1}(M).$$

Theorem IV

- (i) The functor $R\psi_* : D_{\tilde{N},c}^{b,o}(\bar{T}) \rightarrow D_{\text{WT},c}^{+,o}(\bar{T}, R\psi_*(\mathbb{Q}))$ is an equivalence of categories with inverse $L\psi^*$. The derived category on the right is generated by the dg-modules over the dga $R\psi_*(\mathbb{Q})$ of the form $j_{!}j^*R\psi_*(\mathbb{Q}) = R\psi_*j_{!}j^*(\mathbb{Q})$, as $j : \mathcal{O} \rightarrow \bar{T}$ varies over the \tilde{N} -orbits on \bar{T} .
- (ii) The t -structure on the derived category on the left is either the standard one, where the heart is the full subcategory of complexes whose cohomology sheaves are \tilde{N} -equivariant and vanish in all degrees but 0, or the t -structure obtained by gluing as in [1.4, BBD82], where the heart is the full subcategory of \tilde{N} -equivariant perverse sheaves. The corresponding t -structure on the derived category on the right is the one induced by the functor $R\psi_*$ from the t -structure on the derived category on the left by the above adjunction.

Some remarks on the above Theorem

- (i) The toric variety \bar{T} , with the action of $T \times T$, is a non-trivial example of a *toric stack*, since the diagonal torus acts trivially on \bar{T} . Thus, perhaps surprisingly, the theory of toric stacks shows up in the analysis of the equivariant derived categories for toroidal group imbeddings.

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- (ii) The map ψ is induced by the quotient map $\tilde{N} \rightarrow WT$, which is surjective. For dealing with such maps of groups (as opposed to a closed immersion), the commonly adopted geometric models for EG and BG are not suitable. Therefore, throughout the discussion, we will adopt the simplicial models for these and use these to define the corresponding equivariant derived categories. Then one proves a comparison theorem, that shows all the different models of EG and BG give equivalent derived categories.

Separating the W and T actions:I

Currently we have a WT -action on \bar{T} . This has to be separated out into a W -action and a T -action.

Proposition

(i) If X is a space, then giving a WT -action on X is equivalent to providing T and W actions on X so that the T -action $\mu : T \times X \rightarrow X$ is W -equivariant. i.e. The diagram

$$\begin{array}{ccc}
 W \times T \times X & \xrightarrow{id \times \mu_T} & W \times X \\
 \downarrow \mu'_W & & \downarrow \mu_W \\
 T \times X & \xrightarrow{\mu_T} & X
 \end{array}$$

commutes, where μ_W (μ_T) denote the W (T) action and μ'_W denotes the diagonal action of W on $T \times X$.

Separating the W and T actions:II

Proposition

Equivalently, the following relations hold (where \circ denotes generically any of the actions):

$$w \circ (t \circ x) = (w \circ t) \circ (w \circ x), w \in W, t \in T, x \in X.$$

(ii) Moreover, in this case the simplicial space $ET \times_T X$ has an induced action by W .

Let, $p_1 : EWT \times_{WT} (ET \times \bar{T}) \rightarrow EW \times_{W} (ET \times_T \bar{T})$ and

$p_2 : EWT \times_{WT} (ET \times \bar{T}) \rightarrow EWT \times_{WT} (\bar{T})$

denote certain (obvious maps).

The second dg-category:I, Theorem V

Henceforth, we will denote the sheaf of dgas $Rp_{1*}p_2^*(R\psi_*(\mathbb{Q}))$ on $\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}})$ by \mathcal{A} .

The functor $Rp_{1*} \circ p_2^*$ induces equivalences of categories:

$$D_{\text{cart},c}^+(\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}}), \mathcal{A}) \simeq D_{\text{WT},c}^+(\text{EWT} \times_{\text{WT}} \bar{\text{T}}, R\psi_*(\mathbb{Q})) \text{ and}$$

$$D_{\text{cart},c}^{+,o}(\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}}), \mathcal{A}) \simeq D_{\text{WT},c}^{+,o}(\text{EWT} \times_{\text{WT}} \bar{\text{T}}, R\psi_*(\mathbb{Q}))$$

where $D_{\text{cart},c}^{+,o}(\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}}), \mathcal{A})$ is the full subcategory of $D(\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}}), \mathcal{A})$ generated by the functor $Rp_{1*}p_2^*$ applied to the generators $\{R\psi_*(j_{!}j^*(\mathbb{Q}|_{\mathcal{O}})) \mid \mathcal{O} \text{ a WT-orbit on } \bar{\text{T}}\}$ of the subcategory $D_{\text{WT},c}^{+,o}(\bar{\text{T}}, R\psi_*(\mathbb{Q}))$.

The second dg-category:II

Corollary

The t -structures on the derived category on the left is obtained by transferring the t -structures on the derived category on the right. Moreover,

$$Rp_{1*}p_2^*(R\psi_*(IC^{\tilde{N}}(\mathcal{O}_{\tilde{N}}))) = IC^{W,T}(\mathcal{O}_{WT}) \otimes Rp_{1*}p_2^*(R\psi_*(\mathbb{Q})).$$

where $IC^{W,T}(\mathcal{O}_{WT})$ denotes the same equivariant intersection cohomology complex $IC^{WT}(\mathcal{O}_{WT})$ for the action WT , but viewed as an object on $EW \times_{\mathbb{W}} (ET \times_{\mathbb{T}} \bar{T})$.

The third dg category:I

Let $\bar{\bar{T}}/T$ denote the space whose points correspond to the T -orbits on \bar{T} equipped with the topology defined by the closure relations on the T -orbits on \bar{T} . Let $\pi : ET \times_{\bar{T}} \bar{T} \rightarrow \bar{\bar{T}}/T$ denote the obvious map. The fact that the action of T on \bar{T} is equivariant with respect to the W -action shows that one has an induced action of W on the set of T -orbits on \bar{T} . Denoting this W -action on $\bar{\bar{T}}/T$ also by \circ , observe that $w \circ [x] = [w \circ x]$, for any $w \in W$ and $x \in \bar{T}$, where $[x]$ denotes the T -orbit of x . Now W acts on both $ET \times_{\bar{T}} \bar{T}$ and $\bar{\bar{T}}/T$.

Lemma

Assume the above situation. Then, with respect to the above actions, the map π is W -equivariant.

The third dg-category:II

We define $D_{\text{cart},c}^{+,o}(\text{EW}_W \times (\bar{T}/T), R\pi_*(\mathcal{A}))$ to be the full subcategory of $D(\text{EW}_W \times (\bar{T}/T), R\pi_*(\mathcal{A}))$ defined by applying the functor $R\pi_*$ to the generators of the derived category $D_{\text{cart},c}^{+,o}(\text{EW}_W \times (\text{ET}_T \times \bar{T}), \mathcal{A})$ as in Corollary ?.

One may now define

$$R\pi_* : D_{\text{cart},c}^{+,o}(\text{EW}_W \times (\text{ET}_T \times \bar{T}), \mathcal{A}) \rightarrow D_{\text{cart},c}^{+,o}(\text{EW}_W \times (\bar{T}/T), R\pi_*(\mathcal{A}))$$

by once again making use of the canonical Godement resolutions.

The third dg-category:III

A left derived functor

$$L\pi^* : D_{\text{cart},c}^{+,o}(\mathbb{E}W \times_{\mathbb{W}} (\bar{\mathbb{T}}/\mathbb{T}), R\pi_*(\mathcal{A})) \rightarrow D_{\text{cart},c}^{+,o}(\mathbb{E}W \times_{\mathbb{W}} (\mathbb{E}\mathbb{T} \times_{\mathbb{T}} \bar{\mathbb{T}}), \mathcal{A})$$

may be defined by taking flat-resolutions of $M \in D_{\text{cart},c}^+(\mathbb{E}W \times_{\mathbb{W}} (\bar{\mathbb{T}}/\mathbb{T}), R\pi_*(\mathcal{A}))$ and letting

$$L\pi^*(M) = \mathcal{A} \otimes_{\pi^{-1}R\pi_*(\mathcal{A})}^L \pi^{-1}(M).$$

Theorem VI

(i) The natural transformations $K \rightarrow R\pi_* L\pi^*(K)$, $K \in D_{\text{cart},c}^{+,o}(\text{EW}_W \times (\text{ET} \times \bar{\text{T}}), \mathcal{A})$ and $L\pi^* R\pi_*(L) \rightarrow L$ for any $L \in D_{\text{cart},c}^{+,o}(\text{EW}_W(\bar{\text{T}}/\text{T}), R\pi_*(\mathcal{A}))$ are quasi-isomorphisms.

Therefore, the functors $R\pi_*$ and $L\pi^*$ induce an equivalence of categories.

(ii) The restriction functor $D_{G \times G, c}^b(\bar{G}) \rightarrow D_{\tilde{N}, c}^{b,o}(\bar{\text{T}})$ and the functors $R\psi_*$, p_2^* , Rp_{1*} and $R\pi_*$ induce equivalences of derived categories:

$$D_{G \times G, c}^b(\bar{G}) \simeq D_{\text{cart},c}^{+,o}(\text{EW}_W(\bar{\text{T}}/\text{T}), R\pi_*(\mathcal{A})), \text{ where}$$

$$\mathcal{A} = Rp_{1*} p_2^* R\psi_*(\mathbb{Q}).$$

Theorem VI (contd)

Here $D_{cart,c}^{+,o}(\mathbb{E}W \times_{\mathbb{W}}(\bar{\mathbb{T}}/\mathbb{T}), R\pi_*(\mathcal{A}))$ denotes the full subcategory of $D^+(\mathbb{E}W \times_{\mathbb{W}}(\bar{\mathbb{T}}/\mathbb{T}), R\pi_*(\mathcal{A}))$ generated by applying the functor $R\pi_*Rp_{1*}p_2^*$ to the generators of the subcategory $D_{WT,c}^{+,o}(\bar{\mathbb{T}}, R\psi_*(\mathbb{Q}))$.

The t -structure on the derived category on the left is the t -structure whose heart consists of $G \times G$ -equivariant perverse sheaves on \bar{G} , while the corresponding t -structure on the derived category on the right is obtained by transferring the t -structure from the derived category on the left.

Formality of the dgas, $R\tilde{\psi}_*(\mathbb{Q})$, \mathcal{A} and $R\pi_*(\mathcal{A})$

We next consider the commutative diagram:

$$\begin{array}{ccccc}
 E\tilde{N} \times \bar{T} & \xrightarrow{\alpha} & B\tilde{N} & \xrightarrow{\beta} & \text{BWdiag}(T) \\
 \downarrow \psi & & \downarrow \bar{\psi} & & \downarrow \tilde{\psi} \\
 EWT \times_{WT} \bar{T} & \xrightarrow{\bar{\alpha}} & BWT & \xrightarrow{\bar{\beta}} & BW
 \end{array}$$

Formality of the dgas, $R\tilde{\psi}_*(\mathbb{Q})$, \mathcal{A} and $R\pi_*(\mathcal{A})$:II

Proposition

(i) *The sheaf of commutative dgas, $R\tilde{\psi}_*(\mathbb{Q})$ may be replaced by a sheaf of commutative dgas that is quasi-isomorphic to $R\tilde{\psi}_*(\mathbb{Q})$ as a sheaf of dgas, is W -equivariant and formal as sheaf of commutative dgas and whose dga-structure is compatible with the W -action, with the quasi-isomorphisms defining the formality as sheaves of dgas compatible with the W -action.*

(ii) *The corresponding statements also hold for $R\psi_*(\mathbb{Q})$ on $\text{EWT} \times_{\text{WT}} \bar{\text{T}}$, \mathcal{A} on $\text{EW} \times_{\text{W}} (\text{ET} \times_{\text{T}} \bar{\text{T}})$ and $R\pi_*(\mathcal{A})$ on $\text{EW} \times_{\text{W}} (\bar{\text{T}}/\text{T})$.*

Formality of the dgas, $R\tilde{\psi}_*(\mathbb{Q})$, \mathcal{A} and $R\pi_*(\mathcal{A})$: III

The composite map

$\bar{\beta} \circ \bar{\alpha} \circ p_2 : \text{EWT}_{\text{WT}} \times (\text{ET} \times \bar{\text{T}}) \rightarrow \text{EWT}_{\text{WT}} \times \bar{\text{T}} \rightarrow \text{BW}$ factors also

as the composite map $\phi \circ \pi \circ p_1$, where $\phi : \text{EW} \times (\bar{\text{T}}/\text{T}) \rightarrow \text{BW}$

is the map induced sending all of $\bar{\text{T}}/\text{T}$ to a point, and p_i , $i = 1, 2$ are the maps defined earlier. Therefore,

$$\begin{aligned} \mathcal{A} &= Rp_{1*}p_2^*(\bar{\alpha}^*\bar{\beta}^*(R\tilde{\psi}_*(\mathbb{Q}))) = Rp_{1*}p_1^*\pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})) \\ &= \pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})) \otimes_{\mathbb{Q}} Rp_{1*}(\mathbb{Q}) = \pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})). \end{aligned}$$

Formality of the dgas, $R\tilde{\psi}_*(\mathbb{Q})$, \mathcal{A} and $R\pi_*(\mathcal{A})$: IV

The last equality follows from the observation that $Rp_{1*}(\mathbb{Q}) \simeq \mathbb{Q}$. Since $R\tilde{\psi}_*(\mathbb{Q})$ is formal, we may in fact replace this by its cohomology. Therefore, one may also observe that

$$R\pi_*(\mathcal{A}) = \phi^*(R\tilde{\psi}_*(\mathbb{Q})) \otimes_{\mathbb{Q}} R\pi_*(\mathbb{Q}).$$

We will denote $\phi^*(R\tilde{\psi}_*(\mathbb{Q}))$ by $\bar{\mathcal{A}}$.

Next recall that we considered a chain of equivalences of derived categories discussed above.

Correspondences under the Chain of equivalences

If \mathcal{O} denotes an orbit for the $G \times G$ -action on \bar{G} , and $\mathrm{IC}^{G \times G}(\mathcal{O})$ denotes the corresponding equivariant intersection cohomology complex extending the constant sheaf \mathbb{Q} on the orbit \mathcal{O} , then it corresponds to $\mathrm{IC}^{\tilde{N}}(\mathcal{O}_{\tilde{N}})$ under the equivalence of derived categories in Theorem III. Here $\mathcal{O}_{\tilde{N}}$ is the \tilde{N} -orbit on \bar{T} corresponding to the $G \times G$ -orbit \mathcal{O}_G .

Correspondences under the Chain of equivalences: Theorem VII

(i) Under the equivalence of equivariant derived categories considered in Theorem IV, the equivariant intersection cohomology complex $IC^{\tilde{N}}(\mathcal{O}_{\tilde{N}})$ corresponds to $IC^{WT}(\mathcal{O}_{WT}) \otimes R\psi_*(\mathbb{Q})$, where \mathcal{O}_{WT} denotes the same orbit $\mathcal{O}_{\tilde{N}}$ of \tilde{N} viewed as an orbit for the induced action of WT.

(ii) Under the equivalence of derived categories provided by Theorem V, the complex $IC^{WT}(\mathcal{O}_{\tilde{N}}) \otimes R\psi_*(\mathbb{Q})$ corresponds to $IC^{W,T}(\mathcal{O}_{W,T}) \otimes \mathcal{A}$.

(iii) Under the equivalence of derived categories provided by Theorem VI,

$$R\pi_*(IC^{W,T}(\mathcal{O}_{W,T}) \otimes \mathcal{A}) = R\pi_*(IC^{W,T})(\mathcal{O}_{W,T}) \otimes \bar{\mathcal{A}}.$$

Key idea for the rest of the proof of Main Theorem: II

$\mathcal{B}_{G \times G}(\bar{G})$: the corresponding dg-algebra for the equivariant derived category $D_{G \times G, c}^b(\bar{G})$ as in Main Theorem II. (Recall the complex L that shows up in its definition is the sum of the equivariant intersection cohomology complexes $IC^G(\mathcal{O})$ on the orbit-closures for the $G \times G$ -action starting with the constant sheaf \mathbb{Q} on the corresponding orbit.)

$\mathcal{B}_{\tilde{N}}(\bar{T})$ ($\mathcal{B}_{WT}(\bar{T})$): the corresponding dg-algebra for the action of \tilde{N} (WT , respectively) on \bar{T} .

$\mathcal{B}_{W, T}(\bar{T})$: the dg-algebra corresponding to the equivariant derived category $D_W(\bar{T}/T, R\pi_*(\mathcal{A}))$.

Key idea for the rest of the proof of Main Theorem: II (2)

In view of the fact that the equivalences of the above derived categories are induced by functors between these categories, it follows that the corresponding dg-algebras are all quasi-isomorphic as dg-algebras, compatibly with the product structure, which is given by composition.

Therefore, in order to prove the formality of the dg-algebra $\mathcal{B}_{G \times G}(\bar{G})$, it suffices to prove the formality of the dg-algebra $\mathcal{B}_{W, T}(\bar{T})$.

Projective resolutions

Let Y' denote a T -orbit on \bar{T} . Let $W_{Y'}$ denote the stabilizer of Y' in W . Then $P = \bigoplus_{w \in W/W_{Y'}} j_{wY'}! j_{wY'}^*(\mathcal{A})$ is a projective object in $C_{\text{cart},c}^{+,o}(\mathbb{E}W \times_{\mathbb{W}} (\mathbb{E}T \times_{\mathbb{T}} \bar{T}), \mathcal{A})$ in the sense that $\text{Hom}_{\mathcal{A}}(P, \quad)$ preserves quasi-isomorphisms in the second argument.

Similarly, $P = \bigoplus_{w \in W/W_{Y'}} j_{wY'/T}! j_{wY'/T}^*(R\pi_*(\mathcal{A}))$ is a projective object in $C_{\text{cart},c}^{+,o}(\mathbb{E}W \times_{\mathbb{W}} (\bar{T}/T), R\pi_*(\mathcal{A}))$.

(ii) Every sheaf of dg-modules over $\mathcal{H}^*(\mathcal{A})$ has a bounded above resolution by projectives as in (i). The same holds for sheaves of dg-modules over $\mathcal{H}^*(R\pi_*(\mathcal{A}))$.

Grouping the T -orbits into WT -orbits

For this we consider the projective resolution provided above. Let \mathcal{O}_{WT} denote a WT -orbit on \bar{T} . Then \mathcal{O}_{WT} is a disjoint union of T -orbits on \bar{T} permuted under the action of W : therefore, we denote $\mathcal{O}_{WT} = W \cdot \mathcal{O}_T$.

Observe also that the dg-algebra $R\pi_*(\mathcal{A})$ being formal can be viewed as a dg-algebra over $H^*(BT)$. Recall that $IC^{W,T}(\mathcal{O}_{WT})$ denotes the equivariant intersection cohomology complex on the WT -orbit \mathcal{O}_{WT} for the action of the group WT (and extending the constant sheaf \mathbb{Q} on \mathcal{O}_{WT}), but viewed as a complex on $E_W \times (E_T \times \bar{T})$.

Special projective resolutions: I

Proposition

(i) Let $x \in \bar{T}^T$ denote a fixed point. Let W_x denote the stabilizer of x in W and $Wx = W/W_x$ the corresponding W -orbit. Then denoting the sum $\bigoplus_{w \in W/W_x} R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) \otimes \mathcal{A})_{wx}$ by $R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) \otimes \mathcal{A})_{Wx}$, we obtain:

$$R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) \otimes \mathcal{A})_{Wx} = \bigoplus_{w \in W/W_x} R\pi_*(\mathrm{WIC}^T(\mathcal{O}_T))_{wx} \otimes_{\mathbb{Q}} \bar{\mathcal{A}}$$

where $\mathrm{WIC}^T(\mathcal{O}_T)$ denotes a sum of intersection cohomology complexes $\bigoplus \mathrm{IC}^T(\mathcal{O}_T)$ where the sum varies over all the disjoint W -translates of the T -orbit \mathcal{O}_T . Therefore, the cohomology $H^*(R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) \otimes \bar{\mathcal{A}})_{Wx})$ forms a projective module over $H^*(BT) \otimes H^*(\bar{\mathcal{A}})$.

Special projective resolutions:II

Proposition

(ii) Every object $R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ has a projective resolution $\{\dots \rightarrow P^{-n} \rightarrow P^{-n+1} \rightarrow \dots \rightarrow P^0\}$ in $D_{\mathrm{W}}(\bar{\mathrm{T}}/\mathrm{T}, \mathbb{H}^*(R\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}))$ so that the given augmentation $P^0 \rightarrow R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ is a quasi-isomorphism at each stalk of the form $\mathrm{W}x$, $x \in \bar{\mathrm{T}}^{\mathrm{T}}$, it induces a surjection at each stalk $\mathrm{W}x$, $x \in \bar{\mathrm{T}}$ and each P^{-i} is of the form

$$\bigoplus_{w \in \mathrm{W}/\mathrm{W}} j_{U_{wx}}! j_{U_{wx}}^*(H^*(R\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}))[n_{U_{wx}}]$$

as U_{wx} varies over neighborhoods of points $wx \in \bar{\mathrm{T}}/\mathrm{T}$ and where $n_{U_{wx}}$ are integers.

(iii) The complexes P^i , for $i < 0$, are supported at points in $\bar{\mathrm{T}} - \bar{\mathrm{T}}^{\mathrm{T}}$ and hence viewed as modules over $\mathbb{H}^*(\mathrm{BT}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{H}^*(\bar{\mathcal{A}})$ are torsion.

Some remarks on the above proposition

(i) is a straight-forward calculation making use of the following observation. If \mathcal{O}_T denotes an orbit of T on \bar{T} , and $W \cdot \mathcal{O}_T = \mathcal{O}_{WT}$ denotes the orbit for the corresponding WT -action, then

$$\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) = \bigoplus_{w \in W/W_{\mathcal{O}_T}} \mathrm{IC}^T(\mathcal{O}_T),$$

where $W_{\mathcal{O}_T}$ denotes the stabilizer of the \mathcal{O}_T -orbit in W . Applying $R\pi_*$, we therefore obtain:

$$R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}) \otimes \mathcal{A}) = R\pi_*(\mathrm{WIC}^T(\mathcal{O}_T)) \otimes_{\mathbb{Q}} \bar{\mathcal{A}}.$$

Then take the stalks at w_x and the sum $\bigoplus_{w \in W/W_x}$ to obtain the formula in (i).

Some remarks on the above proposition (contd)

Next use the fact that the global equivariant intersection cohomology of a projective toric variety is a free module over the cohomology ring of the classifying space of the torus, and the stalk cohomology of the intersection cohomology complex at a T -fixed point on \bar{T} is isomorphic to the global intersection cohomology of the link at that point: see [(4.0.4) Theorem and (4.2.2), Lu95] and also [Theorem 1.1, BJ04]. This completes the proof of (i).

Then (ii) is an immediate consequence of (i), the projective resolutions (considered earlier) and the formality of the dgas $R\pi_*(\mathcal{A}) = R\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}$. (ii) shows that the augmentation $P^0 \rightarrow R\pi_*(\mathrm{IC}^{\mathrm{W},T}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ induces a quasi-isomorphism at every Wx , for every $x \in \bar{T}^T$. Therefore, it follows that each $\mathcal{H}^i(P^\bullet)$, for $i < 0$, are torsion modules over $H^*(\mathrm{BT}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(\bar{\mathcal{A}})$.

The formality of the dg-algebra $\mathcal{B}_{W,T}(\bar{T})$

Theorem

Assume next that the toroidal imbedding \bar{G} and therefore, the toric variety \bar{T} is projective. Then (i) the dg algebra $\mathcal{B}_{W,T}(\bar{T})$ is formal and (ii) $\mathcal{B}_{W,T}(\bar{T}) = (\mathcal{B}_T(\bar{T}) \otimes R\tilde{\psi}_(\mathbb{Q}))^W$.*

Some remarks on the proof:I

(i) Let $P^\bullet \rightarrow \bigoplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ denote a projective resolution, where each P^{-i} is a sum of terms of the form as in the above Proposition, and where the sum varies over all the WT-orbits in $\bar{\mathrm{T}}$. Observe therefore that the differentials of each P^{-i} are in fact trivial. Now $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}})$ identifies with the total complex of $\mathrm{Hom}(P^\bullet, P^\bullet)$, where the differentials of P^\bullet (i.e. $\{d : P^n \rightarrow P^{n+1} | n\}$) provide the structure of a chain-complex on $\mathrm{Hom}(P^\bullet, P^\bullet)$.

Some remarks on the proof:II

(ii) A *key observation* we make is that the stalk of $\mathcal{H}^*(\tilde{\mathcal{B}}_{W,T}(\bar{T}))$ (at the base point in BW) is a torsion-free module over $H^*(BT, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(R\tilde{\psi}_*(\mathbb{Q}))$. When \bar{G} and hence \bar{T} are projective, this is clear in view of the identification of the stalks of

$$R\phi_*(\mathcal{R}\mathcal{H}om_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(IC^{W,T}(\mathcal{O}_{WT})), \oplus R\pi_*(IC^{W,T}(\mathcal{O}_{WT}))))$$

at the base point in BW with the dg-algebra $\mathcal{B}_T(\bar{T})$, along with the observation that $\bar{\mathcal{A}} = \phi^*(R\tilde{\psi}_*(\mathbb{Q}))$.

Some remarks on the proof:III

Therefore, the properties of the special projective resolutions show that the cohomology of the sheaf of dg-algebras $\tilde{\mathcal{B}}_{W,T}(\bar{T})$, which identifies with the cohomology of the complex $\mathcal{H}om(P^\bullet, P^\bullet)$, vanishes in all degrees except 0. (Here we are using the observation that each P^{-i} has trivial differential so that the spectral sequence for the total complex for $\mathcal{H}om(P^\bullet, P^\bullet)$ degenerates.) Now the diagram of dgas

$$\mathcal{H}^0(\tilde{\mathcal{B}}_{W,T}(\bar{T})) \leftarrow \sigma_{\leq 0}(\tilde{\mathcal{B}}_{W,T}(\bar{T})) \rightarrow \tilde{\mathcal{B}}_{W,T}(\bar{T})$$

(where $\sigma_{\leq 0}$ is the functor that kills the above cohomology in negative degrees) shows that the sheaf of dg-algebras $\tilde{\mathcal{B}}_{W,T}(\bar{T})$ is formal on BW. Finally one simply takes W-invariants, observing that taking W-invariants is an exact functor since we are working with rational coefficients.

Half the obstructions vanish for many spherical varieties

We recall that a well-known theorem of Kadeishvili shows that there are a countable number of obstructions m_i , $i = 3, 4, \dots$, that need to vanish for an A_∞ -dg-algebra to be formal.

An A_∞ -dg-algebra is dg-algebra that is coherently homotopy associative. Implicit in the following theorem is the fact that we make use of the cohomology notation for intersection cohomology complexes, where we start with a local system \mathcal{L} or the constant sheaf \mathbb{Q} on the smooth part of a stratified variety: in [BBD82], they start with $\mathcal{L}[d]$ ($\mathbb{Q}[d]$, respectively) where d is the dimension of the open stratum of the stratified variety.

Theorem VIII

- (i) Suppose the A_∞ -dg-algebra \mathcal{B} has the property that $H^i(\mathcal{B}) = 0$ for all *odd* i . Then all the obstructions m_i , for i odd and $i \geq 3$ vanish.
- (ii) Suppose X is a projective G -spherical variety for a connected reductive group G over a field of characteristic 0. Assume further that X is a *simply-connected spherical variety* in the sense of [BJ04], i.e. the stabilizers in G and B are connected at all points on X , where B is a Borel subgroup of G . Then the conclusions of (i) hold for the dg-algebra $\mathcal{B}_G(X)$ considered in Main Theorem I.

Different models of the Borel construction: the simplicial model

Here $EG \times_G X$ is the simplicial variety given in degree n by $G^n \times X$, with the structure maps induced as follows: $G \times X \rightarrow X$ are the two maps given by the group action and projection to the second factor while, the map $X \rightarrow G \times X$ is given by the map sending $x \in X$ to (e, x) .

Key advantages: This is clearly functorial in X with respect to arbitrary G -equivariant maps.

For example, in section 3, in the context of the analysis of the toroidal imbedding case, it becomes necessary to relate the equivariant derived categories associated to two groups, G and H , where one is provided with a surjective homomorphism $G \rightarrow H$. This is quite difficult and nearly impossible if one uses the geometric models (below).

Geometric models

Here $EG^{\text{gm},m}$ is any space on which G acts so that (i) the action is free and (ii) $EG^{\text{gm},m}$ is *acyclic* up to degree m , i.e. it is simply connected and its (singular) homology groups vanish through degree m .

Then $EG^{\text{gm},m} \times_G X$ is an approximation to the Borel construction by a space. In fact we can choose $EG^{\text{gm},m}$ to be an open subset of a sufficiently high dimensional representation of G , so that the complementary closed subset has high codimension.

Advantage: the approximations are indeed varieties and therefore, much of the machinery from the non-equivariant framework adapts easily. In addition, the geometric models seem to be more popular in the literature. Moreover, the conjecture we are considering in this paper is originally stated in terms of the geometric models.

The comparison

We provide this comparison only for actions of complex reductive groups on complex algebraic varieties, though similar results hold in positive characteristics, using étale cohomology techniques. Here $BG^{\text{gm},m}$ is a degree- m approximation to the classifying space for G and $EG^{\text{gm},m}$ denotes the universal principal G -bundle over $BG^{\text{gm},m}$. Recall this means $U_m = EG^{\text{gm},m}$ is an open G -stable subvariety of a representation W_m of G , so that (i) G acts freely on U_m and a geometric quotient U_m/G exists as a variety and (ii) so that in the family $\{(W_m, U_m) | m \in \mathbb{N}\}$, the codimension of $W_m - U_m$ in W_m goes to ∞ as m approaches ∞ . BG will denote a simplicial model for the classifying space of G and EG will denote its universal principal G -bundle.

The comparison theorem

For each fixed $m \geq 0$, we obtain the diagram of simplicial varieties (where p_1 is induced by the projection $EG^{\text{gm},m} \times X \rightarrow X$ and p_2 is induced by the projection $EG \times (EG^{\text{gm},m} \times X) \rightarrow EG^{\text{gm},m} \times X$):

$$\begin{array}{ccc}
 & EG \times_G (EG^{\text{gm},m} \times X) & \\
 p_1 \swarrow & & \searrow p_2 \\
 EG \times_G X & & EG^{\text{gm},m} \times_G X
 \end{array}$$

(i) For each finite interval $I = [a, b]$ of the integers, with $2m - 2 \geq b - a$,

$$p_1^* : D_G^I(EG \times_G X) \rightarrow D_G^I(EG \times_G (EG^{\text{gm},m} \times X)) \text{ and}$$

$$p_2^* : D_G^I(EG^{\text{gm},m} \times_G X) \rightarrow D_G^I(EG \times_G (EG^{\text{gm},m} \times X))$$

The comparison theorem (contd)

are equivalences of categories. (Here the superscript I denotes the full subcategory of complexes whose cohomology sheaves vanish outside of the interval I and subscript G denotes the full subcategories of complexes whose cohomology sheaves are G -equivariant.)

Moreover, both the functors p_1^* and p_2^* send complexes that are mixed and pure to complexes that are mixed and pure. There exists an equivalence of derived categories:

$$D_G^b(EG^{\text{gm},m} \times_G X) \simeq D_G^b(EG \times_G X)$$

which is natural in X and G .

(ii) Moreover, both the maps p_i , $i = 1, 2$, induce isomorphisms on the fundamental groups.

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