# Equivariant Riemann-Roch for G-quasi-projective varieties

**By**

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Let $G$ denote a complex linear algebraic group. In this paper we establish an equivariant Riemann-Roch theorem valid for the category of $G$-quasi-projective complex varieties. We show that this is particularly suitable for the construction of modules over the Hecke-algebra and work on the p-adic Kazhdan-Lusztig conjecture. We wish to acknowledge [T-3] section 5 as the source of our basic technique; however the equivariant Riemann-Roch theorems of [T-3] are valid only modulo $l^\nu$ ($l$ being a prime) even for complex varieties, while many of our results are valid integrally.

We devote the first section to a quick review of equivariant K-theory, where we recall the definitions and main functorial properties of all the distinct versions of equivariant K-theory we use in the rest of the paper. We consider equivariant algebraic, Atiyah-Segal and topological K-theories and conclude with a preliminary Riemann-Roch theorem.

In the second section we define equivariant homology (with locally compact supports) and provide an equivariant Chern-character and an equivariant Todd-homomorphism into these from (local) equivariant topological K-cohomology. We combine the Riemann-Roch theorem of section 1 with these to provide an equivariant Riemann-Roch via (local) equivariant topological K-cohomology. In the third section, we sketch the construction of equivariant Atiyah-Segal and topological K-homology defined intrinsically. The fourth section contains the final form of the equivariant Riemann-Roch theorem in terms of these $K$-homology theories. The fifth section contains some applications: we discuss various forms of the convolution operation that appear in work on the p-adic Kazhdan-Lusztig conjecture as well as in future applications to representations of quantum groups - see [J-2] and [J-4]. The sixth section contains a discussion of the technique of reduction to a torus as well as the generic slices of Thomason. An appendix develops some background material on equivariant Fredholm complexes.

## 1. Review of Equivariant K-theory

We begin by recalling the definitions of equivariant K-cohomology.

(1.1) *Equivariant algebraic K-theory.* Let $Z$ denote a complex quasi-projective variety provided with the action of a complex linear algebraic group $G$. Now $Mod_{qcoh}(Z)$ ($Mod_{coh}(Z)$, $Mod_{l.f}(Z)$) will denote the category of all quasi-coherent $\mathcal{O}_Z$-modules (coherent $\mathcal{O}_Z$-modules, coherent and locally-free

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$\mathcal{O}_Z$-modules respectively ). A $G$-equivariant quasi-coherent $\mathcal{O}_X$-module will denote a quasi-coherent $\mathcal{O}_X$-module with a $G$-action as in [M-F-K] chapter 1, section 3 or [T-1] (1.2). $Mod^G_{qcoh}(Z)$ ($Mod^G_{coh}(Z)$, $Mod^G_{l.f}(Z)$) will denote the full subcategory of $G$-equivariant quasi-coherent (coherent, coherent and locally-free respectively ) $\mathcal{O}_Z$-modules. If $\mathcal{A}$ denotes any of the symmetric monoidal categories above, $K(\mathcal{A})$ will denote the algebraic K-theory spectrum associated to $\mathcal{A}$. $K_G(Z) = K(Mod^G_{l.f}(Z))$ is the corresponding cohomology theory. $K^0_G(Z) = \pi_0(K_G(Z))$. It is often convenient to identify the above group with the Grothendieck group of bounded complexes in $Mod^G_{l.f}(Z)$; the latter will be denoted $K^0(C_b(Mod^G_{l.f}(Z)))$. (See [SGA-6] for the definition of the Grothendieck group of a triangulated category.)

(1.2) *Equivariant Atiyah-Segal K-theory.* Let $M$ denote a maximal compact subgroup of $G$ and let $K^{A.S}_M(Z(\mathbb{C}))$ denote the $M$-equivariant spectrum of $M$-equivariant continuous maps from $Z(\mathbb{C})$ into the classifying $M$-spectrum of complex vector spaces with $M$-action. (See [Wa] 3.36, 3.41 and also [T-3] p.626 .) The space $Fredh(H_M)$ in (A.3) of the appendix is the 'zero-th space' of the above classifying spectrum. Let $Fredh^M(Z(\mathbb{C}))$ denote the category of $M$-equivariant Fredholm complexes on $Z(\mathbb{C})$ and $M$-equivariant continuous maps as in (A.1) of the appendix. Observe (see (A.5)) that sending an equivariant Fredholm complex to its classifying map induces an isomorphism

(1.3) $\pi_0(K^{A.S}_M(Z(\mathbb{C}))) \simeq K^0(Fredh^M(Z(\mathbb{C})))$

where the right hand side is the group defined as in (A.3.2) (with $Y$ empty). (See [Seg-2] section 5 and the appendix.)

One may observe (see (A.2)) that the obvious functor sending an algebraic vector bundle to the associated topological vector bundle induces a map

(1.4) $\rho : K^0_G(Z) = K^0(C_b(Mod^G_{l.f}(Z))) \rightarrow \pi_0(K^{A.S}_M(Z(\mathbb{C})))$

(1.5) *Equivariant topological K-theory.* Assume the situation in (1.2). Let $|EM \underset{M}{\times} Z(\mathbb{C})|$ be the space obtained as the realization of the simplicial space $EM \underset{M}{\times} Z(\mathbb{C})$ provided by the bar-construction. Equivariant topological K-theory will be the non-equivariant complex K-theory of this space; i.e. $K^{top}_M(Z(\mathbb{C})) = K^{top}(|EM \underset{M}{\times} Z(\mathbb{C})|)$ = the function spectrum of maps from the suspension spectrum of $|EM \underset{M}{\times} Z(\mathbb{C})|_+$ to $KU$, the spectrum representing complex K-theory. $K^{top,0}_M(Z(\mathbb{C})) = \pi_0(K^{top}_M(Z(\mathbb{C})))$. By [Seg-2] section 5, one may indentify $K^{top,0}_M(Z(\mathbb{C}))$ with $K^0(Fredh(|EM \underset{M}{\times} Z(\mathbb{C})|))$, where $Fredh(|EM \underset{M}{\times} Z(\mathbb{C})|)$ denotes the category of Fredholm complexes on $|EM \underset{M}{\times} Z(\mathbb{C})|$ and where the group $K^0(Fredh(|EM \underset{M}{\times} Z(\mathbb{C})|)$ is defined as in (A.3.2) (with $X$ replaced by $|EM \underset{M}{\times} Z(\mathbb{C})|$, $Y$ empty and the group $M$ trivial).

Observe also that there is an obvious functor (which we denote by $B$) that sends an $M$-equivariant vector bundle $\mathcal{E}$ on $Z(\mathbb{C})$ to the vector bundle $EM \underset{M}{\times} \mathcal{E}$ on the space $|EM \underset{M}{\times} Z(\mathbb{C})|$. Corollary (A.5)(ii) is a generalization of this applied to Fredholm complexes and shows :

(1.6) there exists a functor $B$ that sends any $M$-equivariant Fredholm complex $\mathcal{E}^{\cdot}$ on $Z(\mathbb{C})$ to a Fredholm complex $EM \underset{M}{\times} \mathcal{E}^{\cdot}$ on the space $EM \underset{M}{\times} Z(\mathbb{C})$. It is clear that the functor $B$ induces a map:

(1.6') $B : \pi_0(K_M^{A.S}(Z(\mathbb{C}))) \cong K^0(Fredh^M(Z(\mathbb{C}))) \rightarrow K^0(Fredh(|EM \underset{M}{\times} Z(\mathbb{C})|))$

$\overset{\cong}{\rightarrow} K^{top,0}(|EM \underset{M}{\times} Z(\mathbb{C})|)$

(1.6") This map may be extended to a map of spectra $K_M^{A.S}(Z(\mathbb{C})) \rightarrow K_M^{top}(Z(\mathbb{C}))$ by using the natural weak-equivalence: $K_M^{top}(Z(\mathbb{C})) = K^{top}(|EM \underset{M}{\times} Z(\mathbb{C})|) \simeq K_M^{A.S}(|EM| \times Z(\mathbb{C}))$ along with the $M$-equivariant projection $|EM \times Z(\mathbb{C})| \rightarrow Z(\mathbb{C})$. ($M$ acts diagonally on the product $|EM| \times Z(\mathbb{C})$.)

(1.7) *Equivariant cohomology.* Assume the above situation. Now the equivariant cohomology of $Z$ with rational coefficients will be $H^*(|EM \underset{M}{\times} Z(\mathbb{C})|;\ \mathbb{Q})$. This will be denoted $H_M^*(Z(\mathbb{C});\ \mathbb{Q})$. (It will often be convenient to consider the corresponding function spectrum $Map(|EM \underset{M}{\times} Z(\mathbb{C})|_+;\ \underset{i}{\Pi} K(\mathbb{Q}, 2i))$ of maps from the suspension spectrum of $|EM \underset{M}{\times} Z(\mathbb{C})|_+$ to the generalized Eilenberg-Maclane spectrum $\underset{i}{\Pi} K(\mathbb{Q}, 2i)$. We will often abbreviate this to $Map^M(Z(\mathbb{C});\ \underset{i}{\Pi} K(\mathbb{Q}, 2i))$.)

(1.8) *Functoriality* . Let $f : X \rightarrow Y$ denote a $G$-equivariant map between $G$-quasi-projective varieties. Let $M$ denote a maximal compact subgroup of $G$. Assume $f$ also has finite tor dimension; for example $f$ is smooth or factors as the composition of a regular $G$-equivariant immersion followed by a $G$-equivariant smooth map. Now $f$ induces maps:

$f^* : K_G(Y) \rightarrow K_G(X)$, $f^* : K_M^{A.S}(Y(\mathbb{C})) \rightarrow K_M^{A.S}(X(\mathbb{C}))$, $f^* : K_M^{top}(Y(\mathbb{C})) \rightarrow K_M^{top}(X(\mathbb{C}))$ and

$f^* : H_M^*(Y(\mathbb{C});\ \mathbb{Q}) \rightarrow H_M^*(X(\mathbb{C});\ \mathbb{Q})$

These maps are compatible with the maps defined in (1.4), (1.6') as well as with the equivariant Chern-character defined in (2.7).

(1.9) In the rest of this section we proceed to consider the corresponding *cohomology theories with supports contained in a G-equivariant closed subvariety.* We will henceforth assume that $Z$ is *G-quasi-projective*; recall this means that $Z$ imbeds $G$-equivariantly as a locally-closed $G$-invariant subscheme of a projective space $\mathbb{P}^n$ on which $G$ acts linearly. If $G$ is connected, it follows by Sumihiro's theorem that any normal quasi-projective variety with a $G$-action is $G$-quasi-projective. (See [Sum] Theorem 1.) We will presently show that the assumption of connectedness on $G$ may be dropped if $Z$ is normal and quasi-projective. Next assume $G$ is a disconnected group acting linearly on a normal quasi-projective variety

$Z$ and that $i : Z \to \mathbb{P}^n = Proj(V)$ is a $G^o$-equivariant closed immersion into a large projective space as provided by Sumihiro's theorem. (Here $V$ is a suitably large projective space provided with a linear action by $G^o$.) Let $\bar{G} = G/G^o$; choose coset representatives $\{g_1, ..., g_m | g_i \in G\}$ so that $\bar{G} = \{g_1 G^o, ..., g_m G^o\}$. One may view $\underset{\bar{G}}{\oplus} V$ as an induced representation of $G$. Thus $G$ acts on $Proj(\underset{\bar{G}}{\oplus} V)$ linearly extending the action of $G^o$ on $Proj(V)$.

One may utilize the common formulae for the induced action of $G$ (see [C-R] pp. 228-29 for the case of finite groups) to define a $G$-action on $\underset{\bar{G}}{\Pi} Proj(V)$ that makes the obvious closed immersion of the latter into $Proj(\underset{\bar{G}}{\oplus} V)$ $G$-equivariant. i.e. Let $g \in G$, $1 \le i \le n$, so that $g.g_i = g_j.k_0$, for some $j$ and $k_0 \in G^o$. Now we let $(g.(x_{g_1}, ..., x_{g_m}))_{g_j} = k_0.x_{g_i}$, $(x_{g_1}, ..., x_{g_m}) \in \underset{\bar{G}}{\Pi} Proj(V)$. (This defines a $G$ action on $\underset{\bar{G}}{\Pi} Proj(V)$ making the imbedding into $Proj(\underset{\bar{G}}{\oplus} V)$ $G$-equivariant.)

Finally one may also define a closed immersion $I : Z \to \underset{\bar{G}}{\Pi} Proj(V)$, by $(I(x))_{g_j} = i(g_j^{-1}.x)$, $1 \le j \le m$. One checks readily that $I$ is $G$-equivariant with respect to the given action of $G$ on $Z$ and the above action on $\underset{\bar{G}}{\Pi} Proj(V)$. It is clear that the above arguments provide a $G$-equivariant closed immersion of $Z$ in $Proj(\underset{\bar{G}}{\oplus} V)$. (See also [T-3] p.629 where fragments of a similar argument are mentioned.) One may now readily observe that $Z$ imbeds as a closed $G$-invariant sub-variety of an open and $G$-invariant sub-variety $\tilde{Z}$ of $Proj(\underset{\bar{G}}{\oplus} V)$, which is clearly non-singular. Now one obtains the natural weak-equivalence (see [T-1] Theorems (2.7) and (5.7)):

(1.10) $K(Mod^G_{coh}(Z))$ is weakly-equivalent to $K_{G,Z}(\tilde{Z})$ =the canonical homotopy-fiber of the map $K(Mod^G_{l.f}(\tilde{Z})) \to K(Mod^G_{l.f}(\tilde{Z} - Z))$.

$K(Mod^G_{coh}(Z))$ will be denoted $K^G(Z)$ or $K_{G,Z}(\tilde{Z})$ henceforth. This is the equivariant K-homology theory in the algebraic setting; clearly it is independent of the imbedding of $Z$ in $\tilde{Z}$. The weak-equivalence $K_{G,Z}(\tilde{Z}) \xrightarrow{\simeq} K(Mod^G_{coh}(Z))$ will be denoted $P - L_a$ henceforth.

Under the same hypotheses, one may define $K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$ to be the canonical homotopy fiber of the natural map $K^{A.S}_M(\tilde{Z}(\mathbb{C})) \to K^{A.S}_M(\tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))$. We will define $K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$ to be the canonical homotopy-fiber of the map $K^{top}_M(\tilde{Z}(\mathbb{C})) \to K^{top}_M(\tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))$.

(1.11) **Lemma**. Let $Z$ be as before and let $Z \to Y_0 \xrightarrow{i} Y_1$ denote two $G$-equivariant closed immersions so that $Y_0$ and $Y_1$ are *smooth*. Now one obtains the commutative squares:

$$\begin{array}{ccc} \pi_0(K_{G,Z}(Y_0)) & \xleftarrow{\simeq} & \pi_0(K_{G,Z}(Y_1)) \\ \downarrow & & \downarrow \\ \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C}))) & \xleftarrow{\simeq} & \pi_0(K^M_{A.S,Z(\mathbb{C})}(Y_1(\mathbb{C}))) \end{array} \quad \text{and}$$

$$\begin{array}{ccc} \pi_i(K^{A.S}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C}))) & \overset{\simeq}{\longleftarrow} & \pi_i(K^{M}_{A.S,Z(\mathbb{C})}(Y_1(\mathbb{C}))) \\ \downarrow & & \downarrow \\ \pi_i(K^{top}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C}))) & \overset{\simeq}{\longleftarrow} & \pi_i(K^{M}_{top,Z(\mathbb{C})}(Y_1(\mathbb{C}))) \end{array} \qquad \text{for all } i$$

where the horizontal maps are all isomorphisms and are induced by the map $i^*$ as in (1.8).

*Proof.* The commutativity of the diagram follows from the compatibility of the map $i^*$ with respect to the maps defined in (1.4) and (1.6"). Excision shows the horizontal maps are all weak-equivalences.

(1.11') **Corollary** (*Poincaré duality*). If $Z$ is a *smooth* complex variety provided with a $G$-equivariant closed immersion in an ambient smooth $G$-variety $\tilde{Z}$, one obtains weak-equivalences:

$$K_{G,Z}(\tilde{Z}) \overset{\simeq}{\to} K_G(Z),\ K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \overset{\simeq}{\to} K^{A.S}_M(Z(\mathbb{C})) \text{ and } K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \overset{\simeq}{\to} K^{top}_M(Z(\mathbb{C})).$$

*Proof.* Take $Y_0 = Z$ and $Y_1 = \tilde{Z}$ in the above lemma.

Clearly the map $\rho$ in (1.4) induces a map

(1.12) $K_0(Mod^G_{coh}(Z)) \overset{\rho}{\to} \pi_0(K^M_{A.S}(Z(\mathbb{C})))$

Next observe that the map in (1.6") and the above definitions provide a map of spectra:

(1.12') $B : K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$

if $Z \to \tilde{Z}$ is a $G$-equivariant closed immersion into a smooth $G$-variety.

(1.13.1) Let $Z \overset{i_0}{\to} Z' \overset{i_1}{\to} \tilde{Z}$ denote two $G$-equivariant closed immersions of $G$-quasi-projective varieties so that $\tilde{Z}$ is also smooth. Now one obtains natural maps:

$$K_{G,Z}(\tilde{Z}) \overset{i_*}{\longrightarrow} K_{G,Z'}(\tilde{Z}),\ K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}) \overset{i_*}{\longrightarrow} K^{A.S}_{M,Z'(\mathbb{C})}(\tilde{Z}) \text{ and } K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}) \overset{i_*}{\longrightarrow} K^{top}_{M,Z'(\mathbb{C})}(\tilde{Z}).$$

(1.13.2). Let $Z$ denote a $G$-quasi-projective variety and let $\mathbb{P}^n$ denote a projective space with a linear $G$-action. Let $p : Z \times \mathbb{P}^n \to Z$ denote the obvious projection. Let $i : Z \to \tilde{Z}$ denote a $G$-equivariant closed immersion into a smooth $G$-quasi-projective variety. Now one may compute:

$$\pi_0(K_{G,Z\times\mathbb{P}^n}(\tilde{Z}\times\mathbb{P}^n)) \cong \underset{0\le q\le n}{\oplus} \pi_0(K_{G,Z}(\tilde{Z}))[-\mathcal{O}_{\mathbb{P}^n}(-q)],$$

$$\pi_i(K^{A.S}_{M,Z\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}\times\mathbb{P}^n(\mathbb{C}))) \cong \underset{0\le q\le n}{\oplus} \pi_i(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))[-\mathcal{O}_{\mathbb{P}^n}(-q)] \text{ and}$$

$$\pi_i(K^{top}_{M,Z\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}\times\mathbb{P}^n(\mathbb{C}))) \cong \underset{0\le q\le n}{\oplus} \pi_i(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))[-\mathcal{O}_{\mathbb{P}^n}(-q)].$$

See [T-3](5.6) for the first and [Seg-1](3.9) for the others. Therefore one may define

(1.13.3) $p_* : \pi_0(K_{G,Z\times\mathbb{P}^n}(\tilde{Z}\times\mathbb{P}^n)) \to \pi_0(K_{G,Z}(\tilde{Z}))$

to be the projection to the summand indexed by $\mathcal{O}_{\mathbb{P}^n}$. One may also define $p_*$ in a similar manner for the other theories above.

(1.14)**Preliminary Riemann-Roch**. Let $Z$ denote a $G$ quasi-projective variety provided with a $G$-equivariant closed immersion $i : Z \to \tilde{Z}$ into an ambient smooth $G$-quasi-projective variety.

(i) If the map $i$ factors through $i' : Z' \to \tilde{Z}$ (which is the $G$-equivariant closed immersion of another $G$-quasi-projective variety $Z'$), one obtains commutative squares:

$$\begin{array}{ccc} \pi_0(K_{G,Z}(\tilde{Z})) & \longrightarrow & \pi_0(K_{G,Z'}(\tilde{Z})) \\ \rho\downarrow & & \downarrow\rho \\ \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) & \longrightarrow & \pi_0(K^{A.S}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \end{array} \qquad \begin{array}{ccc} \pi_i(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \longrightarrow & \pi_i(K^{A.S}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \\ B\downarrow & & \downarrow B \\ \pi_i(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) & \longrightarrow & \pi_i(K^{top}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \end{array}$$

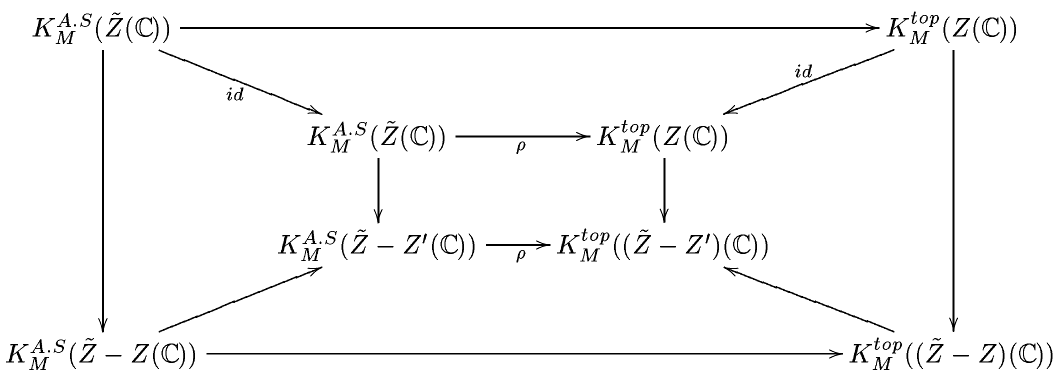
for all $i$.

(ii) Let $\mathbb{P}^n$ denote a projective space provided with a $G$-linear action. Now the projection $p : Z \times \mathbb{P}^n \to Z$ induces maps of commutative squares:

$$\begin{array}{ccc} \pi_0(K_{G,Z\times\mathbb{P}^n}(\tilde{Z}\times\mathbb{P}^n)) & \xrightarrow{p_*} & \pi_0(K_{G,Z}(\tilde{Z})) \\ \rho\downarrow & & \downarrow\rho \\ \pi_0(K^{A.S}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})) & \xrightarrow{p_*} & \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \end{array}$$

$$\begin{array}{ccc} \pi_i(K^{A.S}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C}))) & \xrightarrow{p_*} & \pi_i(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \\ B\downarrow & & \downarrow B \\ \pi_i(K^{top}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})) & \xrightarrow{p_*} & \pi_i(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \end{array}$$

*Proof.* Recall $\pi_0(K_{G,X}(\tilde{Z}))$ is the Grothendieck group of the category of bounded complexes in $Mod^G_{l.f}(\tilde{Z})$ with supports contained in $X$ and $\pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))$ is the group defined in (A.3) associated to the category of equivariant Fredholm complexes on $\tilde{Z}(\mathbb{C})$ with supports contained in $Z(\mathbb{C})$. There is a similar description of the groups associated to $Z'$. This proves the commutativity of the first square in (i). The commutativity of the second square in (i) follows by taking the homotopy fibers of the columns in the following diagram:




(ii). The commutativity of the squares in (ii) are clear by the computation in (1.13.2). This completes the proof of the proposition.

2. **Equivariant Homology (with locally compact supports): the equivariant Todd homomorphism**.

(2.1) Throughout this section $G$ will denote a complex linear algebraic group or a compact lie group and $X$ a $G$-quasi-projective complex variety. Let $BG.$ denote the classifying simplicial space associated to $G$ and given in degree $n$ by $G^n$ with the obvious structure maps. In this situation one may also consider the simplicial space $EG\underset{G}{\times}X$ in the usual manner (See [Fr]p. 4 for example.) Observe that $(EG\underset{G}{\times}X)_n = G^n \times X$ with the usual structure maps. Each of the face maps $d_i : (EG\underset{G}{\times}X)_n \to (EG\underset{G}{\times}X)_{n-1}$ is induced by the group-action $\mu : G \times X \to X$ and the projection $\pi_2 : G \times X \to X$. The obvious map $EG\underset{G}{\times}X \to BG$ induced by the map $X \to$Spec $\mathbb{C}$ will be denoted $\pi$. If $f : X \to Y$ is a $G$-equivariant map between two $G$-varieties, the induced map $EG\underset{G}{\times}X \to EG\underset{G}{\times}Y$ will be still denoted $f$.

Now assume $X.$ is a simplicial space (i.e. a simplicial object in the category of spaces), for example, one of the simplicial spaces obtained above. One puts a *Grothendieck topology* on $X.$ by defining the objects to be maps $u : U \to X_n$, where $u$ is the inclusion of an open set in $X_n$ for some $n$. Given two such open sets $u : U \to X_n$ and $v : V \to X_m$ for some $n$ and $m$, a map $\alpha : u \to v$ is given by a map $\alpha : U \to V$ that lies over a structure map $\alpha' : X_n \to X_m$ of the simplicial space $X.$. The corresponding topology on $X.$ will be denoted by $Top(X.)$.

We will only consider sheaves of $\mathbb{Q}$-vector spaces in this paper. A sheaf $F$ of $\mathbb{Q}$-vector spaces on a simplicial space $X.$ consists of a collection $\{F_n|n\}$, where each $F_n$ is sheaf of $\mathbb{Q}$-vector spaces on $X_n$, provided with a collection of maps $\phi(\alpha) : \alpha^*(F_n) \to F_m$ associated to each structure map $\alpha : X_m \to X_n$ of the simplicial space $X.$ satisfying certain obvious compatibility conditions as in ([Fr] p.14, for example.). The category of such sheaves will be denoted by $Sh(X.)$. We will let $D_b(X.; \mathbb{Q})$ denote the derived category all bounded complexes of sheaves of $\mathbb{Q}$-vector spaces.

(2.2) If $K^\cdot$ is a complex of sheaves of $\mathbb{Q}$-vector spaces on $EG\underset{G}{\times}X$, one defines $\mathbb{H}^*_G(X;\ K^\cdot)$ to be $\mathbb{H}^*(EG\underset{G}{\times}X;\ K^\cdot)$ = *the equivariant hypercohomology* with respect to $K^\cdot$. Now one obtains a spectral sequence:

(2.3) $E_2^{s,t} = H^s(BG;\ R^t\pi_*K^\cdot) \Rightarrow \mathbb{H}_G^{s+t}(X;\ K^\cdot)$

Assume the above situation. Let $\underline{\mathbb{Q}}$ denote the obvious constant sheaf on $BG.$. We let $D_{\mathbb{Q}}^X = R\pi^!(\underline{\mathbb{Q}})$ = the dualizing complex for the category $D_b(EG\underset{G}{\times}X;\ \mathbb{Q})$. (See [J-3] section 6, for a detailed discussion in the étale setting.) We define the *equivariant homology* of $X$ (with locally compact supports) to be

(2.4) $H_*^G(X(\mathbb{C});\ \mathbb{Q}) = \mathbb{H}_G^*(X;\ D_{\mathbb{Q}}^X)$

If $f : X \to Y$ is a $G$-equivariant *proper* map between $G$-varieties, the trace-map $tr(f) : R\pi_*R\pi^!(\underline{\mathbb{Q}}) \to \underline{\mathbb{Q}}$ induces a map $\mathbb{H}_*^G(X(\mathbb{C});\ \mathbb{Q}) \to \mathbb{H}_*^G(Y(\mathbb{C});\ \mathbb{Q})$. Thus equivariant homology is a covariant functor for equivariant proper maps.

Next assume that $X$ is a *smooth* variety of dimension $d$ over $\mathbb{C}$. Now one may identify $R\pi^!(\underline{\mathbb{Q}})$ with $\pi^*(\underline{\mathbb{Q}})[2d]$. One may observe that for the corresponding spectral sequence in (2.3), $E_2^{s,t} = 0$ if $t < -2d$, $t > 0$ or if $s < 0$. Hence

$H^0(BG;\ R^{-2d}\pi_*D_{\mathbb{Q}}^X) \cong E_2^{0,-2d} = E_3^{0,-2d} = ... = E_\infty^{0,-2d} \cong \mathbb{H}_G^{-2d}(X;\ D_{\mathbb{Q}}^X)$

(2.5) One may now observe that if $X$ is smooth of dimension $d$ over $\mathbb{C}$, the fundamental class in $H^0(BG;\ R^{-2d}\pi_*D_{\mathbb{Q}}^X)$ is an infinite cycle in the spectral sequence in (2.3) with $K^\cdot = D_{\mathbb{Q}}^X$ and induces a fundamental class $[X]_G \in H_{2d}^G(X(\mathbb{C});\ \mathbb{Q})$. It follows that, if $X$ is smooth, one obtains a *fundamental class* $[X]_G \in H_{2d}^G(X;\ \mathbb{Q})$.

Next assume that $X$ is a possibly-singular $G$-quasi-projective variety with a $G$-equivariant closed immersion $i : X \to \tilde{X}$ into a smooth $G$-quasi-projective variety. If $D_{\mathbb{Q}}^X$ and $D_{\mathbb{Q}}^{\tilde{X}}$ denote the dualizing complexes on $EG\underset{G}{\times}X$ and $EG\underset{G}{\times}\tilde{X}$, one obtains a pairing of sheaves $D_{\mathbb{Q}}^{\tilde{X}} \otimes i_*Ri^!(\underline{\mathbb{Q}}) \to i_*D_{\mathbb{Q}}^X$ on $EG\underset{G}{\times}\tilde{X}$. This pairing induces a pairing of the associated spectral sequences in (2.3). Cap product with the fundamental class in $H^0(BG;\ R^{-2d}\pi_*D_{\mathbb{Q}}^X)$ induces a map of the spectral sequences:

(2.5.*) $E_2^{s,t}(1) = H^s(BG;\ R^t\pi_*i_*Ri^!(\underline{\mathbb{Q}})) \to E_2^{s,t}(2) = H^s(BG;\ R^{t-2d}\pi_*(D_{\mathbb{Q}}^X))$

where $\pi : EG\underset{G}{\times}\tilde{X} \to BG$ denotes the obvious map. This is induced by a map $H_{X(\mathbb{C})}^*(\tilde{X}(\mathbb{C});\ \mathbb{Q}) \to H_{2d-*}(X(\mathbb{C});\ \mathbb{Q})$ which is cap-product with the fundamental class $[\tilde{X}] \in H_{2d}(\tilde{X};\ \mathbb{Q})$ (= the homology with locally compact supports of $\tilde{X}$). Therefore the map in (2.5.*) is an isomorphism. Since the fundamental class in $H^0(BG;\, R^{-2d}\pi_* D^X_{\mathbb{Q}})$ is an infinite cycle, it follows that the above map induces an isomorphism of the two spectral sequences from the $E_2$-terms onwards and hence on the abutments. We have thereby established *Poincaré-Lefschetz-duality for $G$-equivariant rational homology.* This isomorphism will be denoted

(2.6) $P - L : H^*_{G,X(\mathbb{C})}(\tilde{X}(\mathbb{C});\, \mathbb{Q}) \overset{\cong}{\rightarrow} H^G_*(X(\mathbb{C});\, \mathbb{Q})$

henceforth. If $M \subseteq G$ denotes a maximal compact subgroup, one may replace $G$ by $M$ to obtain Poincaré-Lefschetz-duality in $M$-equivariant homology.

(2.7)*Equivariant Chern classes, Chern character and the Todd homomorphism.* Let $Z$ denote a $G$-quasi-projective variety as in (1.9) and let $M$ denote a maximal compact subgroup of $G$. Let $E$ denote an $M$-equivariant complex topological vector bundle of rank $n+1$ and let $\mathbb{P}(E)$ denote the associated projective space bundle. One may compute $H^*_M(\mathbb{P}(E)(\mathbb{C});\, \mathbb{Q}) \cong H^*_M(Z(\mathbb{C});\, \mathbb{Q})[T]/(T^{n+1})$ with $T$ corresponding to the class of the canonical line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Therefore one may define the equivariant Chern-classes of the vector bundle $E$ as one normally does in the non-equivariant case. This procedure thus defines Chern-classes and Todd-classes for complex topological $M$-equivariant vector bundles on $Z(\mathbb{C})$ with values in the equivariant cohomology ring $H^*_M(Z(\mathbb{C});\, \mathbb{Q})$.

If $Z$ is not projective this method fails to define Chern-classes for all classes in $K^{A.S}_M(Z(\mathbb{C}))$ and for all classes in $K^{top}_M(Z(\mathbb{C}))$ even if $Z$ is projective. Therefore, we adopt the following technique for defining equivariant chern-classes in general. Recall that $K^{top}_M(Z(\mathbb{C})) \simeq Map(|EM \underset{M}{\times} Z(\mathbb{C})|_+, KU)$ - see (1.5); one may view the *universal Chern-character* as a map of spectra $KU \rightarrow \underset{i}{\Pi} K(\mathbb{Q}, 2i))$ that induces a weak-equivalence when $KU$ is localized at $\mathbb{Q}$. Clearly this defines an equivariant Chern-character

$$\widehat{ch}^M : K^{top}_M(Z(\mathbb{C})) \rightarrow Map(|EM \underset{M}{\times} Z(\mathbb{C})|_+,\, \underset{i}{\Pi} K(\mathbb{Q}, 2i))$$

as a map of spectra. One may also define a local equivariant Chern-character as follows. Let $i : Z \rightarrow \tilde{Z}$ denote a $G$-equivariant closed immersion into a smooth $G$-quasi-projective variety. One defines

(2.7.1) $\widehat{ch}^{M,\tilde{Z}}_Z : K^{top}_M(\tilde{Z}(\mathbb{C}),\, \tilde{Z}(\mathbb{C}) - Z(\mathbb{C})) \rightarrow Map(|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|_+ / (|EM \underset{M}{\times} (\tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))|_+);\, \underset{i}{\Pi} K(\mathbb{Q}, 2i))$

to be the obvious map induced by $\widehat{ch}^M$ on $K^{top}_M(\tilde{Z}(\mathbb{C}))$ and on $K^{top}_M(\tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))$. On taking the homotopy groups this defines a local Chern-character

$$ch^{M,\tilde{Z}}_Z : \pi_0(K^{top}_M(\tilde{Z}(\mathbb{C}),\, \tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))) \rightarrow H^*_M(\tilde{Z}(\mathbb{C}),\, \tilde{Z}(\mathbb{C}) - Z(\mathbb{C});\, \mathbb{Q}) \xrightarrow{P-L} H^M_*(Z(\mathbb{C});\, \mathbb{Q})$$

i.e. $ch^{M,\tilde{Z}}_Z = P - L \circ \widehat{ch}^{M,\tilde{Z}}_Z$. Finally one defines an *equivariant Todd homomorphism*

(2.7.2) $\tau_Z^{M,\tilde{Z}} : \pi_0(K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))) \to H_*^M(Z(\mathbb{C});\ \mathbb{Q})$

by $\tau_Z^{M,\tilde{Z}}(\mathcal{E}) = ch_Z^{M,\tilde{Z}}(\mathcal{E}) \cap Td^M(\mathcal{T}_{\tilde{Z}|Z})$, where $\mathcal{T}_{\tilde{Z}|Z}$ denotes the restriction of the tangent bundle of $\tilde{Z}$ to $Z$ and $Td^M(\mathcal{T}_{\tilde{Z}|Z})$ denotes its equivariant Todd-class. We will also consider the Chern-character:

(2.7.3) $ch^{M,\tilde{Z}} : K_M^{top}(\tilde{Z}(\mathbb{C})) \to Map(|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|_+;\ \underset{i}{\Pi} K(\mathbb{Q}, 2i))$

In the rest of this section we discuss properties of the local Chern-character defined above and combine the equivariant Todd homomorphism with (1.14) to obtain an equivariant Riemann-Roch theorem.

(2.8.1) *Additivity and multiplicativity.* Assume the situation above. Let $\mathcal{E}^{\cdot}$ and $\bar{\mathcal{E}}^{\cdot}$ denote Fredholm complexes on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$ with supports on $|EM \underset{M}{\times} Z(\mathbb{C})|$ while $\mathcal{F}^{\cdot}$ denotes a Fredholm complex on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$. Now

$ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot} \oplus \bar{\mathcal{E}}^{\cdot}) = ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot}) + ch_Z^{M,\tilde{Z}}(\bar{\mathcal{E}}^{\cdot})$

$ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot} \otimes \mathcal{F}^{\cdot}) = ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot}) \cap ch^{M,\tilde{Z}}(\mathcal{F}^{\cdot})_{|Z(\mathbb{C})} = ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot}) \cap ch^{M,Z}(\mathcal{F}^{\cdot}_{|Z(\mathbb{C})})$

where $\mathcal{E}^{\cdot} \otimes \mathcal{F}^{\cdot}$ represents the class in $\pi_0(K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C})))$ given by the pairing

$\pi_0(K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))) \otimes \pi_0(K_M^{top}(\tilde{Z}(\mathbb{C}))) \to \pi_0(K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C})))$

These properties follow readily from the corresponding properties of the universal Chern character.

(2.8.2)*Excision.* Let $\tilde{U} \overset{j}{\to} \tilde{Z}$ denote an $M$-stable open sub-space of $\tilde{Z}$ containing $Z$. Let $j^* : K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C})) \to K_M^{top}(\tilde{U}(\mathbb{C}),\ \tilde{U}(\mathbb{C}) - Z(\mathbb{C}))$ denote the obvious restriction map induced by $j$. Now

$ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot}) = ch_Z^{M,\tilde{U}}(j^*(\mathcal{E}^{\cdot}))$

$\mathcal{E}^{\cdot}$ being a Fredholm complex on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$ with supports in $|EM \underset{M}{\times} Z(\mathbb{C})|$.

(2.8.3) *Localization.* Let $Z \overset{i}{\to} Z' \to \tilde{Z}$ denote closed immersions of $G$-stable subvarieties of the smooth $G$-variety $\tilde{Z}$. Let $i_*$ denote the induced maps

$K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z(\mathbb{C})) \to K_M^{top}(\tilde{Z}(\mathbb{C}),\ \tilde{Z}(\mathbb{C}) - Z'(\mathbb{C}))$ and

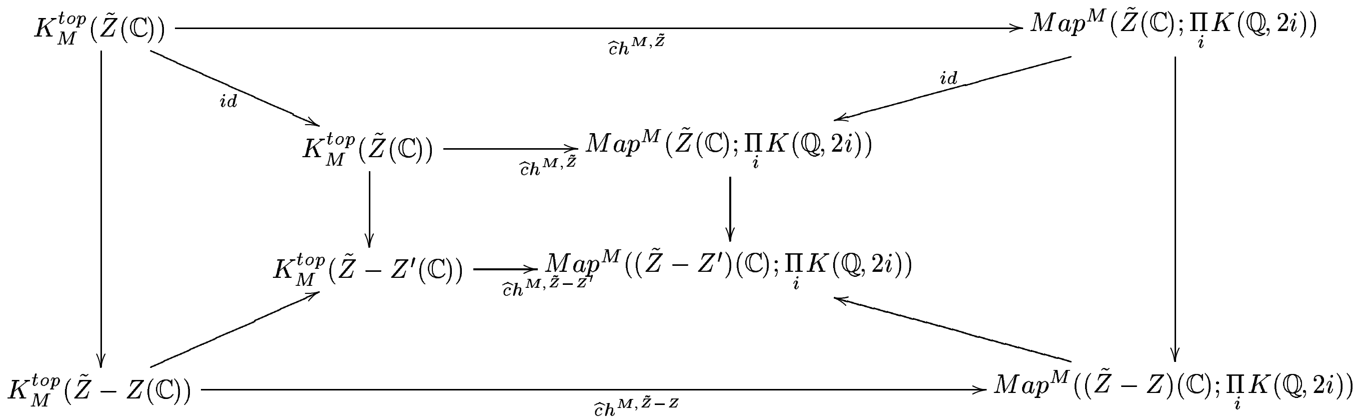
$Map(|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|_+ / |EM \underset{M}{\times} (\tilde{Z}(\mathbb{C}) - Z(\mathbb{C}))|_+,\ \underset{i}{\Pi} K(\mathbb{Q}, 2i))$

$\to Map(|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|_+ / |EM \underset{M}{\times} (\tilde{Z}(\mathbb{C}) - Z'(\mathbb{C}))|_+,\ \underset{i}{\Pi} K(\mathbb{Q}, 2i))$.

Now

(2.8.3.*) $ch_{Z'}^{M,\tilde{Z}}(i_*(\mathcal{E}^{\cdot})) = i_* ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot})$

where $\mathcal{E}^{\cdot}$ is a Fredholm complex on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$ with supports in $|EM \underset{M}{\times} Z(\mathbb{C})|$ and $i_*(\mathcal{E}^{\cdot})$ is the same complex $\mathcal{E}^{\cdot}$ viewed as a complex on the same space but with supports in $|EM \underset{M}{\times} Z'(\mathbb{C})|$. By considering the induced map on the homotopy fibers of the columns in the following diagram




one may first prove that $\widehat{ch}^{M,\tilde{Z}}_{Z'}(i_*(\mathcal{E}^{\cdot})) = i_*\widehat{ch}^{M,\tilde{Z}}_{Z}(\mathcal{E}^{\cdot})$. Now compose this with the Poincaré-Lefschetz-duality to prove (2.8.3.*).

(2.8.4) *Pull-back property.* Assume in addition to the above situation that $p : \bar{Z} \to \tilde{Z}$ is a proper and smooth $G$-equivariant map of $G$-quasi-projective smooth varieties. Let $q : Q = p^{-1}(Z) \to Z$ denote the obvious restriction of $p$. If $\mathcal{E}^{\cdot}$ is a Fredholm complex on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$ with supports in $|EM \underset{M}{\times} Z(\mathbb{C})|$

$$q^*(ch^{M,\tilde{Z}}_{Z}(\mathcal{E}^{\cdot})) = ch^{M,\bar{Z}}_{p^{-1}(Z)}(p^*(\mathcal{E}^{\cdot}))$$

(2.8.5) **Lemma**. Let $Z \overset{i}{\to} \tilde{Z}$ denote a $G$-equivariant closed immersion of $Z$ into a smooth $G$-quasi-projective variety. Let $\pi : N \to \tilde{Z}$ denote a $G$-equivariant vector bundle with $\tilde{Z}$ viewed as the subspace of $N$ by the zero section. Let $\check{N}$ denote the coherent sheaf of sections of the bundle $N$ and let $\Lambda^{\cdot}(\pi^*(\check{N}))$ denote the Koszul-Thom complex on $N$. If $\mathcal{E}^{\cdot}$ is a Fredholm complex on $|EM \underset{M}{\times} \tilde{Z}(\mathbb{C})|$ with supports in $|EM \underset{M}{\times} Z(\mathbb{C})|$, $\Lambda^{\cdot}(\pi^*(\check{N})) \otimes \pi^*(\mathcal{E}^{\cdot})$ has supports in $|EM \underset{M}{\times} Z(\mathbb{C})| \subseteq |EM \underset{M}{\times} N|$ and

$$ch^{M,N}_{Z}(\Lambda^{\cdot}(\pi^*(\check{N})) \otimes \pi^*(\mathcal{E}^{\cdot})) = Td^M(N_{|Z})^{-1} \cap ch^{M,\tilde{Z}}_{Z}(\mathcal{E}^{\cdot})$$

*Proof.* With the above properties of the local Chern character, the proof reduces to the same one in [B-F-M] Proposition (3.4). We will summarize the arguments for the sake of completeness. Imbed $N$ in the projective completion $P = Proj(N \oplus \epsilon_1)$ where $\epsilon_1$ is the trivial one dimensional vector bundle on $\tilde{Z}$. Let $p : P \to \tilde{Z}$ denote the obvious projection and let $q : Q = p^{-1}(Z) \to Z$ denote the obvious restriction of p. On $P$ one obtains an exact sequence $0 \to H \to p^*(\check{N} \oplus \epsilon_1) \to \mathcal{O}_P(1) \to 0$ and $p^*(\check{N} \oplus \epsilon_1) = p^*(\check{N}) \oplus \epsilon_1$. Now projection to the second factor provides a map of sheaves $H \to \mathcal{O}_P$ which is surjective off $\tilde{Z}$. This gives rise to a Koszul complex $\Lambda^{\cdot}H$ on $P$ exact off $\tilde{Z}$ and such that on restriction to $N \subseteq P$, this complex is $\Lambda^{\cdot}(\pi^*(\check{N}))$. By the excision property one obtains: $ch_Z^{M,N}(\Lambda^{\cdot}\pi^*(\check{N}) \otimes \pi^*(\mathcal{E}^{\cdot})) = ch_Z^{M,P}(\Lambda^{\cdot}H \otimes p^*(\mathcal{E}^{\cdot}))$. Let $s : Z \to Q$ denote the zero section. Now $s_*(ch_Z^{M,P}(\Lambda^{\cdot}H \otimes p^*(\mathcal{E}^{\cdot})) = ch_Q^{M,P}(\Lambda^{\cdot}H \otimes p^*(\mathcal{E}))$ by the localization property. Since $p^*(\mathcal{E}^{\cdot})$ is exact off $Q$, the multiplicative property shows $ch_Q^{M,P}(\Lambda^{\cdot}H \otimes p^*(\mathcal{E}^{\cdot})) = ch^{M,Q}(\Lambda^{\cdot}H_{|Q}) \cap ch_Q^{M,P}(p^*(\mathcal{E}^{\cdot}))$. Now $ch_Q^{M,P}(p^*(\mathcal{E}^{\cdot})) = q^*ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot})$ by the pull-back property and $q_* \circ s_* = id$ since $s$ is a section to $q$. It follows that

$$ch_Z^{M,N}(\Lambda^{\cdot}\pi^*(\check{N}) \otimes \pi^*(\mathcal{E}^{\cdot})) = q_* \circ s_*(ch_Z^{M,P}(\Lambda^{\cdot}H \otimes p^*(\mathcal{E}^{\cdot}))$$

$$= q_*(ch^{M,Q}(\Lambda^{\cdot}H_{|Q}) \otimes q^*(ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot}))) = q_*(ch^{M,Q}(\Lambda^{\cdot}H_{|Q})) \cap ch_Z^{M,\tilde{Z}}(\mathcal{E}^{\cdot})$$

Now it suffices to show $Td^M(N_{|Z})^{-1} = q_*(ch^{M,Q}(\Lambda^{\cdot}H_{|Q}))$; this follows from the equality $p_*(ch^{M,P}(\Lambda^{\cdot}H)) = Td^M(N)^{-1}$ which follows by the same argument as on page 112 of [B-F-M].

(2.8.6) **Gysin maps**. Let $Z$ be as before and let $Z \to Y_0 \xrightarrow{i} Y_1$ denote two $G$-equivariant closed immersions so that $Y_0$ and $Y_1$ are also *smooth*. Now any $G$-equivariant complex of vector bundles on $Y_0$ with supports in $Z$ is quasi-isomorphic to an equivariant complex of vector bundles on $Y_1$ with supports in $Z$. This provides a Gysin-map

$$Gysin : K_{G,Z}(Y_0) \xrightarrow{\simeq} K_{G,Z}(Y_1)$$

Let the normal bundle to the closed immersion $Y_0 \to Y_1$ be $N$. One may view the total space of this bundle as a tubular neighborhood of $Y_0$ in $Y_1$. Now cup-product with the Koszul-Thom class $\lambda_N \,\epsilon\, K^{A.S}_{M,Y_0(\mathbb{C})}(N(\mathbb{C}))$ defines a map $K^{A.S}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C})) \to K^{A.S}_{M,Z(\mathbb{C})}(N(\mathbb{C}))$. Using excision the latter is seen to be weakly-equivalent to $K^{A.S}_{M,Z(\mathbb{C})}(Y_1(\mathbb{C}))$. The composite of the above is the Gysin map $Gysin : K^{A.S}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C})) \to K^{A.S}_{M,Z(\mathbb{C})}(Y_1(\mathbb{C}))$. One defines similar Gysin maps in equivariant topological K-theory.

If $Y_1' \to Y_1$ is a map that is suitably transveral to the maps $Y_0 \to Y_1$ and $Z \to Y_1$, and if $Z'$ ($Y_0'$) is defined by an appropriate cartesian square, the above Gysin maps will pull-back to the cooresponding Gysin maps associated to the immersions $Z' \to Y_0' \to Y_1'$. (See [T-4] (3.2) for details.)

(2.8.7) **Homotopy property**. Let $\hat{X} \times \mathbb{A}^1 \xrightarrow{i} \tilde{X}$ denote a $G$-equivariant closed immersion of smooth $G$-quasi-projective varieties over $\mathbb{A}^1$ where $G$ acts trivially on $\mathbb{A}^1$. Assume that the given map $\pi : \tilde{X} \to \mathbb{A}^1$ is smooth. Let $X \to \hat{X}$ denote a closed $G$-equivariant immersion of a $G$-stable subvariety; for each $t \,\epsilon\, \mathbb{A}^1$, let $X_t \to \hat{X}_t \to \tilde{X}_t$ denote the corresponding closed immersions, where $\tilde{X}_t$ is the fiber of $\pi$ over $t$. Let $M$ denote a maximal compact subgroup of $G$.

(i) Now the Gysin-maps $K_{G,X_t}(\hat{X}_t) \to K_{G,X_t}(\tilde{X}_t)$ are all homotopic to the Gysin-map $K_{G,X\times\mathbb{A}^1}(\hat{X} \times$

$\mathbb{A}^1) \to K_{G,X\times\mathbb{A}^1}(\tilde{X})$. Similar results hold in equivariant Atiyah-Segal K-cohomology and equivariant topological K-cohomology with respect to the action of $M$.

(ii) If $\mathcal{E}^\cdot$ is a Fredholm-complex on $|EM\underset{M}{\times}\tilde{X}(\mathbb{C})|$ with supports in $|EM\underset{M}{\times}(X\times\mathbb{A}^1)(\mathbb{C})|$ and $\mathcal{E}_t^\cdot$ is the corresponding complex on $|EM\underset{M}{\times}\tilde{X}(\mathbb{C})_t|$ with supports in $|EM\underset{M}{\times}(X_t(\mathbb{C}))|$, $ch^{M,\tilde{X}}_{X\times\mathbb{A}^1}(\mathcal{E}^\cdot) = ch^{M,\tilde{X}_t}_{X_t}(\mathcal{E}_t^\cdot)$ for all $t\,\epsilon\,\mathbb{A}^1$.

(2.9) **Theorem** (Riemann-Roch via local equivariant K-cohomology)

Let $Z$ denote a $G$ quasi-projective variety provided with a $G$-equivariant closed immersion $i: Z\to\tilde{Z}$ into an ambient smooth $G$-quasi-projective variety.

(i) If the map $i$ factors through $i': Z'\to\tilde{Z}$ (which is the $G$-equivariant closed immersion of another $G$-quasi-projective variety $Z'$), one obtains a commutative diagram:

$$\begin{array}{ccccccc}
\pi_0(K(Mod^G_{coh}(Z))) & \xrightarrow{\rho\circ P-L_a^{-1}} & \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \xrightarrow{B} & \pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) & \xrightarrow{\tau^{M,\tilde{Z}}_Z} & H^M_*(Z(\mathbb{C});\,\mathbb{Q}) \\
\downarrow{i_*} & & \downarrow{i_*} & & \downarrow{i_*} & & \downarrow{i_*} \\
\pi_0(K(Mod^G_{coh}(Z'))) & \xrightarrow{\rho\circ P-L_a^{-1}} & \pi_0(K^{A.S}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \xrightarrow{B} & \pi_0(K^{top}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \xrightarrow{\tau^{M,\tilde{Z}}{}_{Z'}} & H^M_*(Z'(\mathbb{C});\,\mathbb{Q})
\end{array}$$

where the maps denoted $\rho\circ P-L_a^{-1}$ are the compositions :$\pi_0(K(Mod^G_{coh}(Z))) \xrightarrow{P-L_a^{-1}} \pi_0(K_{G,Z}(\tilde{Z})) \xrightarrow{\rho} \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))$ and $\pi_0(K(Mod^G_{coh}(Z'))) \xrightarrow{P-L_a^{-1}} \pi_0(K_{G,Z'}(\tilde{Z})) \xrightarrow{\rho} \pi_0(K^{A.S}_{M,Z'(\mathbb{C})}(\tilde{Z}(\mathbb{C})))$.

(ii) Let $\mathbb{P}^n$ denote a projective space provided with a $G$-linear action. Now the projection $p: Z\times\mathbb{P}^n\to Z$ induces a commutative diagram:

$$\begin{array}{ccccc}
\pi_0(K(Mod^G_{coh}(Z\times\mathbb{P}^n))) & \xrightarrow{\rho\circ P-L_a^{-1}} & \pi_0(K^{A.S}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C}))) & \xrightarrow{B} & \pi_0(K^{top}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})) \\
\downarrow{p_*} & & \downarrow{p_*} & & \downarrow{p_*} \\
\pi_0(K(Mod^G_{coh}(Z\times\mathbb{P}^n))) & \xrightarrow{\rho\circ P-L_a^{-1}} & \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \xrightarrow{B} & \pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))
\end{array}$$

$$\begin{array}{ccc}
\pi_0(K^{top}_{M,Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})) & \xrightarrow{\tau^{M,\tilde{Z}\times\mathbb{P}^n}_{Z\times\mathbb{P}^n}} & H^M_*(Z\times\mathbb{P}(\mathbb{C})^n;\,\mathbb{Q}) \\
\downarrow{p_*} & & \downarrow{p_*} \\
\pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) & \xrightarrow{\tau^{M,\tilde{Z}}_Z} & H^M_*(Z(\mathbb{C});\,\mathbb{Q})
\end{array}$$

where the maps denoted $\rho\circ P-L_a^{-1}$ are the compositions:

$\pi_0(K(Mod^G_{coh}(Z\times\mathbb{P}^n))) \xrightarrow{P-L_a^{-1}} \pi_0(K_{G,Z\times\mathbb{P}^n}(\tilde{Z}\times\mathbb{P}^n)) \xrightarrow{\rho} \pi_0(K_{M,Z\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}\times\mathbb{P}^n)(\mathbb{C}))$ and

$\pi_0(K(Mod^G_{coh}(Z))) \xrightarrow{P-L_a^{-1}} \pi_0(K_{G,Z}(\tilde{Z})) \xrightarrow{\rho} \pi_0(K_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))$.

*Proof* (i). The commutativity of the first two squares are clear from (1.14). The commutativity of the last square in this case follows from the localization property (2.8.3). The localization property (2.8.3) shows $i_* ch_Z^{M,\tilde{Z}}(\mathcal{E}^\cdot) = ch_{Z'}^{M,\tilde{Z}}(i_*(\mathcal{E}^\cdot))$. Now it suffices to observe that

$$\tau_{Z'}^{M,\tilde{Z}}(i_*\mathcal{E}^\cdot) = ch_{Z'}^{M,\tilde{Z}}(i_*(\mathcal{E}^\cdot)) \cap Td(\mathcal{T}_{\tilde{Z}|Z'}) = i_*(ch_Z^{M,\tilde{Z}}(i_*(\mathcal{E}^\cdot))) \cap Td(\mathcal{T}_{\tilde{Z}|Z'})$$

$$= i_*(ch_Z^{M,\tilde{Z}}(i_*(\mathcal{E}^\cdot)) \cap Td(\mathcal{T}_{\tilde{Z}|Z})) = i_*(\tau_Z^{M,\tilde{Z}}(\mathcal{E}^\cdot))$$

Now consider the projection $Z \times \mathbb{P}^n \to Z$. Once again the commutativity of the first two squares follow from (1.14). The computations as in (1.13.2) show

$\pi_0(K^{top}_{Z(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})}(\tilde{Z}(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}))) \cong \pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \underset{\pi_0(K^{top}(|BM|))}{\otimes} \pi_0(K^{top}_M(\mathbb{P}^n(\mathbb{C})))$ and that

$H^M_*(Z(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}); \mathbb{Q}) \cong H^M_*(Z(\mathbb{C}); \mathbb{Q}) \underset{H^*(BM;Q)}{\otimes} H^M_*(\mathbb{P}^n(\mathbb{C}); \mathbb{Q})$.

Therefore one may reduce to the case where $Z$ is a point. Now the description of the map $p_*$ as in (1.13.3) proves the required result as in the non-equivariant case - see [B-S] , Proposition 10.

## 3. **Equivariant Atiyah-Segal and Topological K-homology**.

Let $i : Z \to \tilde{Z}$ denote the $G$-equivariant closed immersion of $G$-quasi-projective varieties with $\tilde{Z}$ smooth. In this setting, one could define the Atiyah-Segal K-homology to be $K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$. However, in order to show that this theory is independent of the imbedding into the ambient smooth $G$-variety $\tilde{Z}$, it is necessary to define Atiyah-Segal K-homology intrinsically (i.e. without the imbedding into $\tilde{Z}$) and obtain a *Poincaré-Lefschetz-duality.* For this we invoke the generalized equivariant homology theories as developed in [LMS]. Let $M$ denote the maximal compact subgroup of $G$. Let the $K^{A.S}_M$ denote the *equivariant spectrum* representing Atiyah-Segal K-theory with respect to $M$ defined as in [LMS] chapter 1.

*Notation.* If $M$ is a compact Lie group acting on a pointed space $Z$ and $E$ is an $M$-equivariant spectrum, $Map_M(Z, E)$ will denote the function spectrum of $M$-equivariant maps from the suspension spectrum of $Z$ to $E$.

If $Z$ is a complex *G-projective* variety we will define

(3.1) $K^M_{A.S}(Z(\mathbb{C})) = Map_M(\Sigma^\infty, Z(\mathbb{C})_+ \wedge K^{A.S}_M)$

where $Map_M$ denotes the function-$M$-spectrum which is the internal Hom-functor in the category of $M$-spectra. $\Sigma^\infty$ is the sphere-$M$-spectrum (with the trivial $M$-action.)

$K^M_{A.S,i}(Z(\mathbb{C})) = \pi_i(Map_M(\Sigma^\infty,\, Z(\mathbb{C})_+ \wedge K^{A.S}_M))$.

(In general the homotopy groups of an $M$-spectrum are indexed by pairs $(n, H)$ where $n$ is an integer and $H$ is a subgroup of $M$; we always let $H = M$, so that the homotopy groups are now indexed only by the integers.) If $Z$ is only $G$-quasi-projective let $Z \to \mathbb{P}^n$ denote a $G$-equivariant locally closed immersion into $\mathbb{P}^n$ with a linear action by $G$. Let $\bar{Z}$ denote the closure of $Z$ in $\mathbb{P}^n$. Now $\bar{Z} - Z$ and $\bar{Z}$ are both $G$-projective varieties and we define

(3.1') $K^M_{A.S}(Z(\mathbb{C}))$ to be the canonical homotopy cofiber of the natural map $K^M_{A.S}(\bar{Z} - Z(\mathbb{C})) \to K^M_{A.S}(\bar{Z}(\mathbb{C}))$.

Next we show, *in outline*, the existence of a fundamental class in $\pi_{2d}(K^M_{A.S}(Z(\mathbb{C})))$ if $Z$ is a $G$-quasi-projective smooth complex variety of dimension $d$ over $\mathbb{C}$. First assume that $Z$ is $G$-projective and smooth. Now one may consider a $G$-equivariant closed immersion $Z \to \mathbb{P}^n$ where $G$ acts linearly on $\mathbb{P}^n$. One may consider an $M$-equivariant imbedding of the $C^\infty$ compact manifold $\mathbb{P}^n$ in a large vector space $\mathbb{R}^N$ on which $M$ acts linearly. Let $\nu$ denote the normal bundle to the composite imbedding $Z \to \mathbb{P}^n \to \mathbb{R}^N$ and let $T(\nu) = D(\nu)/S(\nu)$ be its Thom-space. (Here $D(\nu)$ and $S(\nu)$ are the corresponding disk and sphere bundles.) Now one obtains the Thom-Pontrjagin collapse map $\mu : S^N \to Z(\mathbb{C})_+ \wedge T(\nu)$ - see [LMS] pp. 152-153. One also obtains a Gysin-map $K^{A.S}_M(Z(\mathbb{C})) \to \tilde{K}^{A.S}_M(T(\nu)) = Map_M(T(\nu),\, K^{A.S}_M)$. Slant product with the map $\mu$ provides a map $\tilde{K}^{A.S}_M(T(\nu)) \to K^M_{A.S}(Z(\mathbb{C}))$. Composing these two we obtain a map

(3.2) $P : K^{A.S}_M(Z(\mathbb{C})) \to K^M_{A.S}(Z(\mathbb{C}))$

The image of the class $\mathcal{O}_Z \,\epsilon\, K_G(Z)$ is a class in $K^{A.S}_M(Z(\mathbb{C}))$ which we will denote by $\mathcal{O}_Z$ again. This maps to a class $[Z] \,\epsilon\, K^M_{A.S}(Z(\mathbb{C}))$ under the map $P$; this is *the fundamental class* associated to $Z$. Now one may observe (see [LMS] pp.157-160) that the map in (3.2) is itself obtained by cap-product with the class $[Z]$ and that this map is a weak-equivalence.

Next assume that $Z$ is a $G$-quasi-projective smooth variety. Let $Z \to \bar{Z}$ denote a $G$-equivariant *open immersion* into a *$G$-projective smooth* variety. By blowing up $G$-stable subvarieties in $\bar{Z} - Z$ one may further assume that $\bar{Z} - Z$ is also *smooth*. Let $[\bar{Z}]$ denote the fundamental class associated to $\bar{Z}$ as in (3.2); cap-product with this class induces a homotopy-commutative diagram:

$$
(3.3)\qquad
\begin{array}{ccc}
K^{A.S}_{M,\bar{Z}-Z(\mathbb{C})}(\bar{Z}(\mathbb{C})) & \xrightarrow{\simeq} & K^M_{A.S}(\bar{Z} - Z(\mathbb{C})) \\
\downarrow & & \downarrow \\
K^{A.S}_M(\bar{Z}(\mathbb{C})) & \xrightarrow{\simeq} & K^M_{A.S}(\bar{Z}(\mathbb{C}))
\end{array}
$$

(The above diagram may be obtained as follows. Let $\bar{Z} \to \mathbb{P}^n \to \mathbb{R}^N$ be an $M$-equivariant closed imbedding with normal bundle $\nu_1$ as before; let the normal bundle to the closed immersion $\bar{Z} - Z \to \bar{Z}$

be $N$. Let $\nu_2 = \nu_{1|\bar{Z}-Z} \oplus N$ and let $T(\nu_1)$ and $T(\nu_2)$ denote the respective Thom-spaces. Now one obtains a homotopy commutative diagram:

$$\begin{array}{ccccc} K^{A.S}_{M,\bar{Z}-Z(\mathbb{C})}(\bar{Z}(\mathbb{C})) & \longrightarrow & \tilde{K}^{A.S}_M(T(\nu_2)) & \longrightarrow & K^M_{A.S}(\bar{Z}-Z(\mathbb{C})) \\ \downarrow & & \downarrow & & \downarrow \\ K^{A.S}_M(\bar{Z}(\mathbb{C})) & \longrightarrow & \tilde{K}^{A.S}_M(T(\nu_1)) & \longrightarrow & K^M_{A.S}(\bar{Z}(\mathbb{C})) \end{array}$$

where the two horizontal maps in the right square are obtained as in the paragraph preceding (3.2). The middle vertical map is induced by a Thom-Pontrjagin collapse $T(\nu_1) \to T(\nu_2)$. Now $T(\nu_1)$ is the $M$-equivariant $S$-dual of $\bar{Z}(\mathbb{C})_+$ and $T(\nu_2)$ is the $M$-equivariant $S$-dual of $\bar{Z} - Z(\mathbb{C})_+$ - see [LMS] p. 153. Therefore the right square commutes. The two horizontal maps in the left square are Gysin-maps. The one in the bottom row is cup product with the Koszul-Thom class of the bundle $\nu_1$ on $\bar{Z}$ while the one in the top row is cup product with the Koszul-Thom class of the bundle $\nu_{1|\bar{Z}-Z}$ on $\bar{Z} - Z$; one may also identify the left-most vertical map with the map induced by a Thom-Pontrjagin collapse $\bar{Z}(\mathbb{C}) \to T(N)$. Therefore the left-most square commutes. The horizontal maps in the left-square are clearly weak-equivalences. The diagram in (3.3) is the outer-square of the above diagram. One may observe that the top-row of (3.3) is a special case of Poincaré-Lefschtez-duality.)

One therefore obtains an induced map of the homotopy cofibers of the two columns which will be also a weak-equivalence. i.e. we obtain an induced map $P : K^{A.S}_M(Z(\mathbb{C})) \to K^M_{A.S}(Z(\mathbb{C}))$. The image of the class $\mathcal{O}_Z \,\epsilon\, K_G(Z)$ under the above map defines *the fundamental class* $[Z]$ *of* $Z$ in $K^M_{A.S}(Z(\mathbb{C})$. Moreover the above map $P$ may be realized as cap-product with the fundamental class $[Z]$.

Next we consider *Poincaré-Lefschetz-duality in general.* Let $X$ denote a $G$-quasi-projective variety and let $X \to \tilde{Z}$ denote a $G$-equivariant *closed* immersion into a smooth $G$-quasi-projective variety. Now there exists fundamental classes $[\tilde{Z}]$ in $K^M_{A.S}(\tilde{Z}(\mathbb{C}))$ and $[\tilde{Z} - X]$ in $K^M_{A.S}(\tilde{Z} - X(\mathbb{C}))$. Cap-product with these classes provides us with a homotopy-commutative diagram:

$$\begin{array}{ccc} K^{A.S}_M(\tilde{Z}(\mathbb{C})) & \xrightarrow{\simeq} & K^M_{A.S}(\tilde{Z}(\mathbb{C})) \\ \downarrow & & \downarrow \\ K^{A.S}_M(\tilde{Z} - X(\mathbb{C})) & \xrightarrow{\simeq} & K^M_{A.S}(\tilde{Z} - X(\mathbb{C})) \end{array}$$

(3.4) Therefore, cap-product with $[\tilde{Z}]$ induces a weak-equivalence of the homotopy-fibers of the two columns: i.e. cap-product with $[\tilde{Z}]$ induces a weak-equivalence $K^{A.S}_{M,X(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \xrightarrow{\simeq} K^M_{A.S}(X(\mathbb{C}))$. (Observe that one has a localization sequence: $K^M_{A.S}(X(\mathbb{C})) \to K^M_{A.S}(\tilde{Z}(\mathbb{C})) \to K^M_{A.S}(\tilde{Z} - X(\mathbb{C}))$.) This is the required *Poincaré-Lefschetz-duality.* This will be denoted $P - L_{as}$.

Let $f : X \to Y$ denote a $G$-equivariant *proper* map. Now one may imbed $f$ in a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & \bar{X} & \longleftarrow & \bar{X}-X \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow \\ Y & \longrightarrow & \bar{Y} & \longleftarrow & \bar{Y}-Y \end{array}$$

where $X \to \bar{X}$ ($Y \to \bar{Y}$) is the $G$-equivariant open immersion into a $G$-projective variety. Moreover the two smaller squares are cartesian. It follows that, under the above assumptions, one obtains an induced map

(3.5) $f_* : K^M_{A.S}(X(\mathbb{C})) \to K^M_{A.S}(Y(\mathbb{C}))$.

(3.6) Under the above hypotheses, let

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \tilde{Y} \end{array}$$

denote a pull-back square with the horizontal maps being $G$-equivariant closed immersions into smooth $G$-quasi-projective varieties. By factoring $f$ as the composition of a $G$-equivariant closed immersion into a projective space $\mathbb{P}^n \times Y$ and the projection from the latter to $Y$, we have shown how to define $f_* : K^{A.S}_{M,X(\mathbb{C})}(\tilde{X}(\mathbb{C})) \to K^{A.S}_{M,Y(\mathbb{C})}(\tilde{Y}(\mathbb{C}))$. It remains to show this map is compatible with the map $f_*$ in (3.5) under the weak-equivalence in (3.4); one may prove this separately for $f$ a closed immersion $X \to \mathbb{P}^n \times Y$ and also the projection $\mathbb{P}^n \times Y \to Y$. We skip the direct verification of these.

We will end this section by providing an intrinsic definition of equivariant topological K-homology. Assume as in (1.9) that $G$ is a complex linear algebraic group acting on a $G$-quasi-projective variety $Z$ and that $Z \to \tilde{Z}$ is a $G$-equivariant closed immersion into a $G$-quasi-projective smooth variety. Let $i : Y \to Z$ denote the closed immersion of a $G$-stable subvariety and let $U$ denote its complement. Let $\tilde{U} = \tilde{Z} - Y$; this is a $G$-quasi-projective smooth variety and $U \to \tilde{U}$ is a $G$-equivariant closed immersion. Let $M$ denote the maximal compact subgroup of $G$.

(3.7)**Proposition**. Under the above hypotheses, one obtains fibration-sequences:

$K^{A.S}_{M,Y(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{A.S}_{M,U(\mathbb{C})}(\tilde{U}(\mathbb{C}))$ and

$K^{top}_{M,Y(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{top}_{M,U(\mathbb{C})}(\tilde{U}(\mathbb{C}))$

*Proof.* We will prove the existence of only the second fibration sequence; a similar proof applies to the first. Consider the commutative diagram:

$$\begin{array}{ccccc} K^{top}_{M,Y(\mathbb{C})}(\tilde{Z}(\mathbb{C})) & \longrightarrow & K^{top}_{M}(\tilde{Z}(\mathbb{C})) & \longrightarrow & K^{top}_{M}(\tilde{Z}-Y(\mathbb{C})) \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle id=\gamma}\downarrow & & \downarrow{\scriptstyle \beta} \\ K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) & \longrightarrow & K^{top}_{M}(\tilde{Z}(\mathbb{C})) & \longrightarrow & K^{top}_{M}(\tilde{Z}-Z(\mathbb{C})) \end{array}$$

The two rows are cofibration sequences; therefore the homotopy cofibers provide a cofibration-sequence:

$$cof(\alpha) \to cof(\gamma) \to cof(\beta)$$

However, $cof(\gamma) \simeq *$; therefore $cof(\beta) \simeq \Sigma cof(\alpha)$ or (since we are considering spectra) $\Omega cof(\beta) \simeq cof(\alpha)$. But $U = Z - Y$ is the complement of the open subvariety $\tilde{Z} - Z$ in $\tilde{U} = \tilde{Z} - Y$. Therefore one obtains a fibration sequence:

$$K^{top}_{M,U(\mathbb{C})}(\tilde{U}(\mathbb{C})) \to K^{top}_{M}(\tilde{Z} - Y(\mathbb{C})) \to K^{top}_{M}(\tilde{Z} - Z(\mathbb{C})).$$

It follows that $\Omega cof(\beta) \simeq K^{top}_{M,U(\mathbb{C})}(\tilde{U}(\mathbb{C}))$. Therefore the required fibration-sequence is merely the cofibration-sequence:

$$K^{top}_{M,Y(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \xrightarrow{\alpha} K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to cof(\alpha).$$

This completes the proof.

(3.8) **Theorem**. Assume the above situation. Now the obvious map $K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$ induces an isomorphism: $\pi_i(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))\hat{}_{I_M} \to \pi_i(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})))$ for all $i$. Here $I_M$ is the augmentation ideal in the representation ring $R(M)$ and $\hat{}_{I_M}$ denotes completion with respect to $I_M$.

*Proof.* The naturality of the maps in (6.1.4) and (6.1.5) along with the observation that $R(T_c)$ is a finite module over $R(M)$ (where $T_c$ is the maximal compact torus in $T$) show that it suffices to prove the theorem with $M$ replaced by $T_c$. Observe that the theorem is true if $X$ has Krull dimension 0. This fact and (3.7) show that one may use ascending induction on the Krull dimension to establish the theorem. Moreover, since $M$ has been replaced by $T_c$ it suffices to prove the theorem when $X$ has been replaced by a $T$-stable open subscheme $U$ as in (6.3).

Recall from (6.3) that $U$ is smooth. Therefore one may assume that $\tilde{U} = U$. i.e. One may identify $K^{A.S}_{T_c,U(\mathbb{C})}(\tilde{U}(\mathbb{C}))$ in (3.7) with $K^{A.S}_{T_c}(U(\mathbb{C}))$ and similarly one may identify $K^{top}_{T_c,U(\mathbb{C})}(\tilde{U}(\mathbb{C}))$ in (3.7) with $K^{top}_{T_c}(U(\mathbb{C}))$. Now one may compute (using the Kunneth-formula and the observation that $\pi_*(K^{A.S}_{T_c}(T_c'')) = \pi_*(K^{A.S}_{T_c}(T_c/T_c')) \simeq \pi_*(K^{A.S}_{T_c'})$ is flat over $\pi_*(KU)$ and $\pi_*(K^{top}_{T_c}(T_c'')) = \pi_*(K^{top}_{T_c}(T_c/T_c')) \simeq \pi_*(K^{top}_{T_c'})$ is flat over $\pi_*(KU)$):

$$\pi_*(K^{A.S}_{T_c}(U(\mathbb{C}))) = \pi_*(K^{A.S}_{T_c}(U/T(\mathbb{C}) \times T''(\mathbb{C}))) \simeq \pi_*(K^{top}(U/T(\mathbb{C}))) \underset{\pi_*(KU)}{\otimes} \pi_*(K^{A.S}_{T_c}(T_c'')) \text{ and}$$

$$\pi_*(K^{top}_{T_c}(U(\mathbb{C}))) = \pi_*(K^{top}_{T_c}(U/T(\mathbb{C}) \times T''(\mathbb{C}))) \simeq \pi_*(K^{top}(U/T(\mathbb{C}))) \underset{\pi_*(KU)}{\otimes} \pi_*(K^{top}_{T_c}(T_c''))$$

Clearly $\pi_i(K^{A.S}_{T_c'})\hat{}_{I_{T_c}} \simeq \pi_i(K^{top}_{T_c'})$ for every $i$. The theorem follows.

(3.9)**Remark** . See [A-S] p. 9 for an example of a finite C.W-complex $X$ provided with the action by a finite group where the equivariant topological K-theory fails to be the completion of the corresponding equivariant Atiyah-Segal K-theory. Thus the assumption that $X$ is an algebraic variety (of finite type over $\mathbb{C}$) provided with an algebraic group action seems essential in the above theorem. (This assumption enables us to use the generic-slices in (6.3).)

(3.10)**Definition**. In view of the above theorem we will make the following definition of equivariant topological K-homology. Assume the above situation. Now $K^{top}_{M,i}(Z(\mathbb{C}) = \pi_i(K^{A.S}_M(Z(\mathbb{C}))\hat{}_{I_M}$. Clearly these groups are intrinsically defined and are covariant functors for proper $M$-equivariant maps.

(3.11) **Corollary**. Assume the above situation. Now one obtains a Poincaré-Lefschetz-duality isomorphism: $\pi_*(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{\simeq} K^M_{top,*}(Z(\mathbb{C}))$. This will be denoted $P - L_{top}$.

(3.12) Assume that $Z$ is a $G$-quasi-projective variety as in (3.8). We will conclude this section by showing that the composition

$$K^M_{top,0}(Z(\mathbb{C})) \xrightarrow{P-L^{-1}_{top}} \pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{\tau^{M,\tilde{Z}}_Z} H^M_*(Z(\mathbb{C});\ \mathbb{Q})$$

is *independent* of the closed immersion $Z \to \tilde{Z}$. Similar arguments will prove that the compositions

$$\pi_0(K^M_{A.S}(Z(\mathbb{C}))) \xrightarrow{P-L^{-1}_{as}} \pi_0(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{\tau^{M,\tilde{Z}}_Z \circ B} H^M_*(Z(\mathbb{C});\ \mathbb{Q}) \text{ and}$$

$$\pi_0(K^G(Z)) \xrightarrow{P-L^{-1}_a} \pi_0(K_{G,Z}(\tilde{Z})) \xrightarrow{\tau^{M,\tilde{Z}}_Z \circ B \circ \rho} H^M_*(Z(\mathbb{C});\ \mathbb{Q})$$

are independent of the closed immersion $Z \to \tilde{Z}$. This will follow from the following two propositions.

(3.13)**Proposition**. Let $Z$ be as before and let $Z \to Y_0 \xrightarrow{i} Y_1$ denote two $G$-equivariant closed immersions so that $Y_0$ and $Y_1$ are also *smooth*. Now one obtains the commutative squares:

$$\begin{array}{ccc}
\pi_0(K_{G,Z}(Y_0)) & \underset{\simeq}{\xrightarrow{Gysin}} & \pi_0(K_{G,Z}(Y_1)) \\
\downarrow & & \downarrow \\
\pi_0(K^{A.S}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C}))) & \underset{\simeq}{\xrightarrow{Gysin}} & \pi_0(K^M_{A.S,Z(\mathbb{C})}(Y_1(\mathbb{C}))) \\
\downarrow & & \downarrow \\
\pi_0(K^{top}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C}))) & \underset{\simeq}{\xrightarrow{Gysin}} & \pi_0(K^M_{top,Z(\mathbb{C})}(Y_1(\mathbb{C}))) \\
\tau^{M,Y_0}_Z \downarrow & & \downarrow \tau^{M,Y_1}_Z \\
H^M_*(Z(\mathbb{C});\mathbb{Q}) & \xrightarrow{id} & H^M_*(Z(\mathbb{C});\mathbb{Q})
\end{array}$$

where the horizontal maps are all isomorphisms provided by Gysin-maps.

*Proof.* Let $W$ be the blow-up of $Y_1 \times \mathbb{C}$ along $Y_0 \times 0$. Now one observes as in [B-F-M] pp. 113-114 that one obtains the commutative diagram:

$$\begin{array}{ccccc} Y_0 & \xrightarrow{j_1} & Y_0 \times \mathbb{C} & \xleftarrow[j_0]{} & Y_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{i}} \\ Y_1 & \xrightarrow{k_1} & W & \xleftarrow{k_0} & N \end{array}$$

Here $\bar{i} : Y_0 \to N$ is the closed immersion of $Y_0$ as the zero-section into the normal cone $N$ associated to the closed immersion $i$. One may observe readily that all the maps in the above diagram are $G$-equivariant. The above diagram, along with excision (which all the four cohomology theories above have) and the homotopy property (2.8.6), reduce the problem to the case where $i$ is replaced by $\bar{i}$. Now the commutativity of the top two squares follows from the observation that the Gysin-maps are all compatible. This results from the observation that top three of the horizontal maps above are cup-products with the Koszul-Thom class of the normal bundle associated to the closed immersion $Y_0 \to N$ and the Koszul-Thom classes in the three $K$-cohomology theories are compatible. (See for example [B-F-M] p.166.)

To prove the commutativity of the bottom square, one only needs to make the following observations: $Td^M(\mathcal{T}_{Y_0}) = Td^M(N)^{-1} \cup Td^M(\mathcal{T}_{Y_1})$ and $ch_Z^{M,N}(Gysin(\mathcal{E})) = ch_Z^{M,Y_0}(\mathcal{E}) \cap Td^M(N_{|Z})^{-1}$, where $\mathcal{E} \in \pi_0(K^{top}_{M,Z(\mathbb{C})}(Y_0(\mathbb{C})))$. These are well-known in the non-equivariant context and they extend to the equivariant case by naturality. (The first is a consequence of the multiplicativity of the Todd class. The second follows from (2.8.5).) Excision shows the top-three horizontal maps are isomorphisms. This completes the proof of the proposition.

Next we will assume for simplicity that $Z$ is *G-projective*. If $i_j : Z \to \tilde{Z}_j$, $j = 1, 2$ are two $G$-equivariant closed immersions, in view of (3.13) one may assume that $\tilde{Z}_j$, $j = 1, 2$ are projective spaces $\mathbb{P}^n$ of large enough dimension on which $G$ acts linearly. By taking the product of the two immersions, one obtains a third closed immersion $i_3 : Z \to \mathbb{P}^n \times \mathbb{P}^m$ and this factors as $Z \xrightarrow{\Delta} Z \times \mathbb{P}^m \to \mathbb{P}^n \times \mathbb{P}^m$.

(3.14)**Proposition**. Assume the above situation. Now one obtains a commutative square:

$$\begin{array}{ccccc} K^M_{top,0}(Z(\mathbb{C})) & \xrightarrow{P-L^{-1}_{top}} & \pi_0(K^{top}_{M,Z(\mathbb{C})}(\mathbb{P}^n \times \mathbb{P}^m(\mathbb{C}))) & \xrightarrow{\tau^M} & H^M_*(Z(\mathbb{C});\ \mathbb{Q}) \\ {\scriptstyle \Delta_*}\downarrow & & {\scriptstyle \Delta_*}\downarrow & & {\scriptstyle \Delta_*}\downarrow \\ K^M_{top,0}(Z \times \mathbb{P}^m(\mathbb{C})) & \xrightarrow{P-L^{-1}_{top}} & \pi_0(K^{top}_{M,Z\times\mathbb{P}(\mathbb{C})^m}(\mathbb{P}^n \times \mathbb{P}^m(\mathbb{C}))) & \xrightarrow{\tau^M} & H^M_*(Z \times \mathbb{P}(\mathbb{C})^m;\ \mathbb{Q}) \\ {\scriptstyle p_{1*}}\downarrow & & {\scriptstyle p_{1*}}\downarrow & & {\scriptstyle p_{1*}}\downarrow \\ K^M_{top,0}(Z(\mathbb{C})) & \xrightarrow{P-L^{-1}_{top}} & \pi_0(K^{top}_{M,Z(\mathbb{C})}(\mathbb{P}^n)) & \xrightarrow{\tau^M} & H^M_*(Z(\mathbb{C});\ \mathbb{Q}) \end{array}$$

*Proof.* The commutativity of the two squares on the right follows from (2.9). The left-most vertical maps are defined so as to make the two left-squares commute. The composition of the vertical maps is the identity by (3.6), (3.8) and (3.10).

4. **The equivariant Riemann-Roch theorem (at the level of equivariant homology theories).**

We will presently combine the Riemann-Roch theorem (2.9) with the definition of equivariant K-homology to obtain the following strong form of an equivariant Riemann-Roch theorem.

(4.1) Let $G$ denote a complex linear algebraic group acting on a $G$-quasi-projective variety $Z$ as in (1.9) and let $M$ denote a maximal compact subgroup of $G$. Let $Z \to \tilde{Z}$ denote a $G$-equivariant closed immersion into a smooth $G$-quasi-projective variety. We define natural transformations:

$K^G(Z) \xrightarrow{\rho} K^M_{A.S}(Z(\mathbb{C}))$ and $\pi_*(K^M_{A.S}(Z(\mathbb{C}))) \xrightarrow{B} \pi_*(K^M_{top}(Z(\mathbb{C})))$ as follows. The map $\rho$ will be the composition:

$K^G(Z) \xrightarrow{P-L_a^{-1}} K_{G,Z}(\tilde{Z}) \xrightarrow{\rho} K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \xrightarrow{P-L_{as}} K^M_{A.S}(Z(\mathbb{C}))$. The map $B$ is merely the completion at the augmentation ideal in $R(M)$. In view of (3.8) the map $B$ factors as the composition

$\pi_*(K^M_{A.S}(Z(\mathbb{C}))) \xrightarrow{P-L_{as}^{-1}} \pi_*(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{B} \pi_*(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{P-L_{top}} K^M_{top,*}(Z(\mathbb{C}))$. Finally we define an *M-equivariant Chern-character* $ch^M : K^M_{top,0}(Z(\mathbb{C})) \to H^M_*(Z(\mathbb{C});\ \mathbb{Q})$ as the composition $K^M_{top,0}(Z(\mathbb{C})) \xrightarrow{P-L_{top}^{-1}} \pi_0(K^{top}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \xrightarrow{ch_Z^{M,\tilde{Z}}} H^M_*(Z(\mathbb{C});\ \mathbb{Q})$. One defines a Todd-homomorphism $\tau^M : K^M_{top.0}(Z(\mathbb{C})) \to H^M_*(Z(\mathbb{C});\ \mathbb{Q})$ similarly.

(4.2)**Theorem.** (Equivariant Riemann-Roch).

Assume that $f : X \to Y$ is a $G$-equivariant *proper* map between two $G$-quasi-projective varieties. In this case one obtains the commutative diagram :

$$\begin{array}{ccccccc} \pi_0(K^G(X)) & \xrightarrow{\pi_0(\rho)} & \pi_0(K^M_{A.S}(X(\mathbb{C}))) & \xrightarrow{B} & K^M_{top,0}(X(\mathbb{C})) & \xrightarrow{\tau^M} & H^M_*(X(\mathbb{C});\mathbb{Q}) \\ f_* \downarrow & & \downarrow f_* & & f_* \downarrow & & \downarrow f_* \\ \pi_0(K^G(Y)) & \xrightarrow{\pi_0(\rho)} & \pi_0(K^M_{A.S}(Y(\mathbb{C}))) & \xrightarrow{B} & K^M_{top,0}(Y(\mathbb{C})) & \xrightarrow{\tau^M} & H^M_*(Y(\mathbb{C});\mathbb{Q}) \end{array} \qquad (2.4.*)$$

*Proof.* As observed in (1.14), one may factor $f$ as the composition of a $G$-invariant closed immersion $i$ into a projective space $\mathbb{P}^n_Y$ onto which the $G$-action extends followed by the obvious projection $\pi : \mathbb{P}^n_Y \to Y$. Therefore it suffices to prove the commutativity of the two squares for the two special cases when $f = i$ or $f = \pi$. These follow readily from (2.9) and the definition of the natural transformations in (4.1). This completes the proof of the theorem.

## 5. Applications

One of the main applications of the above Riemann-Roch theorems is to the construction of modules over the affine Hecke algebra associated to a complex reductive group from the equivariant derived category on the unipotent variety. (See [J-2].) We foresee further applications to representations of quantum groups as well. We first state some results in a form suitable for such a variety of applications.

(5.1) **Proposition**. Let $f : X \rightarrow Y$ denote a $G$-equivariant proper map between $G$-quasi-projective varieties. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

denote a commutative cartesian square with $\tilde{X}$ ( $\tilde{Y}$) a $G$-quasi-projective *smooth* variety containing $X$ ($Y$, respectively ) as a $G$-stable closed subvariety and $\tilde{f}$ a $G$-equivariant proper map. Let $M$ denote the maximal compact subgroup of $G$. Now one obtains a commutative diagram:

$$\begin{array}{ccccccc} K^M_{top,0}(X(\mathbb{C})) & \xleftarrow{P-L_{top}} & \pi_0(K^{top}_{M,X(\mathbb{C})}(\tilde{X}(\mathbb{C}))) & \xrightarrow{ch^{M,\tilde{X}}_X} & H^*_{M,X}(\tilde{X};\mathbb{Q}) & \xrightarrow{P-L} & H^M_*(X;\mathbb{Q}) \\ f_*\downarrow & & f_*\downarrow & & \downarrow\tilde{f}_* & & \downarrow\bar{f}_* \\ K^M_{top,0}(Y(\mathbb{C})) & \xleftarrow{P-L_{top}} & \pi_0(K^{top}_{M,Y(\mathbb{C})}(\tilde{Y}(\mathbb{C}))) & \xrightarrow{ch^{M,\tilde{Y}}_Y} & H^*_{M,Y}(\tilde{Y};\mathbb{Q}) & \xrightarrow{P-L} & H^M_*(Y;\mathbb{Q}) \end{array}$$

where $ch^{M,\tilde{X}}_X$ and $ch^{M,\tilde{Y}}_Y$ are the local Chern-characters defined in (2.7), the second-map $f_*$ is defined so as to make the left-most square commute. The map $P-L_{top}$ ($P-L$) is the Poincaré-Lefschetz-duality-map considered in (3.11) ((2.6), respectively .) The last map $\bar{f}_*$ is defined as the composition:

$$H^M_*(X;\mathbb{Q}) \xrightarrow{\cap Td_{\tilde{X}|X}} H^M_*(X;\mathbb{Q}) \xrightarrow{f_*} H^M_*(Y;\mathbb{Q}) \xrightarrow{\cap Td_{\tilde{Y}|Y}^{-1}} H^M_*(Y;\mathbb{Q})$$

and the third vertical map $\bar{f}_*$ is defined so as to make the last square commute.

*Proof.* This is clear from the Riemann-Roch theorem (4.2).

Many of the above-mentioned applications involve the convolution in K-theory considered by Lusztig, Ginzburg (see [C-G]) and others.

(5.2) **Convolution**. Let $\tilde{Z}_i$, $i = 1,2,3$ denote $G$-quasi-projective smooth varieties and let $Z_{1,2} \subseteq \tilde{Z}_1 \times \tilde{Z}_2$, $Z_{2,3} \subseteq \tilde{Z}_2 \times \tilde{Z}_3$, $Z_{1,3} \subseteq \tilde{Z}_1 \times \tilde{Z}_3$ denote $G$-stable closed subvarieties. Assume further that the following condition holds. Let $p_{i,j} : \tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3 \rightarrow \tilde{Z}_i \times \tilde{Z}_j$ denote the projections to the $(i,j)$-th factor for $i < j$, $1 \leq i,j \leq 3$. Now the restriction of $p_{1,3}$ to $p^{-1}_{1,2}(Z_{1,2}) \cap p^{-1}_{2,3}(Z_{2,3}) \rightarrow \tilde{Z}_1 \times \tilde{Z}_3$ is *proper and maps into* $Z_{1,3}$.

One may now define a convolution

$* : \pi_0(K_{G,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2)) \otimes \pi_0(K_{G,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3)) \rightarrow \pi_0(K_{G,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3))$ as the composition

(5.1.1) $\pi_0(K_{G,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2)) \otimes \pi_0(K_{G,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3))$

$\xrightarrow{p_{1,2}^* \otimes p_{2,3}^*} \pi_0(K_{G,p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3})}(\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3))$

$\rightarrow \pi_0(K_{G,p_{1,3}^{-1}(Z_{1,3})}(\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3)) \xrightarrow{p_{1,3*}} \pi_0(K_{G,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3))$

By pre-composing the above operation with the inverse of the Poincaré-Lefschetz-duality

$\pi_0(K_{G,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2)) \otimes \pi_0(K_{G,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3)) \xrightarrow{P-L_a \otimes P-L_a} \pi_0(K^G(Z_{1,2})) \otimes \pi_0(K^G(Z_{2,3}))$

and by following it with the Poincaré-Lefschetz-duality $\pi_0(K_{G,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3)) \xrightarrow{P-L_a} \pi_0(K^G(Z_{1,3}))$ one obtains a convolution:

$* : \pi_0(K^G(Z_{1,2})) \otimes \pi_0(K^G(Z_{2,3})) \rightarrow \pi_0(K^G(Z_{1,3}))$

If $M$ denotes the maximal compact subgroup of $G$, one may define similar convolutions

(5.2.2) $* : \pi_0(K^{A.S}_{M,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2)) \otimes \pi_0(K^{A.S}_{M,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3)) \rightarrow \pi_0(K^{A.S}_{M,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3))$

(5.2.3) $* : \pi_0(K^{top}_{M,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2)) \otimes \pi_0(K^{top}_{G,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3)) \rightarrow \pi_0(K^{top}_{G,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3))$

as well as $* : \pi_0(K^M_{A.S}(Z_{1,2}(\mathbb{C}))) \otimes \pi_0(K^M_{A.S}(Z_{2,3}(\mathbb{C}))) \rightarrow \pi_0(K^M_{A.S}(Z_{1,3}(\mathbb{C})))$ and $* : K^M_{top,0}(Z_{1,2}(\mathbb{C})) \otimes K^M_{top,0}(Z_{2,3}(\mathbb{C})) \rightarrow K^M_{top,0}(Z_{1,3}(\mathbb{C}))$. One may also define a convolution

(5.2.4) $* : H^*_{M,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2;\, \mathbb{Q}) \otimes H^*_{M,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3;\, \mathbb{Q})$

$\xrightarrow{p_{1,2}^* \otimes p_{2,3}^*} H^*_{M,p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3})}(\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3;\, \mathbb{Q})$

$\rightarrow H^*_{M,p_{1,3}^{-1}(Z_{1,3})}(\tilde{Z}_1 \times \tilde{Z}_2 \times \tilde{Z}_3;\, \mathbb{Q}) \xrightarrow{\hat{p}_{1,3*}} H^*_{M,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3;\, \mathbb{Q})$

where $\hat{p}_{1,3*}$ is *either the map* $p_{1,3*}$ or the map $\bar{p}_{1,3*}$ defined in (5.1). By pre-composing this operation with the inverse of the Poincaré-Lefschetz-duality $H^*_{M,Z_{1,2}}(\tilde{Z}_1 \times \tilde{Z}_2;\, \mathbb{Q}) \otimes H^*_{M,Z_{2,3}}(\tilde{Z}_2 \times \tilde{Z}_3;\, \mathbb{Q}) \xrightarrow{P-L \otimes P-L} H^M_*(Z_{1,2};\, \mathbb{Q}) \otimes H^M_*(Z_{2,3};\, \mathbb{Q})$ and following it with the Poincaré-Lefschetz-duality $H^*_{M,Z_{1,3}}(\tilde{Z}_1 \times \tilde{Z}_3;\, \mathbb{Q}) \xrightarrow{P-L} H^M_*(Z_{1,3};\, \mathbb{Q})$ one may obtain a convolution $* : H^M_*(Z_{1,2};\, \mathbb{Q}) \otimes H^M_*(Z_{2,3};\, \mathbb{Q}) \rightarrow H^M_*(Z_{1,3};\, \mathbb{Q})$.

(5.2.5) **Proposition**. The last two convolutions, with $\hat{p}_{1,3*} = \bar{p}_{1,3*}$, are compatible with the convolutions defined in (5.2.3).

*Proof.* This follows readily from Proposition (5.1). (The local chern-character clearly commutes with inverse-images; (5.1) shows it also commutes with the direct-image maps when $\hat{p}_{1,3*} = \bar{p}_{1,3*}$.)

To apply this to the case considered in [J-2], (i.e. to the construction of modules over the affine Hecke algebra associated to a reductive group) we first recall the basic set-up. Let $G$ denote a complex reductive group; $\mathcal{U}$ will denote the variety of all unipotent elements in $G$ and $\mu : \Lambda \rightarrow \mathcal{U}$ will denote the $G$-equivariant resolution of singularities due to Steinberg. Let $\mathcal{B}$ denote the variety of all Borel subgroups of $G$. (Recall that $\Lambda = \{(u, B) \,\epsilon\, \mathcal{U} \times \mathcal{B} | u \,\epsilon\, B\}$ and that $\mu$ is now simply the projection to the first factor.) One obtains an action of $G \times \mathbb{C}^*$ on $\mathcal{U}$ on the right by: $u.(g, q) = g^{-1}u^q g$. There is a similar action on $\Lambda$ defined by: $(u, B).(g, q) = (g^{-1}u^q g, g^{-1}Bg)$. Finally $Z = \Lambda \underset{\mathcal{U}}{\times} \Lambda = \{(u, B', B) | u \,\epsilon\, \mathcal{U}, B', B \,\epsilon\, \mathcal{B}, u \,\epsilon\, B' \cap B\}$. Now there are two possible situations where we may define a convolution as in (5.2). In both cases the group that acts on the varieties will be $G \times \mathbb{C}^*$ and $M$ will denote its maximal compact subgroup.

(5.3.1) We will let $\tilde{Z}_i$ (as in (5.2)) $= \Lambda$ for all $i = 1, 2, 3$. Now each $\tilde{Z}_i$ is a $G$-quasi-projective *smooth* variety and $Z$ imbeds $G \times \mathbb{C}^*$-equivariantly as a closed subvariety of $\Lambda \times \Lambda$. We let $Z_{i,j} = Z$ for $(i, j) = (1, 2)$, $(i, j) = (2, 3)$ or $(i, j) = (1, 3)$. Now the map $p_{1,3} : p_{1,2}^{-1}(Z) \cap p_{2,3}^{-1}(Z) \rightarrow \Lambda^3 \xrightarrow{p_{1,3}} \Lambda^2$ is proper and maps into $Z$. Therefore the hypotheses of (5.2) are met and we obtain convolution-products

$* : \pi_0(K^{G \times \mathbb{C}^*}(Z)) \otimes \pi_0(K^{G \times \mathbb{C}^*}(Z)) \rightarrow \pi_0(K^{G \times \mathbb{C}^*}(Z))$, $* : \pi_0(K^M_{A.S}(Z(\mathbb{C}))) \otimes \pi_0(K^M_{A.S}(Z(\mathbb{C}))) \rightarrow \pi_0(K^M_{A.S}(Z(\mathbb{C})))$, $* : K^M_{top,0}(Z(\mathbb{C})) \otimes K^M_{top,0}(Z(\mathbb{C})) \rightarrow K^M_{top,0}(Z(\mathbb{C}))$, and $* : H^M_*(Z(\mathbb{C}); \mathbb{Q}) \otimes H^M_*(Z(\mathbb{C}); \mathbb{Q}) \rightarrow H^M_*(Z(\mathbb{C}); \mathbb{Q})$ .

(5.3.2) In the second situation we let $\tilde{Z}_i$ (as in (5.2)) $= \mathcal{B}$ for all $i = 1, 2, 3$. We let $Z_{i,j} = \mathcal{B}^2$ for $(i, j) = (1, 2)$, $(i, j) = (2, 3)$ and $(i, j) = (1, 3)$. Once again the hypotheses of (5.2) are satisfied and we obtain convolution-products

$* : \pi_0(K^{G \times \mathbb{C}^*}(\mathcal{B}^2)) \otimes \pi_0(K^{G \times \mathbb{C}^*}(\mathcal{B}^2)) \rightarrow \pi_0(K^{G \times \mathbb{C}^*}(\mathcal{B}^2))$, $* : \pi_0(K^M_{A.S}(\mathcal{B}^2(\mathbb{C}))) \otimes \pi_0(K^M_{A.S}(\mathcal{B}^2(\mathbb{C}))) \rightarrow \pi_0(K^M_{A.S}(\mathcal{B}^2(\mathbb{C})))$, $* : K^M_{top,0}(\mathcal{B}^2(\mathbb{C})) \otimes K^M_{top,0}(\mathcal{B}^2(\mathbb{C})) \rightarrow K^M_{top,0}(\mathcal{B}^2(\mathbb{C}))$ and $* : H^M_*(\mathcal{B}^2(\mathbb{C}); \mathbb{Q}) \otimes H^M_*(\mathcal{B}^2(\mathbb{C}); \mathbb{Q}) \rightarrow H^M_*(\mathcal{B}^2(\mathbb{C}); \mathbb{Q})$.

(5.3.3) **Theorem**. (i) The maps $\pi_0(K^{G \times \mathbb{C}^*}(Z)) \xrightarrow{\pi_0(\rho)} \pi_0(K^M_{A.S}(Z(\mathbb{C}))) \xrightarrow{B} \pi_0(K^M_{top}(Z(\mathbb{C})))$ and the maps $\pi_0(K^{G \times \mathbb{C}^*}(\mathcal{B}^2)) \xrightarrow{\pi_0(\rho)} \pi_0(K^M_{A.S}(\mathcal{B}^2(\mathbb{C}))) \xrightarrow{B} \pi_0(K^M_{top}(\mathcal{B}^2(\mathbb{C})))$ are homomorphisms that preserve the convolution products in (5.3.2).

(ii) The maps

$$K^M_{top,0}(Z(\mathbb{C})) \xrightarrow{P-L^{-1}_{top}} \pi_0(K^{top}_{M,Z(\mathbb{C})}(\Lambda^2(\mathbb{C}))) \xrightarrow{ch^{M,\Lambda^2}_Z} H^*_{M,Z}(\Lambda^2; \mathbb{Q}) \xrightarrow{P-L} H^M_*(Z; \mathbb{Q}) \text{ and}$$

$$K^M_{top,0}(\mathcal{B}^2(\mathbb{C})) \xrightarrow{P-L^{-1}_{top}} \pi_0(K^{top}_M(\mathcal{B}^2(\mathbb{C}))) \xrightarrow{ch^{M,\mathcal{B}^2}} H^*_M(\mathcal{B}^2; \mathbb{Q}) \xrightarrow{P-L} H^M_*(\mathcal{B}^2; \mathbb{Q})$$

also preserve the convolution products in (5.3.1).

*Proof* These are clear from (5.1) and the definitions in (5.3.1) and (5.3.2).

We will next show that the map $\rho$ (as in (4.1)) induces an isomorphism when the variety $X$ is $Z$ or if $X$ is the flag-manifold $\mathcal{B}^n$ $(n \geq 1)$ both provided with the diagonal action of $G \times \mathbb{C}^*$. (Recall that $\mathbb{C}^*$ acts trivially on $\mathcal{B}$.) Let $W$ denote the Weyl group of $G$; for each $y \,\epsilon\, W$ one defines

(5.4) $Z_y = \{(u, B, B') \,\epsilon\, Z | (B, B') \,\epsilon\, O(y) = \text{the orbit of } G \text{ on } \mathcal{B} \times \mathcal{B} \text{ indexed by } y\}$

and $Z_I = \underset{y \,\epsilon\, I}{\cup} Z_y$, for each subset $I$ of $W$. Throughout $M$ will denote a maximal compact subgroup of $G \times \mathbb{C}^*$.

(5.5) **Theorem**. The map $\rho$ in (4.1) induces isomorphisms:

(i) $\pi_0(K(Mod_{coh}^{G\times\mathbb{C}^*}((\mathcal{B})^n)))) \xrightarrow{\rho} \pi_0(K^M_{A.S}(\mathcal{B}(\mathbb{C})^n))$ for all $n \geq 0$

(ii)$\pi_0(K(Mod_{coh}^{G\times\mathbb{C}^*}(Z_I))) \xrightarrow{\rho} \pi_0(K^M_{A.S}(Z_I(\mathbb{C})))$ for each subset $I$ of $W$.

(iii)In particular $K_0(Mod_{coh}^{G\times\mathbb{C}^*}(Z_I)) \otimes Q$ is a projective $R_{G\times\mathbb{C}^*} \otimes Q$-module of rank = (cardinality $I$).(cardinality $W$).

*Proof.* The proof of (i) is essentially an application of the localization sequence to the natural filtration on $\mathcal{B}$ by Schubert cells along with an application of the *five lemma* in a strong-form. (To apply the five lemma, one may first prove, using the above filtration, that $\pi_1(K^M_{A.S}(\mathcal{B}(\mathbb{C})^n) = 0$ for all $n$.) The third statement of the proof follows from the second statement using the computation of $\pi_0(K^M_{A.S}(Z_I(\mathbb{C}))) \otimes Q$ as in ([K-L] (3.1.6)). If $I$ consists of only a single element $y \,\epsilon\, W$, observe that the obvious map $Z_y \rightarrow \mathcal{B}$ given by $(u, B, B') \rightarrow B$ is a locally trivial fibration with fibers isomorphic to complex affine spaces $\mathbb{C}^\nu$, $\nu$ = the dimension of $\mathcal{B}$. Therefore, in this case, we reduce to proving $\rho$ induces an isomorphism $\pi_0(K(Mod_{coh}^{G\times\mathbb{C}^*}(\mathcal{B}))) \otimes Q \simeq \pi_0(K_M^{A.S}(\mathcal{B}(\mathbb{C}))) \otimes Q$, which is proven in (i).

In general, if $I$ is as above and $y \,\epsilon\, I$, $Z_{\{y' \leq y | y' \,\epsilon\, I\}}$ is closed in $Z_I$ and its complement $Z_{\{y' > y | y' \,\epsilon\, I\}}$ is open in $Z_I$. One may now use the obvious localization sequences along with the strong form of the five lemma to obtain the conclusion. (To apply the five lemma, one needs to first prove that $\pi_1(K^M_{A.S}(Z_I(\mathbb{C})) = 0$. For this, clearly one may use induction on the cardinality of $I$ along with an obvious localization sequence as above.)

Next we consider *a convolution product that arises often in the context of quantum groups.* (See [Lusz] section 3, or [J-4] for example.) Let $\tilde{X}_i$, $= 1, 2, 3, 4, 5$ denote smooth quasi-projective varieties. Let $G_i$, $i = 1, 2, 3$ denote three complex linear algebraic groups so that $G_i$ acts on $\tilde{X}_i$ for $i = 1, 2$. $G_3$ acts on $\tilde{X}_3$, $\tilde{X}_4$ and $\tilde{X}_5$. Assume that $X_i \subseteq \tilde{X}_i$ are closed subvarieties that are stable under the corresponding actions. We may assume that $G_1 \times G_2 \times G_3$ acts on all of the varieties by letting $G_3$ act trivially on $\tilde{X}_1$ and $\tilde{X}_2$ while $G_1$ and $G_2$ will act trivially on $\tilde{X}_4$ and $\tilde{X}_5$. Let $p_1 : \tilde{X}_3 \rightarrow \tilde{X}_1 \times \tilde{X}_2$, $p_2 : \tilde{X}_4 \rightarrow \tilde{X}_3$ and $p_3 : \tilde{X}_4 \rightarrow \tilde{X}_5$ denote *smooth* maps that are equivariant for the action of $G_1 \times G_2 \times G_3$. Assume further that $p_3$ is *proper* and that $p_2$ is a principal $G_1 \times G_2$-bundle. Now one may define a convolution-product $*$ as the composition:

(5.6.1) $* : \pi_0(K_{G_1,X_1}(\tilde{X}_1)) \otimes \pi_0(K_{G_2,X_2}(\tilde{X}_2)) \xrightarrow{p_1^*} \pi_0(K_{G_1\times G_2\times G_3,X_3}(\tilde{X}_3)) \cong \pi_0(K_{G_3,X_4}(\tilde{X}_4)) \xrightarrow{p_{3*}}$ $\pi_0(K_{G_3,X_5}(\tilde{X}_5))$

The isomorphism $\pi_0(K_{G_1\times G_2\times G_3,X_3}(\tilde{X}_3)) \cong \pi_0(K_{G_3,X_4}(\tilde{X}_4))$ is induced by $p_2^*$ and uses the fact that $p_2$ is a principal $G_1 \times G_2$-bundle. Let $M_i$ denote the maximal compact subgroup of $G_i$. Now one may also define similar convolution-products

$* : \pi_0(K^{A.S}_{M_1,X_1(\mathbb{C})}(\tilde{X}_1(\mathbb{C}))) \otimes \pi_0(K^{A.S}_{M_2,X_2(\mathbb{C})}(\tilde{X}_2(\mathbb{C}))) \rightarrow \pi_0(K^{A.S}_{M_3,X_3(\mathbb{C})}(\tilde{X}_3(\mathbb{C})))$ and

$* : \pi_0(K^{top}_{M_1,X_1(\mathbb{C})}(\tilde{X}_1(\mathbb{C}))) \otimes \pi_0(K^{top}_{M_2,X_2(\mathbb{C})}(\tilde{X}_2(\mathbb{C}))) \rightarrow \pi_0(K^{top}_{M_3,X_3(\mathbb{C})}(\tilde{X}_3(\mathbb{C})))$.

Making use of Poincaré-Lefschetz-duality one may obtain similar convolution products on equivariant $K$-homology. One may also define a convolution-product:

(5.6.2) $* : H^*_{M_1}(X_1; \mathbb{Q}) \otimes H^*_{M_2}(X_2; \mathbb{Q}) \xrightarrow{p_1^*} H^*_{M_1\times M_2\times M_3}(X_3; \mathbb{Q}) \cong H^*_{M_3}(X_4; \mathbb{Q}) \xrightarrow{\bar{p}_{3*}} H^*_{M_3}(X_5; \mathbb{Q})$

where $\bar{p}_{3*}$ is the map defined in (5.1).

(5.6.3) **Proposition**. The convolution-products in (5.6.1) are all compatible under the natural transformations in (1.4), (1.12) and (1.12'). The local equivariant Chern-character maps the convolution-products in (5.6.1) to the convolution-product in (5.6.2).

*Proof.* This is clear from (5.1).

## 6. Special Techniques

In this section we discuss briefly two techniques that are often quite handy in studying equivariant homology theories: namely that of reduction to the action by a maximal torus and the existence of generic slices for torus actions. As in (1.9) we let $G$ denote a complex linear algebraic group $G$ acting on a $G$-quasi-projective variety $Z$. Let $B$ denote a Borel subgroup (i.e. a maximal connected solvable subgroup) and let $T$ denote a maximal torus contained in $B$. Let $M$ denote a maximal compact subgroup of $G$ and let $T_c$ denote the maximal compact torus in $T$. Now one obtains the weak-equivalences (natural in $Z$):

(6.1.1) $K_G(G\underset{B}{\times}Z) \simeq K_B(Z) \simeq K_T(Z)$ , $K^G(G\underset{B}{\times}Z) \simeq K^B(Z) \simeq K^T(Z)$

See [T-3] (1.10). Now the projection map $\pi : G\underset{B}{\times}\tilde{Z} \to \tilde{Z}$ is a proper and smooth $G$-equivariant map and the inverse image of $Z \subseteq \tilde{Z}$ under $\pi$ is $G\underset{B}{\times}Z$; therefore it induces maps

(6.1.2) $p_* : K^T(Z) \simeq K^G(G\underset{B}{\times}Z) \to K^G(Z)$ and $p^* : K^G(Z) \to K^G(G\underset{B}{\times}Z) \simeq K^T(Z)$

Next observe the $M$-equivariant maps $M\underset{T}{\times}Z(\mathbb{C}) \to G\underset{T}{\times}Z(\mathbb{C}) \to G\underset{B}{\times}Z(\mathbb{C})$ are homotopy-equivalences (with equivariant homotopy inverses) and hence induce weak-equivalences:

(6.1.3) $K^{A.S}_{T_c,c}(Z(\mathbb{C})) \simeq K^{A.S}_{M,c}(M\underset{T_c}{\times}Z(\mathbb{C}) \xrightarrow{\simeq} K^{A.S}_{M,c}(G\underset{T}{\times}Z(\mathbb{C})) \xrightarrow{\simeq} K^{A.S}_{M,c}(G\underset{B}{\times}Z(\mathbb{C}))$

and similarly:

(6.1.3') $K^{T_c}_{A.S}(Z(\mathbb{C})) \simeq K^{M}_{A.S}(M\underset{T_c}{\times}Z(\mathbb{C}) \xrightarrow{\simeq} K^{M}_{A.S}(G\underset{T}{\times}Z(\mathbb{C})) \xrightarrow{\simeq} K^{M}_{A.S}(G\underset{B}{\times}Z(\mathbb{C}))$

Clearly the same argument applies to provide similar weak-equivalences in $K^{top}_{M,c}$ and $K^{M}_{top}$.

Assume that $Z \overset{i}{\to} \tilde{Z}$ is the $G$-equivariant closed immersion of two $G$-quasi-projective varieties with $\tilde{Z}$ *smooth*. The map $\pi$ induces maps

(6.1.4) $p_* : K^{A.S}_{T_c,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \simeq K^{A.S}_{M,G\underset{B}{\times}Z(\mathbb{C})}(G\underset{B}{\times}\tilde{Z}(\mathbb{C})) \to K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$

and

(6.1.5) $p^* : K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C})) \to K^{A.S}_{M,G\underset{B}{\times}Z(\mathbb{C})}(G\underset{B}{\times}\tilde{Z}(\mathbb{C})) \simeq K^{A.S}_{T_c,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))$

Similar results hold in equivariant topological K-cohomology of $\tilde{Z}(\mathbb{C})$ with supports in $Z(\mathbb{C})$.

(6.2)**Proposition**. Assume the above situation. Now the composition of the maps $p^*$ and $p_*$ in (6.1.2), in (6.1.5) and (6.1.4) as well as the corresponding maps on equivariant topological K-cohomology induce the identity on taking the homotopy groups.

*Proof.* For the maps in (6.1.2) this is established in [T-3] Theorem (1.13). We will sketch a proof for the maps in (6.1.5) and (6.1.4). The proof for the remaining cases are similar. Observe that the fibers of the projection $\pi$ are all isomorphic to $G/B$; moreover there is a section to this map given by sending a point $\tilde{z}$ of $\tilde{Z}$ to $(e, \tilde{z}) \,\epsilon\, G\underset{B}{\times}\tilde{Z}$. It follows one obtains a natural map:

(6.2.1) $\pi_*(K^{A.S}_M(G/B(\mathbb{C}))) \underset{\pi_*(K^{A.S}_M)}{\otimes} \pi_*(K^{A.S}_{M,Z(\mathbb{C})}(\tilde{Z}(\mathbb{C}))) \to \pi_*(K^{A.S}_{M,G\underset{B}{\times}Z(\mathbb{C})}(G\underset{B}{\times}\tilde{Z}(\mathbb{C}))$

Now one may compute (use the filtration by the Schubert-cells and a localization sequence) $\pi_i(K_M^{A.S}(G/B(\mathbb{C}))$ to be a projective module over $R(M) = \pi_0(K_M^{A.S})$ if $i$ is even and trivial if $i$ is odd. One may similarly compute the right-hand-side (6.2.1) using the filtration by the Bruhat cells in $G$ and a localization sequence; it will readily follow that the above map is an *isomorphism*. Therefore it suffices to show that the composition of the two maps $R(M) = \pi_0(K_M^{A.S}) \xrightarrow{\pi^*} \pi_0(K_M^{A.S}(G/B(\mathbb{C}))) \xrightarrow{\pi_*} \pi_0(K_M^{A.S}) = R(M)$ is the identity. This is clear since $R(M) \cong R(G) \cong \pi_0(K_G(Spec\ \mathbb{C}))$ and $\pi_0(K_M^{A.S}(G/B(\mathbb{C}))) \cong \pi_0(K_G(G/B))$ and since composition of the maps $p^*$ and $p_*$ in (6.1.2) is already observed to be the identity. This completes the proof in equivariant Atiyah-Segal K-cohomology. To obtain the corresponding result for equivariant topological K-cohomology, one first observes that there exists an isomorphism similar to the one in (6.2.1) in this theory. Now the observation that $\pi_0(K_M^{top}(G/B(\mathbb{C}))) \cong$ the completion of $\pi_0(K_M^{A.S}(G/B(\mathbb{C})))$ at the augmentation ideal completes the proof.

(6.3) **Existence of generic slices for torus actions**. (See [T-2] Proposition (4.10).) Let $T$ denote a complex algebraic torus acting on a quasi-projective variety $Z$. Then there exists a non-empty $T$-invariant open subscheme $U$ of $Z$ with the following properties.

i) $U$ is an affine scheme that is non-singular

ii) The geometric quotient $U/T$ exists, it is affine and non-singular and the map $U \to U/T$ is smooth

iii) There exists a diagonalizable subgroup $T'$ of $T$ with quotient torus $T'' = T/T'$ and an action of $T''$ on $U$ so that $T$ acts via the map $T \to T''$. Further $T''$ acts freely on $U$ and there is an isomorphism

$$U \cong T'' \times U/T \cong T/T' \times U/T$$

of schemes with $T$-action.

**Appendix. Equivariant Fredholm complexes.** Here we re-examine the basic setup in [Seg-2].

(A.1) A topological vector space over the complex numbers is admissible if it is Hausdorff, locally convex, complete and has a neighborhood of the origin which contains no half-line. (See [Seg-2] p. 387.) If $X$ is a paracompact space, *a vector bundle* on $X$ is a topological space $E$ with a projection $p : E \to X$ so that the fibers $E_x$ are admissible topological vector spaces and so that $p$ is locally trivial. If $X$ is provided with the action of a compact Lie group $M$, an $M$ vector bundle on $X$ is an $M$-space $E$ provided with an $M$-equivariant projection $p : E \to X$ so that the action $m : E_x \to E_{mx}$ is linear, for each $m \epsilon M$. One defines the category of $M$-vector bundles on $X$ in the obvious manner. A morphism $f : E \to E'$ of $M$-vector bundles is compact if there is a neighborhood $U$ of the zero-section in $E$ and a subspace $K$ of $E'$ proper over $X$ so that $f(U) \subseteq K$. A bounded complex $E^{\cdot}$ of $M$-equivariant vector bundles (where the differentials are equivariant and of degree $+1$) is called an *$M$-equivariant Fredholm complex* if there exist $M$-equivariant maps $h^k : E^k \to E^{k-1}$ so that

(A.1.1) $h^{k+1}d^k + d^{k-1}h^k = id - \kappa^k$,

where $\kappa^k : E^k \to E^k$ is a compact homomorphism. The category of $M$-equivariant Fredholm complexes on $X$ will be denoted $Fredh^M(X)$. The set of points $x \epsilon X$ at which the complex $E^{\cdot}$ fails to be exact is the support of the complex $E^{\cdot}$. (It is shown in [Seg-2] that the support of an $M$-equivariant Fredholm complex is a closed $M$-invariant subspace of $X$.)

(A.2) Observe that if $f : E \to E'$ is an $M$-equivariant map with *finite rank*, then $f$ is *compact* and one may assume without loss of generality that $U$ and $K$ as above are stable under the action of $M$. To see this, one argues as follows: since $f$ is $M$-equivariant, one may assume that the image of $f$ is contained in an $M$-invariant finite dimensional subbundle $V$ of $E'$. Moreover since $M$ is compact, one may assume that the action of $M$ on $V$ is unitary; now one may take the closed unit ball in $V$ for $K$. The inverse-image $f^{-1}(int(K))$, where $int(K)$ denotes the interior of $K$, is an open neighborhood of the zero-section in $E$, which is also stable under the action of $M$. It follows that a bounded complex of finite dimensional $M$-equivariant vector bundles on $X$ is an $M$-equivariant Fredholm complex.

(A.3) Now we recall the following basic result from [Seg-2]. Let $M$ denote a compact group as before and let $H_M$ denote a separable Hilbert space with a linear $M$-action so that every irreducible representation of $M$ occurs with infinite multiplicity. We may assume that $M$ acts on the right on $H_M$. Let $Fredh(H_M)$ denote the space of all Fredholm-operators on $H_M$; $M$ acts on $Fredh(H_M)$ by, $m.T(v) = m(T(v.m^{-1}))$, $v \epsilon H_M$, $m \epsilon M$ and $T \epsilon Fredh(H_M)$. Clearly the fixed point sub-space

$Fredh(H_M)^M$ = the sub-space of all $M$-equivariant Fredholm operators. Now it is shown in [Seg-2] that

(A.3.1) $\pi_0(K_M^{A,S}(X)) = [X, Fredh(H_M)]_M$,

which is the set of all $M$-homotopy classes of $M$-maps of $X$ into $Fredh(H_M)$. Moreover it is shown in [Seg-2] section 5 that if $Y$ is a closed $M$-invariant subspace of $X$, $\pi_0(K_{M,Y}^{A,S}(X))$ is isomorphic to the group of equivalence classes of $M$-equivariant Fredholm-complexes on $X$ with supports contained in $Y$ where the equivalence relation $\sim$ is defined as follows:

(A.3.2) $\mathcal{E}_0^{\cdot} \sim \mathcal{E}_1^{\cdot}$ if there are acyclic $M$-equivariant Fredholm complexes $F_0^{\cdot}$ and $F_1^{\cdot}$ so that $\mathcal{E}_0^{\cdot} \oplus F_0^{\cdot} \simeq \mathcal{E}_1^{\cdot} \oplus F_1^{\cdot}$, where $\simeq$ denotes $M$-homotopy. This group will be denoted $K^0(Fredh_Y^M(X))$

Moreover any continuous $M$-map $\tilde{d} : X \rightarrow Fredh(H_M)$ defines a complex

(A.3.3) $H^{\cdot}(\tilde{d}) = (... \rightarrow 0 \rightarrow X \times H_M \xrightarrow{d} X \times H_M \rightarrow 0....)$

where $d(x, \psi) = (x, \tilde{d}(x)(\psi))$, $x \in X$, $\psi \in H_M$. (See [Seg-2] p. 398.) Now observe that the map $d$ is $M$-equivariant for the action of $M$ on $X$ and $Fredh(H_M)$ as above. It is shown by ([Seg-2] Proposition 2.3) that this is in-fact a Fredholm complex. However it is not clear from [Seg-2] section 5 that the maps $h^k$ and $\kappa^k$ as in (A.1.1) will be $M$-equivariant. We will rectify this problem by the following result.

(A.4) **Lemma**. There is a continuous $M$-equivariant map

$P : Fredh(H_M) \rightarrow Fredh(H_M)$

so that $A \circ P(A) - id$ and $P(A) \circ A - id$ are $M$-equivariant operators of finite rank (and hence compact) for all $A \in Fredh(H_M)$.

*Proof.* Observe first of all that $Fredh(H_M)$ is an infinite-dimensional manifold; for each $N$ ($N'$), a closed $M$-invariant subspace of $H_M$ of finite codimension (finite dimension respectively ), let $U_{N,N'}$ be the open sub-set of $Fredh(H_M)$ consisting of all $A$ so that the composition $N \xrightarrow{i} H_M \xrightarrow{A} H_M \xrightarrow{p} (H_M)/N'$ is an isomorphism. Now $\{U_{N,N'} | N, N'\}$ forms an $M$-invariant open cover of $Fredh(H_M)$. To see this one may proceed as follows: let $A : H_M \rightarrow H_M$ be a Fredholm operator, let $\tilde{N}$ = a finite dimensional subspace of $H_M$ stable under the action of $M$ and containing $ker(A)$. Since $M$ is compact every representation of $M$ breaks up into the sum of irreducible representations which are all finite dimensional. Let $N$ be a complement to $\tilde{N}$. Let $N' = \tilde{N}$. Now it is clear that the composition $N \xrightarrow{i} H_M \xrightarrow{A} H_M \xrightarrow{p} (H_M)/N'$ is an isomorphism.

Now we define $P_{N,N'} : U_{N,N'} \rightarrow Fredh(H_M)$ by $P_{N,N'}(A) = i(pAi)^{-1}p$. One patches together these $P_{N,N'}$ defined locally to define a $P : Fredh(H_M) \rightarrow Fredh(H_M)$, using an $M$-invariant partition of unity; one may obtain this from a partition of unity by averaging with respect to a Haar measure. (Observe that the above argument is merely a slight modification of the proof of proposition (2.3) in [Seg-2].) This completes the proof of the lemma.

Observe that in this case the condition in (A.1.1) reduces to: there exists an $M$-equivariant map $h : X \times H_M \rightarrow X \times H_M$ (over $X$) so that both $d \circ h$ and $h \circ d$ differ from the identity by a compact operator. If $P$ denotes the operator in (A.4), one may let $h(x, \psi) = (x, P(\tilde{d}(x))(\psi))$.

(A.5)**Corollary**. Assume the situation of (A.3).

(i) Now each class in $\pi_0(K_M^{A.S}(X))$ is represented by an $M$-equivariant Fredholm-complex of the form $H^{\cdot}(\tilde{d})$ in (A.3.3). Moreover given any $M$-equivariant Fredholm complex $\mathcal{E}^{\cdot}$, one can find an $M$-equivariant Fredholm complex of the form $H^{\cdot}(\tilde{d})$ and acyclic complexes $A_0^{\cdot}$, $A_1^{\cdot}$ so that $\mathcal{E}^{\cdot} \oplus A_0 \simeq H^{\cdot}(\tilde{d}) \oplus A_1$.

(ii) The functor $H^{\cdot}(\tilde{d}) \rightarrow EM \underset{M}{\times} H^{\cdot}(\tilde{d})$ defines a map

$B : K_M^{A.S,0}(X) \rightarrow K_M^{top,0}(X)$

*Proof.* These are now obvious from (A.4) and [Seg-2] Theorem (5.1).

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