EQUIVARIANT MOTIVIC HOMOTOPY THEORY

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Abstract. In this paper, we develop the theory of equivariant motivic homotopy theory, both unstable and stable. While our main interest is the case when the group is pro-finite, we discuss our results in a more general setting so as to be applicable to other contexts, for example when the group is in fact a smooth group scheme. We also discuss how \( \mathbb{A}^1 \)-localization behaves with respect to ring and module spectra and also with respect to mod-\( l \)-completion, where \( l \) is a fixed prime. In forthcoming papers, we apply the theory developed here to produce a theory of Spanier-Whitehead duality and Becker-Gottlieb transfer in this framework, and explore various applications of the transfer.

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1. Introduction

The purpose of this paper is to set up a suitable framework for a series of papers exploring the notions of Spanier-Whitehead duality and Becker-Gottlieb transfer in the setting of stable motivic homotopy theory and their applications. Some of the notable applications we have in mind are the conjectures due to the first author on equivariant algebraic K-theory, equivariant with respect to the action of a Galois group, see [Carl1]. Due to the rather versatile nature of the classical Becker-Gottlieb transfer, other potential applications also exist which will be explored in the sequel. It has become clear that a framework suitable for such applications would be that of equivariant motivic stable homotopy, equivariant with respect to the action of a profinite group, which typically will be the Galois group of a field extension.

The study of algebraic cycles and motives using techniques from algebraic topology has been now around for about 15 years, originating with the work of Voevodsky on the Milnor conjecture and the paper of Morel and Voevodsky on $\mathbb{A}^1$-homotopy theory: see [Voev] and [MV]. The paper [MV] only dealt with the unstable theory, and the stable theory was subsequently developed by several authors in somewhat different contexts, for example, [Hov-3] and [Dund2]. An equivariant version of the unstable theory already appears in [MV] and also in [Guill], but both are for the action of a finite group. Equivariant stable homotopy theory, even in the classical setting, needs a fair amount of technical machinery. As a result, and possibly because concrete applications of the equivariant stable homotopy framework in the motivic setting have been lacking till now, equivariant stable motivic homotopy theory has not been worked out or even considered in any detail up until now. The present paper hopes to change all this, by working out equivariant stable motivic homotopy theory in detail.

One of the problems in handling the equivariant situation in stable homotopy is that the spectra are no longer indexed by the non-negative integers, but by representations of the group. One way of circumventing this problem that is commonly adopted is to use only multiples of the regular representation. This works when the representations are in characteristic 0, because then the regular representation breaks up as a sum of all irreducible representations. However, when considering the motivic situation, unless one wants to restrict to schemes in characteristic 0, the above decomposition of the regular representation is no longer true in general. As a result there seem to be serious technical issues in adapting the setting of symmetric spectra to study equivariant stable motivic homotopy theory. On the other hand the technique of enriched functors as worked out in [Dund1] and [Dund2] only requires as indexing objects, certain finitely presented objects in the unstable category. As result this approach requires the least amount of extra work to cover the equivariant setting, and we have chosen to adopt this as the appropriate framework for our work.

The following is an outline of the paper. Section 2 is devoted entirely to the unstable theory. One of the main results in this section is the following.

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Throughout the paper $G$ will denote one of the following: (i) either a discrete group, (ii) a profinite group or (iii) a smooth group scheme, viewed as a presheaf of groups. The profinite group will be a Galois group defined as follows. We will fix a field $k_0$ and let $Gal$ denote the Galois group of an algebraic closure $k$ over $k_0$.

Then $PSh(Sm/k_0, G)$ will denote the category of simplicial presheaves on the category $Sm/k_0$ of smooth schemes of finite type over $k_0$ provided with an action by $G$. Next we will fix a family, $W$, of subgroups of $G$, so that if $H \in W$ and $H' \supset H$ is a subgroup of $G$, then $H' \in W$. In the case $G$ is a finite group, $W$ will denote all subgroups of $G$ and when $G$ is profinite, it will denote all subgroups of finite index in $G$. When $G$ is a smooth group-scheme, we will leave $W$ unspecified, for now.

$PSh_c(Sm/k_0, G)$ will denote the full subcategory of such simplicial sheaves where the action of $G$ is continuous with respect to $W$, i.e. if $s \in \Gamma(U, P)$ and $P \in PSh(Sm/k_0, G)$, then the stabilizer of $s$ belongs to $W$. Then we define the following structure of a cofibrantly generated simplicial model category on $PSh(Sm/k_0, G)$ and on $PSh_c(Sm/k_0, G)$ starting with the projective model structure on $PSh(Sm/k_0)$. (The projective model structure is where the fibrations and weak-equivalences are defined object-wise and the cofibrations are defined using the left-lifting property with respect to trivial fibrations.)

(i) The generating cofibrations are of the form

$I_G = \{G/H \times i \mid i \text{ a cofibration of } PSh(Sm/k_0), H \in W\}$

(ii) the generating trivial cofibrations are of the form
Then we obtain: cofibrantly generated model categories. The above categories are symmetric monoidal with the smash-product, or maps of the form $f: P' \to P$ in PSh(Sm/k_0, G) so that $f^H: P'^H \to P^H$ is a weak-equivalence (fibration, respectively) in PSh(Sm/k_0) for all $H \in W$.

**Proposition 1.1.** The above structure defines a cofibrantly generated simplicial model structure on PSh(Sm/k_0, G) and on PSh_c(Sm/k_0, G) that is proper. In addition, the smash product of pointed simplicial presheaves makes these symmetric monoidal model categories.

When $G$ denotes a finite group or profinite group, the category PSh_c(Sm/k_0, G) is locally presentable, i.e. there exists a set of objects, so that every simplicial presheaf $P \in$ PSh_c(Sm/k_0, G) is filtered colimit of these objects. In particular, it also is a combinatorial model category.

One also considers the following alternate framework for equivariant unstable motivic homotopy theory. First one considers the orbit category $O_G = \{G/H \mid H \in W\}$. A morphism $G/H \to G/K$ corresponds to $\gamma \in G$, so that $\gamma.H \gamma^{-1} \subseteq K$. One may next consider the category $\text{PSh}^G$ of $O_G$-diagrams with values in PSh(Sm/k_0). The two categories $\text{PSh}_c(Sm/k_0, G)$ and $\text{PSh}^G$ are related by the functors:

$$\Phi: \text{PSh}_c(Sm/k_0, G) \to \text{PSh}^G,$$

$$F \mapsto \Phi(F) = \{\Phi(F)(G/H) = F^H\}$$

and

$$\Theta: \text{PSh}^G \to \text{PSh}_c(Sm/k_0, G), \quad M \mapsto \Theta(M) = \lim_{\{H|H \in W\}^\gamma} M(G/H).$$

A key result then is the following.

**Proposition 1.2.** When $G$ denotes a discrete group or a profinite group, the functors $\Phi$ and $\Theta$ are Quillen-equivalences.

Section 3 is entirely devoted to the stable theory and the following is one of the main results in this section. We let $C$ denote one of the categories $\text{PSh}_c(Sm/k_0)$, $\text{PSh}_c(Sm/k_0, G)$, or $\text{PSh}^G$ provided with the above structure of cofibrantly generated model categories. The above categories are symmetric monoidal with the smash-product, $\wedge$, of simplicial presheaves as the monoidal product. The unit for the monoidal product will be denoted $S^0$.

Since not every object in Sm/k_0 has an action by $G$, it is not possible to consider localization of $\text{PSh}_c(Sm/k_0, G, ?)$ or $\text{PSh}(Sm/k_0, G, ?)$ by inverting maps associated to an elementary distinguished square in the Nisnevich topology or maps of the form $U'_s \to U$ where $U \in \text{Sm/k}_0$ and $U'_s \to U$ is a hypercovering in the given topology. Therefore, we instead localize PSh by inverting such maps and consider the category of diagrams of type $O_G$ with values in this category. $\text{PSh}^G_{\text{mot}}$ ($\text{PSh}^G_{\text{des}}$) will denote the category of all diagrams of type $O_G$ with values in the localized category $\text{PSh}_{\text{mot}}$ ($\text{PSh}_{\text{des}}$, respectively).

Let $C'$ denote a $C$-enriched full-subcategory of $C$ consisting of objects closed under the monoidal product $\wedge$, all of which are assumed to be cofibrant and containing the unit $S^0$. Let $C'_0$ denote a $C$-enriched sub-category of $C'$, which may or may not be full, but closed under the monoidal product $\wedge$ and containing the unit $S^0$. Then the basic model of equivariant motivic stable homotopy category will be the category $[C'_0, C]$. This is the category whose objects are $C$-enriched covariant functors from $C'_0$ to $C$, see [Dund1, 2.2]. There are several possible choices for the category $C'_0$, which are discussed in detail in Examples 3.3.

We let $\text{Sph}(C'_0)$ denote the $C$-category defined by taking the objects to be the same as the objects of $C'_0$ and where $\text{Hom}_{\text{Sph}(C'_0)}(T_V, T_V) = T_W$ if $T_V = T_W \wedge T_U$ and $\ast$ otherwise. Since $T_W$ is a sub-object of $\text{Hom}_C(T_V, T_V)$, it follows that $\text{Sph}(C'_0)$ is a sub-category of $C'_0$. Now an enriched functor in $[\text{Sph}(C'_0), C]$ is simply given by a collection $\{X(T_V)|T_V \in \text{Sph}(C'_0)\}$ provided with a compatible collection of maps $T_W \wedge X(T_V) \to X(T_W \wedge T_V)$. We let $\text{Spectra}(C) = [\text{Sph}(C'_0), C]$.

We provide two model structures on $\text{Spectra}(C)$: (i) the level model structure and (ii) the stable model structure. A map $f: X' \to X$ in $\text{Spectra}(C)$ is a level equivalence (level fibration, level trivial fibration, level cofibration, level trivial cofibration) if each $\text{ev}_{T_V}(f)$ is a weak-equivalence (fibration, trivial fibration, cofibration, trivial cofibration, respectively) in $C$. Such a map $f$ is a projective cofibration if it has the left-lifting property with respect to every level trivial fibration. Then we obtain:
Proposition 1.3. The projective cofibrations, the level fibrations and level equivalences define a cofibrantly generated model category structure on \( \text{Spectra}(\mathcal{C}) \) with the generating cofibrations (generating trivial cofibrations) being \( I_{\text{Sp}} \) (\( J_{\text{Sp}} \), respectively). This model structure (called the projective model structure on \( \text{Spectra}(\mathcal{C}) \)) is left-proper (right proper) if the corresponding model structure on \( \mathcal{C} \) is left proper (right proper, respectively). It is cellular if the corresponding model structure on \( \mathcal{C} \) is cellular.

See Corollary 3.8 for further details. Next we localize the above model structure suitably to define the stable model structure. A spectrum \( X \in \text{Spectra}(\mathcal{C}) \) is an \( \Omega \)-spectrum if it is level-fibrant and each of the natural maps \( X(T_V) \to \text{Hom}_\mathcal{C}(T_W, X(T_V \wedge T_W)) \), \( T_V, T_W \in C'_0 \) is a weak-equivalence in \( \mathcal{C} \).

Proposition 1.4. (See Proposition 3.10 for more details.) (i) The corresponding stable model structure on \( \text{Spectra}(\mathcal{C}) \) is cofibrantly generated, left proper and cellular when \( \mathcal{C} \) is assumed to have these properties.

(ii) The fibrant objects in the stable model structure on \( \text{Spectra}(\mathcal{C}) \) are the \( \Omega \)-spectra defined above.

The above results then enable us to define the various equivariant stable motivic homotopy categories that we consider: these are discussed in Definition 3.11.

In the last section we study how \( \mathbb{A}^1 \)-localization behaves with respect to ring and module spectra and also with respect to \( Z/l \)-completion, for a fixed prime \( l \). We make use of ring and module spectra in an essential way in this paper and the sequels: therefore it is important to show that \( \mathbb{A}^1 \)-localization preserves ring and module structures on spectra. Since comparisons are often made with the \( \acute{e} \)tale homotopy types, completions at a prime also play a major role in our work. Unfortunately the Bousfield-Kan completion as such may not commute with \( \mathbb{A}^1 \)-localization. This makes it necessary for us to redefine a Bousfield-Kan completion at a prime \( l \) that behaves well in the \( \mathbb{A}^1 \)-local setting: this is a combination of the usual Bousfield-Kan completion and \( \mathbb{A}^1 \)-localization. Since the resulting completion is in general different from the traditional Bousfield-Kan completion, we denote our completion functor by \( \tilde{Z}/l_\infty \). The main properties of the resulting completion functor are discussed in 4.0.7.

2. The basic framework of equivariant motivic homotopy theory: the unstable theory

In this section we define a framework of equivariant motivic stable homotopy theory, where the group \( G \) will be either finite, profinite or a smooth group scheme. The main motivation for this formulation is so as to provide a framework for work on the conjectures in [Carl1], where the group involved will be the profinite Galois group associated to a field extension. One main point to observe in this context is that there are two different kinds of equivariant cohomology and homotopy theories: one in the sense of Borel and one in the sense of Bredon with the latter being a finer variant than the former. Classical equivariant homotopy theory is set in the framework of Bredon-style theories and this is also the framework for work on the above mentioned conjectures. These observations should show the necessity for developing the following framework.

We let \( k_0 \) denote a fixed field of arbitrary characteristic, with \( k \) denoting a fixed algebraic closure of \( k_0 \). While we allow the possibility that \( k_0 \) be algebraically or separably closed, for the most part this will not be case. \( \text{Sm}/k_0 \) will denote the category of separated smooth schemes of finite type over \( k_0 \). If \( X \) denotes a separated smooth scheme of finite type over \( k_0 \), \( (\text{Sm}/X) \) will denote the corresponding subcategory of objects defined over \( X \). \( (\text{Sm}/k_0) \) will denote the category \( \text{Sm}/k_0 \) provided with a Grothendieck topology with \( ? \) denoting one of the big Zariski, Nisnevich or \( \acute{e} \)tale topologies. If \( X \) denotes a separated smooth scheme of finite type over \( k_0 \), \( (\text{Sm}/X) \), will denote the corresponding big site over \( X \). If \( p : X \to \text{Spec} k_0 \) denotes the structure map of \( X \), one has the obvious induced functors:

\[
\begin{align*}
(p^{-1} : (\text{Sm}/k_0) & \to (\text{Sm}/X), p^{-1}(Z) = \underset{\text{Spec } k}{\times} X, \text{ and} \\
p_! : (\text{Sm}/X) & \to (\text{Sm}/k_0), p_!(Y) = Y
\end{align*}
\]

We will let \( S \) denote any one of the above categories or sites: since the schemes we consider are of finite type over \( k_0 \), it follows that these categories are all skeletally small. \( \text{PSH}(S) \) (\( \text{SSH}(S) \)) will denote the category of all simplicial presheaves (sheaves, respectively) on \( S \): observe that to any scheme \( X \) one may associate a (simplicial) presheaf represented by \( X \).
2.1. Model structures and localization. Though the main objects considered in this paper are simplicial presheaves, we will often need to provide them with different model structures model structures (in the sense of [Qui]). The same holds for the stable case, where different variants of spectra will be considered. In this paper we will work with several different kinds of objects: in addition to simplicial presheaves and simplicial sheaves, we will need to consider pro-objects of simplicial presheaves and sheaves. It will also become necessary to consider what may be called spectral objects (spectra, for short) in these categories. Therefore, the basic framework will be that of model categories, often simplicial (symmetric) monoidal model categories, i.e. simplicial model categories that have the structure of monoidal model categories as in [Hov-2]. We will further assume that these categories have other nice properties, for example, are closed under all small limits and colimits, are proper and cellular. (See [Hov-1] and [Hirsh] for basic material on model categories.)

2.1.1. The category of simplicial presheaves. We fix a scheme $S \in \text{Sm}/k_0$ and consider the category $\text{PSh}(\text{Sm}/S)$ of pointed simplicial presheaves on $\text{Sm}/S$. We will need to consider different Grothendieck topologies on $\text{Sm}/S$: this is one reason that we prefer to work for the most part with pointed simplicial presheaves rather than with pointed simplicial sheaves. We will consider pointed simplicial sheaves only when it becomes absolutely essential to do so. If $?_i$ denotes any one of the Grothendieck topologies, ét, Nis or Zar, and $X \in \text{Sm}/S$, $\text{Sh}(\text{Sm}/X_?)$ will denote the category of pointed simplicial sheaves on the corresponding big site.

There are at least three commonly used model structures on $\text{PSh}(\text{Sm}/S)$: (i) the object-wise model structure, (ii) the injective model structure and (iii) the projective model structure. We will mostly consider the last structure in the paper but will provide a few remarks on the other two here for comparison purposes.

By viewing any simplicial presheaf as a diagram of type $\text{Sm}/S$ with values in simplicial sets, we obtain the model-structure where weak-equivalences and cofibrations are defined object-wise and fibrations are defined using the right-lifting property with respect to trivial cofibrations. Observe as a consequence, that all objects are cofibrant and cofibrations are simply monomorphisms. This is the object-wise model structure.

All the above sites have enough points: for the étale site these are the geometric points of all the schemes considered, and for the Nisnevich and Zariski sites these are the usual points (i.e. spectra of residue fields) of all the schemes considered. Therefore, it is often convenient to provide the simplicial topoi above with the injective model structure where the cofibrations are defined as before, but the weak-equivalences are maps that induce weak-equivalences on the stalks and fibrations are defined by the lifting property with respect to trivial cofibrations. By providing the appropriate topology $?$ on the category $\text{Sm}/S$, one may define a similar injective model structure on $\text{PSh}(\text{Sm}/S)$. Since this depends on the topology $?$, we will denote this model category structure by $\text{PSh}(\text{Sm}/S)$. 

The projective model structure on $\text{PSh}(\text{Sm}/S)$ is defined by taking fibrations (weak-equivalences) to be maps that are fibrations (weak-equivalences, respectively) object-wise, i.e of pointed simplicial sets on taking sections over each object in the category $\text{Sm}/S$. The cofibrations are defined by the left-lifting property with respect to trivial fibrations. This model structure is cofibrantly generated, with the generating cofibrations $I$ (generating trivial cofibrations $J$) defined as follows.

\begin{align}
J &= \{ h_X \land \Delta[n]_+ \to h_X \land \Delta[n]_+ | n \geq 0 \} \text{ and} \\
I &= \{ h_X \land \Delta[n]_+ \to h_X \land \Delta[n]_+ | n \geq 0 \}
\end{align}

Here $X$ denotes an object of the category $\text{Sm}/S$, $h_X$ denotes the corresponding pointed sheaf of sets represented by $X_+$ and $\land$ is provided by the simplicial structure on this category. Observe that every map that is a cofibration in the projective model structure is a cofibration in the other two model structures. Weak-equivalences in the object-wise model structure and the projective model structure are the same and they are both weak-equivalences in the injective model structure. It follows that any fibration in the object-wise model structure is a fibration in the projective model structure and that a fibration in the injective model structure is a fibration in the other two model structures. Next we summarize a couple of useful observations in the following proposition.

Proposition 2.1. (i) If $P \in \text{PSh}(\text{Sm}/X)$ is fibrant in the object-wise model structure or projective model structure, $\Gamma(U, GP) = \text{holim}_{\Delta} \Gamma(U, G^s P)$ is a fibrant pointed simplicial set, where $G^s P$ is the cosimplicial object defined by the Godement resolution and $U$ is any object in the site.

(ii) The functor $a : \text{PSh}(\text{Sm}/k_0) \to \text{Sh}(\text{Sm}/k_0)$ that sends a presheaf to its associated sheaf sends fibrations to maps that are fibrations stalk-wise and weak-equivalences to maps that are weak-equivalences stalk-wise, when
PSh(Sm/\mathcal{O}_k) is provided with the projective model structure. The same functor sends cofibrations to injective maps when PSh(Sm/\mathcal{O}_k) is provided with the injective model structure.

Proof. The proof of (i) amounts to the observation that if F is a pointed simplicial presheaf that is a fibration object-wise, then the stalks of F are fibrant pointed simplicial sets. It follows that, therefore, \Gamma(U,G^n F) are all fibrant pointed simplicial sets. (See [B-K, Chapter XI, 5.5 Fibration lemma,]) The first statement in (ii) follows from the observation that fibrations and weak-equivalences are defined object-wise and these are preserved by the filtered colimits involved in taking stalks and that the stalks of a presheaf and associated sheaf are isomorphic. The second statement in (ii) follows from the observation that a map of presheaves is a cofibration (i.e. a monomorphism) if and only if the induced maps on all the stalks are cofibrations (i.e. monomorphisms).

Let X, Y denote two smooth schemes over Spec \mathcal{O}_k and let f : X \to Y denote a map. Then f induces functors

\[ f_* : \text{PSh}(\text{Sm/X}) \to \text{PSh}(\text{Sm/Y}) \quad \text{and} \quad f^* : \text{PSh}(\text{Sm/Y}) \to \text{PSh}(\text{Sm/X}) \]

with the former right-adjoint to the latter. (One also obtains similar functors at the level of sheaves.) In case Y = Spec \mathcal{O}_k with f = p the structure map of X, then f* is simply the restriction of a presheaf (sheaf) to Sm/X. Therefore if F \in \text{PSh}(\text{Sm/Spec(\mathcal{O}_k)}), we will denote this restriction of F to Sm/X by F|_X. In general, neither of these functors preserve any of the model category structures so that one needs to consider the appropriate derived functors associated to these functors. The left-derived functor of f*, Lf*, may be defined using representables as in [MV, Chapter 2] and the right derived functor of f*, Rf*, may often be defined using the Godement resolution in view of the above result. In case f itself is a smooth morphism, one can readily show that f* preserves cofibrations and weak-equivalences in all three of the above model structures. In the case f is again assumed to be a smooth morphism, Rf* preserves fibrations and weak-equivalences in the projective model structures. (To see this one first observes that filtered colimits of fibrations of simplicial sets are fibrations, so that each of the maps G^n f will be a fibration object-wise. Now homotopy inverse limits preserve fibrations.)

One may observe that the above model categories are all simplicial model categories in the sense of [Qu]. If A,B are objects in the above categories, Map(A,B) will denote the corresponding simplicial mapping space: Map(A,B)_n = Hom(A \land \Delta[n]_+, B) and one may observe that (A \land \Delta[n]_+)_s = \bigvee_{n \in \Delta[n]_s} A_s.

One defines a symmetric monoidal structure on any of the above categories by sending (P,Q) to P \land Q. One may verify readily that, with this monoidal structure, the above categories of pointed simplicial presheaves and sheaves are monoidal simplicial model categories which are proper and cellular. The projective model category structure is also weakly finitely generated in the sense of [Dund1, Definition 3.4].

2.1.4. Convention. By default we will assume the projective model structure with respect to one of the given Grothendieck topologies. Objects of PSh(Sm/\mathcal{O}_k)\gamma will be referred to as spaces.

2.2. \text{\mathbb{A}^1}-localization and Simplicial sheaves up to homotopy. Model structures on the category PSh(Sm/\mathcal{O}_k)\gamma are discussed in detail in the G-equivariant context below: the discussion there applies to the present setting by taking the group-action to be trivial.

One may apply \text{\mathbb{A}^1}-localization to each of the above model structures as in [MV]. The resulting model categories will be denoted with the subscript \text{\mathbb{A}^1}. In view the basic results on localization as in [Hirsh, Chapter 4], they inherit the basic properties of the above model categories. The fibrant objects F in these categories are characterized by the property that they are fibrant in the underlying model structure and that Map(P \times \text{\mathbb{A}^1},F) \simeq Map(P,F), for all P \in \text{PSh}(\mathcal{S}). These will be called \text{\mathbb{A}^1}-local. PSh_{\mathbb{A}^1}(\text{Sm/\mathcal{O}_k})\gamma will denote the \text{\mathbb{A}^1}-localization of PSh(Sm/\mathcal{O}_k)\gamma. Objects of PSh_{\mathbb{A}^1}(\text{Sm/\mathcal{O}_k})\gamma will be referred to as \text{\mathbb{A}^1}-spaces.

Next, there are two means of passing to sheaves or rather sheaves up to homotopy. When the topology is the Nisnevich topology, one defines a presheaf P \in PSh(Sm/\mathcal{O}_k)\text{\mathbb{N}is} to be motivically fibrant if (i) P is fibrant in PSh(Sm/\mathcal{O}_k)\text{\mathbb{N}is}, (ii) \Gamma(\phi,P) is contractible (where \phi denotes the empty scheme), (iii) sends an elementary distinguished squares as in [MV] to a homotopy cartesian square and (iv) the obvious pull-back \Gamma(U,P) \to \Gamma(U \times \text{\mathbb{A}^1},P) is a weak-equivalence. Then a map f : A \to B in PSh(Sm/\mathcal{O}_k)\text{\mathbb{N}is} is a motivic weak-equivalence if the induced map Map(f,P) is a weak-equivalence for every motivically fibrant object P, with Map denoting the simplicial mapping space. One then localizes such weak-equivalences. As shown in [Dund2, Section 2], this localization produces a model structure which is also weakly finitely generated and also which is symmetric monoidal model. The resulting model structure will be denoted PSh_{mot}(Sm/\mathcal{O}_k).
An alternate approach that applies in general is to localize by inverting hypercovers as in [DHI]. Following
[DHI], a simplicial presheaf has the descent property for all hypercovers if for $U$ in $(\text{Sm}/k_0)^\circ$, and all hypercoverings
$U^n \to U$, the induced map $P(U) \to \text{holim} \{ \Gamma(U_n, P)|n \}$ is a weak-equivalence. By localizing with respect to maps
of the form $U^n \to U$ where $U^n$ is a hypercovering of $U$ and also maps of the form $U \times k^1 \to U$, it is proven in
[Dug, Theorem 8.1] and [DHI, Example A.10] that we obtain a model category which is Quillen equivalent to the
Voevodsky-Morel model category of simplicial sheaves on $(\text{Sm}/k_0)^\circ$, as in [MV]. Though the resulting localized
category is cellular and left-proper (see [Hirsh, ] it is unlikely to be weakly finitely generated: the main issue is that
the hypercoverings, being simplicial objects, need not be small. Nevertheless this seems to be the only alternative
available in the étale setting. This localized category of simplicial presheaves will be denoted by $\mathcal{P}Sh_{\text{des}}(S)$. Often
the subscript $\text{des}$ will be omitted if there is no chance for confusion.

**Remark 2.2.** $\Delta^1$-localization is better discussed in detail in the stable setting, which we consider below. Moreover,
since the étale homotopy type of affine spaces is trivial after completion away from the characteristic of the base
field, $\Delta^1$-localization in the étale setting is often simpler than the corresponding motivic version as shown below.

### 2.3. Equivariant presheaves and sheaves.

As we pointed out earlier we allow the group $G$ to be one of the following:

(i) a finite group

(ii) a profinite group or

(iii) a smooth group scheme viewed as a presheaf of groups on $\text{Sm}/k_0$.

In all these cases, one defines a $G$-equivariant presheaf of pointed sets on $\text{Sm}/k_0$ to be a presheaf of sets $P$ on
$\text{Sm}/k_0$ taking values in the category of pointed sets provided with an action by $G$. One defines $G$-equivariant
presheaves of simplicial sets, pointed simplicial sets and $G$-equivariant sheaves of simplicial sets and pointed
simplicial sets similarly. Henceforth $\mathcal{P}sh(\text{Sm}/k_0, G)$ ($\mathcal{S}h(\text{Sm}/k_0, G, ?)$) will denote the category of $G$-equivariant
pointed simplicial presheaves (sheaves on the $\cdot$-topology, respectively). Clearly this category is closed under all
small limits and colimits.

**Definition 2.3.** The category $(\text{Sm}/k_0, \text{Gal})$ will denote the subcategory of $\text{Sm}/k_0$ consisting of schemes of finite
type over $k'$, where $k'$ is some finite Galois extension of $k_0$ and provided with an action of $\text{Gal}$ through the quotient
$\text{Gal}_{k_0}(k')$. The morphisms between such schemes will be morphisms over $k_0$ compatible with the action of $\text{Gal}$.

The first case, when $G$ is a finite group, is already discussed in some detail in the literature, at least in the
unstable case: see [MV] and [Guill]. Therefore, presently we will only consider the last two cases.

One obtains the Galois-equivariant framework as follows. Let $k'$ denote a fixed algebraically (or separably) closed
field containing the given field $k_0$ with $\text{Gal} = \text{Gal}(k/k_0)$. We will view the Galois group $\text{Gal}$ as an inverse system of
constant presheaves (of finite groups) on $\text{Sm}/k_0$.

The main examples of Galois-equivariant pointed simplicial sheaves will be the representable ones and filtered
colimits of such representable ones. Here a representable sheaf is one of the form $h_X$, i.e. represented by $X'$ where
$X' = X_0 \times_{\text{Spec } k_0} \text{Spec } k'$. Here $X_0$ is a smooth scheme of finite type over $k_0$ and $k'$ which is a finite Galois extension
of the base field $k_0$ and contained in the fixed algebraic closure $k$.

**Remark 2.4.** Our discussion through 2.8 will only consider the category $\mathcal{P}sh(\text{Sm}/k_0, G)$ explicitly though everything
that is said here applies equally well to the $\Delta^1$-localized category $\mathcal{P}sh_{\Delta^1}(\text{Sm}/k_0, G)$ as well.

#### 2.3.1. Pointed simplicial sets with action by $G$.

A related context that comes up, especially in the context of étale realization is the following: let $(\text{simp}.\text{sets})_*$ denote the category of pointed simplicial sets and let $(\text{simp}.\text{sets}, G)_*$ denote the category of all pointed simplicial sets with actions by the group $G$ with the morphisms in this case being equivariant maps. By considering the constant presheaves associated to a pointed simplicial set, we may imbibe this category into $\mathcal{P}sh(\text{Sm}/k_0, G)$.

#### 2.3.2. Simplicial presheaves with continuous action by the group $G$.

We let $W$ denote a family of subgroups of $G$ so that if $H \in W$ and $H'$ is a subgroup of $G$ containing $H$, the $H' \in W$. As pointed out earlier, when
the group \( G \) is finite, \( W \) will denote the family of all subgroups and when \( G \) is profinite, \( W \) will denote the family of all subgroups of finite index in \( G \). When \( G \) is a smooth group scheme, we will leave the family \( W \) unspecified for now.

For the most part, we will only consider \( G \)-equivariant presheaves of pointed simplicial sets \( P \) on which the action of \( G \) is continuous, i.e. for each object \( U \in \mathcal{S} \) and each section \( s \in \Gamma(U, F)_n \), the stabilizer \( Z(s) \) of \( s \) belongs to \( W \). This category will be denoted \( \text{PSh}_c(\text{Sm}/k_0, G) \).

Clearly the above terminology is taken from the case where the group \( G \) is profinite. Observe that, in this case the intersection of the conjugates \( \cap g^{-1} Z(s)g \) of this stabilizer as \( g \) varies over a set of coset representatives of \( Z(s) \) in \( G \) is a normal subgroup of \( G \), contained in \( Z(s) \) and of finite index in \( G \). Therefore \( G \) acts continuously on \( P \) if and only if \( \Gamma(U, P) = \lim_{\{H \mid [G : H] < \infty, H \in G\}} \Gamma(U, P)^H \).

For each subgroup \( H \in W \), let \( P^H \) denote the sub-presheaf of \( P \) of sections fixed by \( H \), i.e. \( \Gamma(U, P^H) = \Gamma(U, P)^H \). If \( H = G/H \), then \( \Gamma(U, P)^H = \{ s \in \Gamma(U, P) \mid \text{the action of } G \text{ on } s \text{ factors through } H \} \).

Lemma 2.5. (i) Let \( \phi : P' \to P \) denote a map of simplicial presheaves. Then \( \phi \) induces a map \( \phi^H : P'^H \to P^H \) for each subgroup \( H \in W \). The association \( \phi \mapsto \phi^H \) is functorial in \( \phi \) in the sense that if \( \psi : P'' \to P' \) is another map, then the composition \( (\phi \circ \psi)^H = \phi^H \circ \psi^H \).

(ii) The full subcategory \( \text{PSh}_c(\text{Sm}/k_0, G) \) of simplicial presheaves with continuous action by \( G \) is closed under all small colimits, with the small colimits the same as those computed in \( \text{PSh}(\text{Sm}/k_0, G) \).

For the remaining statements, we will assume the group \( G \) is profinite.

(iii) The full subcategory \( \text{PSh}_c(\text{Sm}/k_0, G) \) is also closed under all small limits, where the limit of a small diagram \( \{P_\alpha\} \) is

\[
\lim_{\{H \mid [G : H] < \infty \}} \lim_{\alpha} P^H \alpha.
\]

When the inverse limit above is finite, it commutes with the filtered colimit over \( H \), so that in this case, the inverse limit agrees with the inverse limit computed in \( \text{PSh}(\text{Sm}/k_0, G) \).

(iv) Let \( \{Q_\alpha\} \) denote a diagram of simplicial sub-presheaves of a simplicial presheaf \( Q \in \text{PSh}(\text{Sm}/k_0, G) \) indexed by a small filtered category. If \( K \) is any subgroup of \( G \), then \( \lim Q_\alpha^K = \lim Q_\alpha^K \).

(v) Let \( \{P_\alpha\} \) denote a diagram of objects in \( \text{PSh}_c(\text{Sm}/k_0, G) \) indexed by a small filtered category. Let \( K \) denote a subgroup of \( G \) with finite index. Then \( \lim P_\alpha^K = \lim P_\alpha^K \).

Proof. (i) and (ii) are clear. In order to prove (iii), we show that giving a compatible collection of maps \( \phi_\alpha : P \to P_\alpha \) from a simplicial presheaf \( P \) with a continuous action by \( G \) to the given inverse system \( \{P_\alpha\} \) corresponds to giving a map \( \phi : P \to \lim_{\alpha} P^H_\alpha \). Observe that the maps \( \phi_\alpha \) induce a compatible collection of maps \( \phi^H : P^H \to P^H_\alpha \) for each fixed \( H \). Therefore, one may now take the limit of these maps over \( \{\alpha\} \) to obtain a compatible collection of maps \( \phi^H : P^H \to \lim P^H_\alpha \).

One may next take the colimit over \( H \) of this collection to obtain a map \( \phi : P = \lim_{\{H \mid [G : H] < \infty \}} P^H \to \lim_{\alpha} P^H_\alpha \). The latter then projects to \( \lim_{\{H \mid [G : H] < \infty \}} P^H_\alpha \).

Clearly the latter maps naturally to \( \lim_{\{H \mid [G : H] < \infty \}} P^H_\alpha \). The latter then projects to \( \lim_{\alpha} P^H_\alpha \). Moreover, the composition of the above maps \( P \to P_\alpha \) identifies with the given map \( \phi_\alpha \) since the corresponding maps identify after applying \( (\ )^H \).
The action of $G$ on $\lim_{\alpha} P^H_\alpha$ is continuous. The above arguments show that the latter is in fact the inverse limit of $\{P_\alpha^0\}$ in the category $\text{PSh}(\text{Sm}/k_0, G)$. This completes the proof of (iii). The last statement in (iii) is clear since filtered colimits commute with finite limits.

(iv) follows readily in view of the observation that for a simplicial sub-presheaf $Q'$ of $Q$, with $Q' \in \text{PSh}(\text{Sm}/k_0, G)$, $(Q')^K = Q' \cap Q^K$.

Next we consider (v). Since each $P_\alpha$ has a continuous action by $G$, we first observe that $P_\alpha = \lim_{\alpha} P^H_\alpha$ for each $\alpha$. Therefore, $\lim_{\alpha} P_\alpha^H = \lim_{\alpha} P_\alpha^H = \lim_{\alpha} \lim_{\alpha} P^H_\alpha$. Let $Q_H = \lim_{\alpha} P_\alpha^H$, then we see that each $Q_H$ is a sub-simplicial presheaf of $P_\alpha$. Then the collection $\{Q_H | H, |G/H| < \infty, H \triangleleft G\}$ satisfies the hypotheses in (iv) so that we obtain:

$$\left( \lim_{\alpha} \lim_{\alpha} P_\alpha^H \right)^K = \lim_{\alpha} \left( \lim_{\alpha} P_\alpha^H \right)^K = \lim_{\alpha} \lim_{\alpha} \left( P_\alpha^H \right)^K = \lim_{\alpha} \left( P_\alpha^H \right)^K.$$

Next observe that, for any simplicial presheaf $Q$ and any fixed normal subgroup $H$ of finite index in $G$, the action of $G$ on $Q^H$ is through the finite group $G/H$. Therefore, $(Q^H)^K = (Q^K)^K$, where $K$ is the image of $K$ in $G/H$. Therefore we obtain the isomorphism

$$\lim_{\alpha} \left( P_\alpha^H \right)^K \simeq \left( \lim_{\alpha} P_\alpha^H \right)^K \simeq \lim_{\alpha} \left( P_\alpha^H \right)^K \simeq \lim_{\alpha} \left( P_\alpha^H \right)^K.$$

Making use of this identification and commuting the two colimits, we therefore obtain the identification

$$\lim_{\alpha} \left( \lim_{\alpha} P_\alpha^H \right)^K = \lim_{\alpha} \lim_{\alpha} \left( P_\alpha^H \right)^K = \lim_{\alpha} \left( P_\alpha^H \right)^K.$$

The last identification follows from (iv) and the observation that for each fixed $\alpha$, each $P_\alpha^H \subseteq P_\alpha$ as simplicial presheaves. \qed

### 2.3.4. Finitely presented objects.

Recall an object $C$ in a category $\mathcal{C}$ is finitely presented if $\text{Hom}_\mathcal{C}(C, -)$ commutes with all small filtered colimits in the second argument.

**Lemma 2.6.** Let $G$ denote a profinite group. Let $P \in \text{PSh}(\text{Sm}/k_0, G)$ be such that in $\text{PSh}(\text{Sm}/k_0)$ it is finitely presented and $P = P^H$ for some normal subgroup $H$ of finite index in $G$. Then $P$ is a finitely presented object in $\text{PSh}(\text{Sm}/k_0, G)$.

**Proof.** Let $\text{Hom}$ denote the external Hom in the category $\text{PSh}(\text{Sm}/k_0)$ and let $\text{Hom}_G$ denote the external Hom in the category $\text{PSh}(\text{Sm}/k_0, G)$. Then $\text{Hom}_G(P, Q) = \text{Hom}(P, Q)^G$, where $G$ acts on the set $\text{Hom}(P, Q)$ through its actions on $P$ and $Q$. Next suppose $P = P^H$ for some normal subgroup $H$ of $G$ with finite index. Then $\text{Hom}_G(P, Q) \cong \text{Hom}_G(P, Q^H) \cong \text{Hom}(P, Q^H)^G/H$.

Next let $\{Q_\alpha\}$ denote a small collection of objects in $\text{PSh}(\text{Sm}/k_0, G)$ indexed by a small filtered category. Then,

$$\lim_{\alpha} \text{Hom}_G(P, Q_\alpha) = \lim_{\alpha} \text{Hom}(P, Q_\alpha^H)^G/H = \left( \lim_{\alpha} \text{Hom}(P, Q_\alpha^H) \right)^G/H$$

where the last equality follows from the fact that taking invariants with respect to the finite group $G$ is a finite inverse limit which commutes with the filtered colimit over $\alpha$. Next, since $P$ is finitely presented as an object in $\text{PSh}(\text{Sm}/k_0)$, the last term identifies with $\left( \text{Hom}(P, \lim_{\alpha} Q_\alpha^H) \right)^G/H$. By (v) of the last lemma, this then identifies with $\left( \text{Hom}(P, \lim_{\alpha} Q_\alpha) \right)^G/H$. This clearly identifies with $\text{Hom}_G(P, \lim_{\alpha} Q_\alpha)$. \qed

Next we define the following structure of a cofibrantly generated simplicial model category on $\text{PSh}(\text{Sm}/k_0, G)$ and on $\text{PSh}(\text{Sm}/k_0, G)$ starting with the projective model structure on $\text{PSh}(\text{Sm}/k_0)$?:
(i) The generating cofibrations are of the form
\[ I_G = \{ G/H \times i \mid i \text{ a cofibration of } PSh(Sm/k_0), \ H \in W \}, \]

(ii) the generating trivial cofibrations are of the form
\[ J_G = \{ G/H \times j \mid j \text{ a trivial cofibration of } PSh(Sm/k_0), H \subseteq G, \ H \in W \} \]

(iii) where weak-equivalences (fibrations) are maps \( f : P^l \to P \) in \( PSh(Sm/k_0, G) \) so that \( f^H : P^lH \to PH \) is a weak-equivalence (fibration, respectively) in \( PSh(Sm/k_0) \), for all \( H \in W \).

**Proposition 2.7.** The above structure defines a cofibrantly generated simplicial model structure on \( PSh(Sm/k_0, G) \) and on \( PSh_c(Sm/k_0, G) \) that is proper. In addition, the smash product of pointed simplicial presheaves makes these symmetric monoidal model categories.

If \( G \) denotes a profinite group, the category \( PSh_c(Sm/k_0, G) \) is locally presentable, i.e. there exists a set of objects, so that every simplicial presheaf \( P \in PSh_c(Sm/k_0, G) \) is filtered colimit of these objects. In particular, it also a combinatorial model category.

**Proof.** The key observation is the following: Let \( H \subseteq G \) denote a subgroup with finite index. Then the functor \( P \mapsto PH, \ PSh(Sm/k_0, G) \to PSh(Sm/k_0) \) has as left-adjoint, the functor \( Q \mapsto G/H \times Q \). It follows from this observation that the domains of \( I \) (\( J \)) are small with respect to the subcategory \( I \) (\( J \), respectively) so that the small object argument (see [Hirsh, ]) is possible. It follows readily then that the categories \( PSh(Sm/k_0, G) \) and \( PSh_c(Sm/k_0, G) \) are cofibrantly generated model categories. We skip the verification that they are proper, that they are simplicial model categories and that the smash-product of two pointed simplicial presheaves makes these monoidal model categories. (See, for example, [Fausk].) In view the assumption of continuity it suffices to show that the category of simplicial presheaves on \( Sm/k_0 \) with the action of finite group is locally presentable, which follows readily from the corresponding assertion when the group is trivial. \( \square \)

The following is an alternative approach to providing a model structure on the category of pointed simplicial presheaves with continuous action by the \( G \). For this one considers first the orbit category \( O_G = \{ G/H \mid H \in W \} \). A morphism \( G/H \to G/K \) corresponds to \( \gamma \in G \), so that \( \gamma.H\gamma^{-1} \subseteq K \). One may next consider the category \( PSh^G_G \) of \( O_G \)-diagrams with values in \( PSh(Sm/k_0) \). This category, being a category of diagrams with values in \( PSh(Sm/k_0) \) readily inherits the structure of a cofibrantly generated model category. Moreover it is left-proper (right-proper, cellular, simplicial) since the model category \( PSh \) is.

Recall that the generating cofibrations of this diagram category are defined as follows: \( I_{O_G} = \{ (G/H) \times i \mid i \text{ a cofibration in } Psh, \ H \in W \} \). The corresponding generating trivial cofibrations \( J_{O_G} = \{ (G/H) \times j \mid j \text{ a trivial cofibration in } Psh, \ H \in W \} \). The diagram \( (G/H) \times i \) is the \( O_G \)-diagram defined by \( (G/H) \times i \to (G/K) \times i = Hom_G(G/H, G/K) \times i \).

Next we restrict to the case where \( G \) is profinite. Then the two categories \( PSh_c(Sm/k_0, G) \) and \( PSh^G_G \) are related by the functors:

\[ \Phi : PSh_c(Sm/k_0, G) \to PSh^G_G, \quad F \mapsto \Phi(F) = \{ \Phi(F)(G/H) = F^H \} \]

\[ \Theta : PSh^G_G \to PSh_c(Sm/k_0, G), \quad M \mapsto \Theta(M) = \lim_{\{H \in W \}} M(G/H). \]

For any such \( H \), there is a largest subgroup \( H_G \subseteq H \) so that \( H_G \) is normal in \( G \) and of finite index in \( G \): this is often called the core of \( H \). The obvious quotient map \( G/H_G \to G/H \) induces a map \( M(G/H) \to M(G/H_G) \) so that \( \{ M(G/H) \} \) is cofinal in the direct system used in the above colimit. Now \( G/H \) acts on \( G/K \) by translation and this induces an action by \( G/K \) on \( M(G/K) \). Therefore, the above colimit has a natural action by \( G \) which is clearly continuous. The natural transformation \( \Theta \circ \Phi \to id \) may be shown to be an isomorphism readily. Moreover \( \Theta \) is left-adjoint to \( \Phi \). It follows, therefore, that the functor \( \Phi \) is full and faithful, so that \( \Phi \) is an imbedding of the category \( PSh_c(Sm/k_0, G) \) in \( PSh^G_G \).

**Proposition 2.8.** Assume the above situation. Then the following hold

(i) If \( P \in PSh^G_G \) is cofibrant, the natural map \( \eta : P \to \Phi \Theta(P) \) is an isomorphism.
(ii) The two functors $\Phi$ and $\Theta$ are Quillen-equivalences.

(iii) Let $P \in \text{PSh}(\text{Sm}/k_0, G)$ be such that in $\text{PSh}(\text{Sm}/k_0)$ it is finitely presented and $P = P^H$ for some normal subgroup $H$ of finite index in $G$. Then $\Phi(P)$ is a finitely presented object in $\text{PSh}^{G\text{c}}$.

Proof. A key observation is that the the functor $\Theta$ sends the generating cofibrations, i.e. diagrams of the form $(G/H) \times i$ to $G/H \times i \in J_G$ and similarly sends the generating trivial cofibrations, i.e. diagrams of the form $(G/H) \times j$ to $G/H \times j \in J_G$. $\Phi$, on the other hand, sends the generating cofibrations in $J_G$ (generating trivial cofibrations in $J_G$) to the generating cofibrations in $\text{I}_{G\text{c}}$ (respectively). Since any cofibrant object in the model category $\text{PSh}^{G\text{c}}$ is a retract of an $\text{I}_{G\text{c}}$-cell, and both $\Theta$ and $\Phi$ preserve retracts, it suffices to prove (i) when $P$ is an $\text{I}_{G\text{c}}$-cell. Since $\Theta$ obviously preserves colimits and $\Phi$ also does the same as proven in Lemma 2.5, it suffices to consider the case when $P = (G/H) \times i$, which is already observed to be true. Therefore, the first statement follows.

Observe that it suffices to prove the following in order to establish the second statement. Let $X \in \text{PSh}^{G\text{c}}$ be cofibrant and let $Y \in \text{PSh}_c(\text{Sm}/k_0, G)$ be fibrant. Then a morphism $f : \Theta(X) \to Y$ in $\text{PSh}_c(\text{Sm}/k_0, G)$ is a weak-equivalence if and only if the corresponding map $g : X \to \Phi(Y)$ in $\text{PSh}^{G\text{c}}$ is a weak-equivalence. Now the induced map $g : X \to \Phi'(Y)$ induced by $f$ by adjunction factors as the composition $X \xrightarrow{\eta} \Phi(\Theta(X)) \xrightarrow{\Phi(f)} \Phi(Y)$. The first map is an isomorphism since $X$ is cofibrant, so that the map $g : X \to \Phi(Y)$ is a weak-equivalence if and only if the map $\Phi(f)$ is a weak-equivalence. But the map $\Phi(f)$ is a weak-equivalence if and only if each of the maps $f^H : X^H = (\Theta(X))^H \to Y^H$ is a weak-equivalence, which is equivalent to $f$ being a weak-equivalence. This proves (ii). The proof of (iii) is similar to the proof of Lemma 2.6 and is therefore skipped. (One needs to first observe that $P = P^H$ for all subgroups $H'$ of $G$ for which $H' \subseteq H$.)

Remarks 2.9. (i) One may observe that the functor $\Phi$ evidently commutes with all small limits while the functor $\Theta$ evidently commutes with all small colimits. Lemma 2.5 (v) shows that the functor $\Phi$ also commutes with all small colimits.

(ii) The above proposition proves that, instead of $\text{PSh}_c(\text{Sm}/k_0, G)$, it suffices to consider the diagram category $\text{PSh}^{G\text{c}}$ which is often easier to handle.

Proposition 2.10. The categories $\text{PSh}_c(\text{Sm}/k_0, G)$ (when the $G$ is profinite) and $\text{PSh}^{G\text{c}}$ as well as the corresponding $\mathbb{A}^1$-localized model categories $\text{PSh}^{G\text{c}}_{\mathbb{A}^1}$ are weakly finitely generated in the sense of [Dund1, Definition 3.4], i.e. the following hold: (i) The domains and co-domains of the maps in the classes of the generating cofibrations and generating trivial cofibrations are finitely presented and (ii) there exists a subset $J'$ of the generating trivial cofibrations in either of the two model categories, $\text{PSh}_c(\text{Sm}/k_0, G)$ and $\text{PSh}^{G\text{c}}$ so that a map $f : A \to B$ with fibrant co-domain is a fibration if and only if it has the right lifting property with respect to $J'$.

Moreover, the above categories are locally presentable and therefore combinatorial model categories.

Proof. The first statement needs the adjunction between the functor $G/H \times (\ )$ and the fixed points functor $(\ )^H$ for the category $\text{PSh}_c(\text{Sm}/k_0, G)$ and between the free functor $G/H \times (\ )$ and the functor of evaluating a diagram at $G/H$ for the category $\text{PSh}^{G\text{c}}$. As shown in Lemma 2.5(v), the fixed point functor $(\ )^H$ commutes with filtered colimits in $\text{PSh}_c(\text{Sm}/k_0, G)$. Since colimits in $\text{PSh}^{G\text{c}}$ are calculated point-wise, it is clear that taking colimits commutes with evaluating at $G/H$. The second statement follows from the corresponding property of the category $\text{PSh}_c(\text{Sm}/k_0)$ making use of the same adjunctions. The last statement reduces first to the case of finite group actions and then to the case where the group is trivial.

2.3.5. Coverings and hypercoverings in the $G$-equivariant setting, where $G$ is profinite. As shown in [DH], one way to consider simplicial presheaves that are simplicial sheaves up to homotopy is to consider a localization of the category of simplicial presheaves by inverting hypercoverings. Therefore, $G$-equivariant hypercoverings are important which we proceed to consider in some detail next. Since we often need to consider rigid coverings and rigid hypercoverings, we will define these first.

Definition 2.11. (Rigid coverings and hypercoverings) Let $X \in \text{Sm}/k_0$, denote a fixed scheme. Then a rigid covering of $X$ in the topology $\mathcal{O}$ is the following data: let $X$ denote a chosen conservative family of points of $X$ appropriate to the site. (For example, if $\mathcal{O} = \acute{e}t$, these form a conservative family of geometric points of $X$ and if $\mathcal{O} = Nis$ or $\mathcal{O} = Zar$, these form a conservative family of points of $X$ in the usual sense.) A rigid covering of $X$ then
is a disjoint union of pointed separated maps \( U_x, u_x \to X, x \) in the chosen site with each \( U_x \) connected and indexed by \( x \in X \). If the topology \( ? = \text{ét} \) or \( \text{Nis} \), observe that that the maps \( U_x \to X \) are étale and if \( ? = \text{ét} \) with \( u_x \) a point of \( U_x \) lying above \( x \) so that induced map \( k(x) \to k(u_x) \) is an isomorphism. If \( ? = \text{Zar} \), each \( U_x \to X \) is an open immersion containing the chosen point \( x \), so that \( u_x = x \) in this case.

A \textit{rigid hypercovering} of \( X \) is then a simplicial scheme \( U \to X \) together with a map \( U \to X \) so that \( U_0 \) is a rigid covering of \( X \) in the given topology \( ? \) and so that for each \( n \geq 0 \), the induced map \( U_n \to (\cosk_{n-1}U_\bullet)_n \) is a rigid covering. (Here we use the convention that \( \text{cosk}^0 \).) One may extend this definition readily to define rigid hypercoverings of simplicial schemes. We will let \( HRR(X) \) denote the category of all rigid hypercoverings of \( X \) in the given topology \( ? \). We skip the proof that this is a directed category, i.e. there is at most one map between two objects and is filtered.

Let \( X \) denote a smooth scheme of finite type over \( k_0 \) and provided with an action by a finite quotient of the group \( G \). We will denote this finite quotient of \( G \) by \( G \).

**Proposition 2.12.** (i) If \( \phi : U \to X \) is a covering in the étale (Nisnevich) site, then \( \psi : V = U \times G = U \times \bigsqcup G \) is a \( G \)-equivariant cover in the étale (Nisnevich site, respectively). (Here \( \bigsqcup G \) denotes a copy \( U \) indexed by \( G \).) If \( \phi \) is a rigid covering so is \( \psi \).

(ii) If \( \phi_\bullet : U_\bullet \to X \) is a hypercovering in the étale (Nisnevich) site, then \( \psi_\bullet : V_\bullet = U_\bullet \times \cosk_0{G} \to X \) is a \( G \)-equivariant hypercovering of \( X \) in the étale (Nisnevich site, respectively). Here the map \( U_0 \times (\cosk_0{G})^0 = U_0 \times G \to X \) is the obvious map sending \( U_0 \to X \) and \( G \to \text{Spec} k_0 \). If \( \phi_\bullet \) is a rigid hypercovering, so is \( \psi_\bullet \).

(iii) Assume \( U \to X \) a \( G \)-equivariant étale map which is a covering in the étale site (Nisnevich site) which factors through the structure map \( V_\bullet \to X \) of a \( G \)-equivariant hypercovering \( V_\bullet \), for some fixed \( k \geq 0 \). Then there exists a \( G \)-equivariant hypercovering \( W_\bullet \to X \) together with a map \( W_k \to V_k \) so that the map \( W_k \to V_k \) factors through \( U \to V_k \). In case \( U \to X \) is a rigid covering, \( W_\bullet \) can be chosen to be a rigid hypercovering.

**Proof.** (i) is obvious. For the following discussion we use the following convention: if \( W_\bullet \to X \) is a simplicial scheme over \( X \), then \( (\cosk_{-1}(W_\bullet)) = X \). Since \( V_n = U_n \times (\cosk_0(G))_n \) for all \( n \), the map \( V_{n+1} \to (\cosk_n(V_\bullet))_{n+1} \) is given by \( U_{n+1} \to (\cosk_0(G))_{n+1} \to (\cosk_n(V_\bullet))_{n+1} \) (forming part of the structure of the hypercovering \( U_\bullet \) and the identity map of \( (\cosk_0(G))_{n+1} \). Therefore this is a covering map in the appropriate topology, so that \( V_\bullet \) is a hypercovering. In case \( U_\bullet \) is a rigid hypercovering, then the map \( U_{n+1} \to (\cosk_n(U_\bullet))_{n+1} \) is also a rigid covering in the appropriate topology, so that so that the induced map \( V_{n+1} \to (\cosk_n(V_\bullet))_{n+1} \) for all \( n \geq 1 \). One may also verify that all the structure maps of the hypercovering \( V_\bullet \) are \( G \)-equivariant for the diagonal action of \( G \) on each \( V_n \). These prove (ii).

The proof of (iii) follows from the standard construction of \( W_\bullet \): \( W_n \) is defined as the pull-back of the two maps \( V_n \to \Pi V_k \leftarrow \Pi U \) where the map sending \( V_n \) to the factor of \( V_k \) indexed by \( \alpha \) is just \( \alpha:|k|\to[n] \) in \( \Delta \). If \( \alpha:|k|\to[n] \) in \( \Delta \). If \( \alpha:|k|\to[n] \) in \( \Delta \) and the maps \( U \to V_k \) are all the given map. It follows readily that \( W_\bullet \) is a hypercovering, which is rigid if all the objects involved in its definition are rigid. It is also \( G \)-equivariant since all the objects involved in its definition are \( G \)-equivariant. \( HRR(X,G) \) will denote the collection of all \( G \)-equivariant rigid hypercoverings of \( X \).

Let \( G \) denote a finite fixed group. Let \( \text{Sm}/k_0, G, ? \) denote the site whose objects are all smooth schemes of finite type over \( k_0 \) provided with an action by \( G \). The morphisms are morphisms in the given site that are also \( G \)-equivariant. The coverings of an object \( U \) are given by coverings \( V \to U \) in this site so that this map is also \( G \)-equivariant. The points on this site are \( \bigsqcup G/H \), where \( g \in G/H \) is any subgroup of \( G \). Let \( \text{AbSh}(\text{Sm}/k_0, G, ?) \) denote the category of \( G \)-equivariant abelian sheaves on the big site \( \text{Sm}/k_0 \). An equivariant abelian presheaf \( P \) on the site \( \text{Sm}/k_0, G, ? \) is additive if \( H \Gamma(U \cup V, P) = \gamma(U, P) \times H \Gamma(V, P) \). The category of all such abelian presheaves that are \( G \)-equivariant will be denoted \( \text{AbPsh}(\text{Sm}/k_0, G, ?) \).

Given any smooth scheme \( U \) of finite type over \( k_0 \), observe that \( G/H \times U \) defines a smooth scheme of finite type over \( k_0 \) with a \( G \)-action. (Here \( H \) is any subgroup of \( G \).) Then, \( \text{Hom}_G(G/H \times U, F) = \text{Hom}(U, F), \) where \( \text{Hom}_G \) denotes the external Hom in the category of \( G \)-equivariant sheaves (sheaves, respectively). Therefore, \( F^H \) denotes a sheaf on the big site \( \text{Sm}/k_0 \) by the rule \( \Gamma(U, F^H) = \text{Hom}(U, F^H) \). Now the following lemma is clear.
Lemma 2.13. AbSh(Sm/k₀, G, ?) is a Grothendieck category and hence has enough injectives. A sequence of abelian sheaves 0 → \mathcal{F}' → \mathcal{F} → \mathcal{F}'' → 0 in the above category is exact if and only if 0 → (\mathcal{F}')^H → \mathcal{F}^H → (\mathcal{F}'')^H → 0 is exact as sheaves on Sm/k₀, for every subgroup H of G.

Proposition 2.14. (Cohomology from equivariant hypercoverings) Let P ∈ AbPsh(Sm/k₀, G, ?) and let aP denote the associated abelian sheaf. Given any X ∈ (Sm/k₀, G, ?), we obtain the isomorphism

\[ H^*_i(X, aP) \cong \lim_{U_i ∈ \mathcal{HRR}(X, G)} H^*(\Gamma(U_i, aP)) \cong \lim_{U_i ∈ \mathcal{HRR}(X, G)} H^*(\Gamma(U_i, P)) \]

that is functorial in P.

Proof. We review the standard proof here. First one observes that if U_∗ → X is a G-equivariant hypercovering, then ZU_∗ → Z_X is a (simplicial) resolution in AbPsh(Sm/k₀, G, ?), where for any scheme W, Z_W denotes the abelian sheaf representing W. Therefore, if aP → I_∗ is an injective resolution by G-equivariant abelian sheaves, (2.36)

\[ \Gamma(X, I^*) \cong \text{Hom}_G(Z_X, I^*) \cong \text{Hom}_G(Z_{U_•}, I^*) = \Gamma(U_•, I^*) \]

In particular \( H^0(X, aP) \cong H^0(\Gamma(U_•, aP)) \) and if aP is an injective object of AbSh(Sm/k₀, G, ?), \( H^i(\Gamma(U_•, aP)) = 0 \) for all \( i > 0 \). Therefore, it now suffices to show that \( F \mapsto \lim_{U_i ∈ \mathcal{HRR}(X, G)} H^*(\Gamma(U_i, F)) \) is a \( \delta \)-functor and that one obtains the last isomorphism in the statement of the proposition.

Let 0 → F_1 → F_2 → F_3 → 0 denote a short exact sequence of G-equivariant abelian sheaves on the site Sm/k₀, G, ?). (Recall that the exactness of the above sequence is equivalent to the exactness of the sequences 0 → \( F'^H_1 \rightarrow F'^H_2 \rightarrow F'^H_3 \rightarrow 0 \) of abelian sheaves on Sm/k₀ for all subgroups H of G.) Let \( G_1 = \text{coker}(F_1 \rightarrow F_2) \) and \( G_3 = \text{coker}(G_1 \rightarrow F_3) \) where the cokernels are taken as abelian presheaves on the site (Sm/k₀, G, ?). Then, for any G-equivariant hypercovering \( U_• \rightarrow X \), we obtain the two short exact sequences of complexes:

\[ 0 → \Gamma(U_•, F_1) → \Gamma(U_•, F_2) → \Gamma(U_•, G_1) → 0 \quad \text{and} \quad 0 → \Gamma(U_•, G_1) → \Gamma(U_•, F_3) → \Gamma(U_•, G_3) → 0. \]

Presently we will show that \( \lim_{U_i ∈ \mathcal{HRR}(X, G)} (\Gamma(U_•, G_3)) = 0 \). The key observation here is that the sheaf associated to \( G_3 \), i.e. \( aG_3 \) is trivial. Therefore, given any class \( \alpha ∈ \Gamma(U_n, G_3) \) for some fixed G-equivariant rigid hypercovering \( U_• \) and some integer \( n ≥ 0 \), there exists some G-equivariant cover \( V → U_n \) so that \( \alpha → 0 \) in \( \Gamma(V, G_3) \). One may then find a G-equivariant rigid hypercovering \( V_• \) that dominates \( U_• \) and for which the map \( V_n → U_n \) factors through the given map \( V → U \).

The last isomorphism in the proposition follows essentially from the observation that the stalks of \( P \) and \( aP \) are isomorphic.

Corollary 2.15. Let \( P ∈ AbPsh(Sm/k₀, G, ?) \) and let \( Y ∈ Sm/k₀ \) be fixed scheme. Then one also obtains an isomorphism

\[ \text{Ext}^2_G(Z(X_+ ∧ S^n ∧ Y), aP) \cong \lim_{U_• ∈ \mathcal{HRR}(X, G)} H^*(\text{Hom}_G(Z(U_• ∧ S^n ∧ Y), P)), \]

where \( \text{Hom}_G \) denotes Hom in the category AbPsh(Sm/k₀, G, ?) and \( \text{Ext}^2_G \) denotes the derived functor of the corresponding \( \text{Hom}_G \) for G-equivariant abelian sheaves.

Proof. One defines a new abelian presheaf \( \bar{P} \) by \( \Gamma(U, \bar{P}) = \text{Hom}(Z(U_+ ∧ S^n ∧ Y), P) = \text{Hom}(Z(U_+) ⊗ Z(S^n) ⊗ Z(Y), P). \) (The identification \( Z(U_+ ∧ S^n ∧ Y) = Z(U_+) ⊗ Z(S^n) ⊗ Z(Y) \) is clear, see [Wei, p. 6] for example.) Then one may observe that the functor \( P → \bar{P} \) is exact and sends an additive abelian presheaf to an additive abelian presheaf. Therefore, the conclusion follows by applying the last proposition to the abelian presheaf \( P \).

Remark 2.16. Since not every object in Sm/k₀ has an action by G, it is not possible to consider localization of PSh_k(Sm/k₀, G, ?) or PSh_k(Sm/k₀, G, ?) by inverting maps associated to an elementary distinguished square in the Nisnevich topology or maps of the form \( U_• → U \) where \( U ∈ Sm/k₀ \) and \( U → U \) is a hypercovering in the given topology. Therefore, we instead localize PSh by inverting such maps and consider the category of diagrams of type \( O^G_\text{des} \) with values in this category.

Definition 2.17. \( \text{PSh}_{\text{mot}}^\text{O^G_\text{des}} \) \( (\text{PSh}^\text{O^G_\text{des}}_{\text{des}}) \) will denote the category of all diagrams of type \( O^G_\text{des} \) with values in the localized category \( \text{PSh}_{\text{mot}} \) (\( \text{PSh}^\text{O^G_\text{mot}}_{\text{des}} \), respectively).
Lemma 2.18. $\text{PSh}_{\text{mot}}^{\mathbb{G}_{m}}$ is a cofibrantly generated symmetric monoidal model category which is also weakly finitely generated in the sense of [Dund1, Definition 3.4] whereas $\text{PSh}_{\text{des}}^{\mathbb{G}_{m}}$ is a cofibrantly generated symmetric monoidal model category which is also left proper and cellular. Both categories are locally presentable and therefore also combinatorial model categories.

Proof. The symmetric monoidal structure is inherited from the one on $\text{PSh}_{\text{mot}}^{\mathbb{G}_{m}}$. Recall from [Hirsch, Theorem 4.3.1] that the generating cofibrations in the localized model category are the same as those in the original category. Now all the statements except the one claiming the monoidal model structure follow by standard arguments. (See [Dund2, 2.14, Lemma 2.15] for a proof of these for the category $\text{PSh}_{\text{mot}}$.)

We will sketch an argument to show this for $\text{PSh}_{\text{des}}$ from which the same conclusions follow for $\text{PSh}_{\text{des}}^{\mathbb{G}_{m}}$. The generating trivial cofibrations in the localized model category $\text{PSh}_{\text{des}}$ are obtained as pushout-products $c\bar{f}\bar{g}$ where $\bar{c}: \bar{U}_{*} \to \bar{c}U$ is the cofibrant approximation to the obvious structure map $f: U_{*} \to U$ of a hypercovering and $g: A \to B$ is a generating cofibration. (Observe that the domains of all the generating cofibrations $X \times \delta\Delta[n]$ are cofibrant, so that $\land$ with these preserve weak-equivalences in $\text{PSh}$.) Now it suffices to show that if $g': A' \to B'$ is also a generating cofibration, then $(c\bar{f}\bar{g})\square g'$ is weak-equivalence. First one uses the properness and monoidal model structure of $\text{PSh}$ to observe that the induced maps $(c\land id_{B})\square g'$ and $(c\land id_{A})\square g'$ are weak-equivalences. Then one obtains the pushout:

$$
\begin{array}{ccc}
\text{U}_{*} \land A \land B' & \lor & c\text{U} \land A \land A' \\
\downarrow & & \downarrow \\
\text{U}_{*} \land B \land B' & \lor & c\text{U} \land B \land A'
\end{array}
$$

which maps to $c\text{U} \land B \land B'$. The top row is $c\land id_{A}\square g'$ and so is a weak-equivalence. The left column is a cofibration and so the pushout is a homotopy-pushout. Therefore the bottom row is also a weak-equivalence. Now one may observe that the map $(c\land id_{B})\square g'$ from the left-most vertex of the bottom row to $c\text{U} \land B \land B'$ is a weak-equivalence and that it factors through the above pushout. Therefore, the induced map from the above pushout to $c\text{U} \land B \land B'$ given by $(c\square g')\square g'$ is also a weak-equivalence. This proves the the monoidal model structure for $\text{PSh}_{\text{des}}$ and the proof for $\text{PSh}_{\text{mot}}$ is similar.

For the remainder of this section, we will assume that $k_{0}$ is a fixed field, $k$ a fixed algebraic closure of $k_{0}$, $G = \text{Gal}_{k_{0}}(k)$.

Example 2.19. Then one obtains the following pushout square of objects in $\text{PSh}_{\text{mot}}^{\mathbb{G}_{m}}$:

$$
\begin{array}{ccc}
\mathbb{G}_{m} & \to & \mathbb{A}^{1} \\
\downarrow & & \downarrow \\
\mathbb{A}^{1} & \to & \mathbb{P}^{1}
\end{array}
$$

where all the schemes above are obtained by extension from $k_{0}$ to $k'$ which is a finite Galois extension $k'_{0}$ of $k_{0}$. Therefore, all of them have a natural $G_{k_{0}}(k')$-action. This shows that $\mathbb{P}^{1}/\mathbb{A}^{1} \cong \mathbb{A}^{1}/\mathbb{G}_{m}$ as objects in $\text{PSh}_{\text{mot}}^{\mathbb{G}_{m}}$. Moreover the obvious map $\mathbb{P}^{1} \to \mathbb{P}^{1}/\mathbb{A}^{1}$ (obtained by sending $k_{0}$ to the origin in $\mathbb{A}^{1}$) is $G$-equivariant and also an $\mathbb{A}^{1}$-equivariance. One may prove similarly using ascending induction on $n \geq 1$ that $\mathbb{P}^{n}/\mathbb{P}^{n-1} \cong \mathbb{A}^{n}/(\mathbb{A}^{n} - \{0\}) \cong \land^{n}\mathbb{P}^{1}$ in $\text{PSh}_{\text{mot}}^{\mathbb{G}_{m}}$.

2.4. Sheaves with transfer and correspondences. Given two smooth schemes $X, Y$ in $(\text{Sm}/k_{0}, G)$, let

$$
\text{Cor}_{G}(X, Y) = \{ Z \subseteq X \times Y \mid \text{closed, integral so that the projection } Z \to X \text{ is finite} \}.
$$

Observe that since $X$ and $Y$ are assumed to have actions through some finite quotient of $G$, there is natural induced action of $G$ on $\text{Cor}_{G}(X, Y)$; hence the presence of the subscript $G$. One may also define the category of correspondences on $S$, by letting the objects be the smooth schemes in $(\text{Sm}/k_{0}, G)$ and where morphisms are elements of $\text{Cor}_{G}(X, Y)$. This category will be denoted $\text{Cor}_{G}$. An abelian $G$-equivariant presheaf with transfers is a contravariant functor $\text{Cor}_{G} \to (\text{abelian groups}, G)$, where the category on the right consists of abelian groups with a continuous action by $G$, continuous in the sense that the stabilizers are all subgroups with finite quotients.
An abelian $G$-equivariant sheaf with transfers is an abelian $G$-equivariant presheaf with transfers which is a sheaf on the site $(\text{Sm}/k)_\text{Nis}$. This category will be denoted by $\text{ASH}_{tr}(S,G)$.

One obtains an imbedding of the category $(\text{Sm}/k,G)$ into $\text{Cor}_G$ by sending a scheme to itself and a $G$-equivariant map of schemes $f : X \rightarrow Y$ to its graph $\Gamma_f$. Given a scheme $X$ in $(\text{Sm}/k,G)$, $\text{Z}_{tr,G}(X)$ will denote the sheaf with transfers defined by $\Gamma(U,\text{Z}_{tr,G}(X)) = \text{Cor}_G(U,X)$. One extends this to define the $G$-equivariant motive of $X$ as the complex associated to the simplicial abelian sheaf $n \mapsto \text{Cor}_G(\ast \times \Delta[n],X)$, where $\ast$ takes any object in the big Nisnevich site as argument and $\Delta[n] = \text{Spec}(k_0[x_0, \cdots, x_n]/\Sigma_i^nx_i = 1)$. We will denote this by $\mathcal{M}_G(X)$.

It is important to realize that $\mathcal{M}_G(X)$ is a complex of sheaves on the site $(\text{Sm}/k)_\text{Nis}$. The global sections of this complex, i.e. sections over $\text{Spec} k_0$ will be denoted $\mathcal{M}_G(X)$.

2.41. A key property. Let $F$ denote an abelian $G$-equivariant sheaf with transfers. Then $\Gamma(X,F) \cong \text{Hom}_{\text{ASH}_{tr}(S,G)}(\mathcal{M}_G(X),F)$.

3. The basic framework of equivariant motivic homotopy theory: the stable theory

The theory below is a variation of the theory of enriched functors and spectra as in [Dund1], [Dund2] and also [Hov-3] modified so as to handle equivariant spectra for the action of a group $G$ where $G$ denotes a group as before.

3.1. Enriched functors and spectra. Let $\mathcal{C}$ denote a symmetric monoidal cofibrantly generated model category: recall this means $\mathcal{C}$ is a cofibrantly generated model category, also has the additional structure of a symmetric monoidal category, compatible with the model structure as in [Hov-2]. We will assume that $\mathcal{C}$ is pointed, i.e. the initial object identifies with the terminal object, which will be denoted $\ast$. The monoidal product will be denoted $\wedge$; the unit of the monoidal structure will be denoted $S^0$ and this is assumed to be cofibrant. We will let $I$ ($J$) denote the generating cofibrations (generating trivial cofibrations, respectively). We will make the following additional assumptions:

(i) The model structure on $\mathcal{C}$ is cellular: see [Hirsh, ]. In practice this means the domains and co-domains of both $I$ and $J$ are small with respect to $I$ and that the cofibrations are effective monomorphisms. We will assume the model structure is also left-proper.

(ii) We will further assume that the domains and co-domains of the maps in $I$ are cofibrant.

(iii) Though this assumption is not strictly necessary, we will further assume that $\mathcal{C}$ is simplicial or at least pseudo-simplicial so that there is a bi-functor $\text{Map} : \mathcal{C}^{op} \times \mathcal{C} \rightarrow (\text{simplicial, sets})$. We will also assume that there exists an internal Hom functor $\text{Hom}_C : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$, so that for any finitely presented object $C \in \mathcal{C}$, one obtains:

\[
\text{Map}(C, \text{Hom}_C(K,M)) \cong \text{Map}(C \wedge K, M)
\]

(In fact if $\text{Map}(K,N)_n = \text{Hom}_C(K \otimes \Delta[n],N)$ where $K \otimes \Delta[n]$ denotes an object in $\mathcal{C}$ defined using a pairing $\otimes : C \times (\text{simplicial, sets}) \rightarrow \mathcal{C}$, then it suffices to assume that $(K \otimes \Delta[n])_k$ is finitely presented for all $k \geq 0$ and that the analogue of the isomorphism in (3.1.1) holds with $\text{Map}$ replaced by $\text{Hom}_C$.)

Let $\mathcal{C}'$ denote a $\mathcal{C}$-enriched full-subcategory of $\mathcal{C}$ consisting of objects closed under the monoidal product $\wedge$, all of which are assumed to be cofibrant and containing the unit $S^0$. Let $\mathcal{C}'_0$ denote a $\mathcal{C}$-enriched sub-category of $\mathcal{C}'$, which may or may not be full, but closed under the monoidal product $\wedge$ and containing the unit $S^0$. (In [Dund1, section 4], there are a series of additional hypotheses put on the category $\mathcal{C}'$ to obtain stronger conclusions. These are not needed for the basic conclusions as in [Dund1, Theorem 4.2] and therefore, we skip them for the time being.) In view of the motivating example considered below, we will denote the objects of $\mathcal{C}'_0$ as $\{T_V\}$. Observe that $T_W \cong \text{Hom}(S^0, T_W)$ and that $S^0 \rightarrow \text{Hom}_C(T_V, T_V)$. Now the pairing $\text{Hom}_C(T_V, T_V) \wedge \text{Hom}_C(S^0, T_W) \rightarrow \text{Hom}_C(T_V \wedge T_V, T_V \wedge T_W)$, pre-composed with $S^0 \wedge \text{Hom}_C(S^0, T_W) \rightarrow \text{Hom}_C(T_V \wedge T_V) \vee \text{Hom}_C(S^0, T_W)$, sends $T_W \rightarrow \text{Hom}_C(S^0, T_W)$ to a sub-object of $\text{Hom}_C(T_V \wedge T_V, T_V \wedge T_W)$.

Definitions 3.1. (i) $[\mathcal{C}'_0, \mathcal{C}]$ will denote the category whose objects are $\mathcal{C}$-enriched covariant functors from $\mathcal{C}'_0$ to $\mathcal{C}$: see [Dund1, 2.2]. An enriched functor $\mathcal{X}$ sends $\text{Hom}_C(T_V, T_W) \mapsto \text{Hom}_C(\mathcal{X}(T_V), \mathcal{X}(T_W))$ for objects $T_V$ and $T_W \in \mathcal{C}'$ with this map being functorial in $T_V$ and $T_W$. The adjunction between $\wedge$ and $\text{Hom}_C$ shows that in this case, one is provided with a compatible family of pairings $\text{Hom}_C(T_V, T_W) \wedge \mathcal{X}(T_V) \mapsto \mathcal{X}(T_V \wedge T_W)$. On replacing $T_W$ with $T_V \wedge T_W$ and pre-composing the above pairing with $S^0 \wedge \text{Hom}_C(S^0, T_W) \rightarrow \text{Hom}_C(T_V \wedge T_V) \vee \text{Hom}_C(S^0, T_W)$, one sees that, an enriched functor $\mathcal{X}$ comes provided with the structure maps $T_W \wedge \mathcal{X}(T_V) \mapsto \mathcal{X}(T_W \wedge T_V)$ that are compatible as $T_W$ and $T_V$ vary in $\mathcal{C}'_0$.) A morphism $\phi : \mathcal{X}' \rightarrow \mathcal{X}$ between two such enriched
functors is given by a $C$-natural transformation, which means that one is provided with a compatible collection of maps $\{X'(T_V) \to X(T_V) \mid T_V \in \mathcal{C}_0\}$, compatible with the pairings $\text{Hom}_C(T_V, T_W) \land X'(T_V) \to X'(T_W)$ and $\text{Hom}_C(T_V, T_W) \land X'(T_V) \to X'(T_W)$.

(ii) **Spectra**($\mathcal{C}$). We let $\text{Sph}(\mathcal{C}_0)$ denote the $\mathcal{C}$-category defined by taking the objects to be the same as the objects of $\mathcal{C}_0$ and where $\text{Hom}_{\text{Sph}(\mathcal{C}_0)}(T_V, T_W) = T_W$ if $T_V = T_W \land T_U$ and $*$ otherwise. Since $T_W$ is a sub-object of $\text{Hom}_C(T_V, T_W)$, it follows that $\text{Sph}(\mathcal{C}_0)$ is a sub-category of $\mathcal{C}_0$. Now an enriched functor in $\text{Sph}(\mathcal{C}_0, \mathcal{C})$ is simply given by a collection $\{X(T_V) \mid T_V \in \text{Sph}(\mathcal{C}_0)\}$ provided with a compatible collection of maps $T_W \land X'(T_V) \to X'(T_W)$. We let $\text{Spectra}(\mathcal{C}) = \text{Sph}(\mathcal{C}_0, \mathcal{C})$.

(iii) For each fixed $T_V \in \mathcal{C}_0$, we associate the free $\mathcal{C}$-functor $\mathcal{F}_T : \mathcal{C} \to [\mathcal{C}_0, \mathcal{C}]$ defined by $\mathcal{F}_T(X) = X \land \text{Hom}_C(T_V, \cdot)$ and the free $\mathcal{C}$-functor $\mathcal{F}_T : \mathcal{C} \to \text{Spectra}(\mathcal{C})$ that sends each $X \in \mathcal{C}$ to the spectrum $\mathcal{F}_T(X)$ defined by

\[(3.1.2) \quad \mathcal{F}_T(X)(T_W) = X \land T_U, \text{ if } T_W = T_U \land T_V \text{ and } = * \text{ otherwise.} \]

(iv) For each $T_V \in \mathcal{C}_0$, we let $\mathcal{R}_T : \mathcal{C} \to [\mathcal{C}_0, \mathcal{C}]$ denote the $\mathcal{C}$-enriched functor $\mathcal{R}_T(P) = \text{Hom}_C(\text{Hom}_C(T_V, T_U), P)$. Similarly we let $\mathcal{R}_T : \mathcal{C} \to \text{Spectra}(\mathcal{C})$ denote the functor defined by $\mathcal{R}_T(P)(T_W) = \text{Hom}_C(T_U, T_V)$ if $T_U \land T_W = T_V$ and $S^0$ otherwise.

Let $\text{Eval}_{\mathcal{C}_0} : [\mathcal{C}_0, \mathcal{C}] \to \mathcal{C}$ denote the $\mathcal{C}$-enriched functor sending $X \mapsto X(T_V)$. Similarly, let $\text{Eval}_\mathcal{T}_V : \text{Spectra}(\mathcal{C}_0) \to \mathcal{C}$ denote the $\mathcal{C}$-enriched functor that sends a spectrum $X \in \text{Spectra}(\mathcal{C})$ to $X(T_V)$.

**Lemma 3.2.** Now $\mathcal{R}_T$ is right-adjoint to $\text{Eval}_\mathcal{T}_V$ while $\mathcal{F}_T$ is left-adjoint to $\text{Eval}_\mathcal{T}_V$. Similarly, $\mathcal{R}_T$ is right-adjoint to $\text{Eval}_\mathcal{T}_V$ while $\mathcal{F}_T$ is left-adjoint to $\text{Eval}_\mathcal{T}_V$.

**Example 3.3.** The main examples of the above are the following. Let $\mathcal{C}$ denote one of the categories $\text{PSh}_h(\text{Sm}/k_0, \mathcal{G})$, $\text{PSh}(\text{Sm}/k_0, \mathcal{G})$, $\text{PSh}^{\mathcal{G}}$ or the corresponding $\mathcal{A}^1$-localized categories of simplicial sheaves up to homotopy. We may also restrict to the full subcategories of constant pointed simplicial presheaves with action by $\mathcal{G}$ in the first two categories and to the full subcategory of diagrams of type $\mathcal{O}_\mathcal{G}$ in the category of pointed simplicial sets. One has the following choices for the category $\mathcal{C}_0$.

(i) Let $V$ denote an affine space over $k$ (where $k$ is some finite Galois extension of $k_0$) and provided with a linear action by $\mathcal{G}$ so that $V^H = V$ for some $H \in \mathcal{W}$. We let $T_V = V/V - 0 = \text{Thom-space of the corresponding } \mathcal{G}/K\text{-equivariant vector bundle over } k$. We let $\mathcal{C}_0 = \mathcal{C}'$ denote the collection $\{T_V\}$ as one varies over all such representations of $\mathcal{G}$. Observe that $\mathcal{C}'$ now contains the sphere $S^0$ (by taking the $V$ to be the 0-dimensional vector space) and is closed under smash products. The resulting category of spectra will be denoted $\text{Spectra}(\mathcal{C})$ (and by $\text{Spectra}(\mathcal{C}, \mathcal{C}')$ to clarify the choice of the sub-category $\mathcal{C}'$.)

A variant of the above framework is the following: let $F : \mathcal{C} \to \mathcal{C}$ denote a functor that is compatible with the monoidal product $\land$ (i.e. there is a natural map $F(A) \land F(B) \to F(A \land B)$) and with all small colimits. We may now let $\mathcal{C}_0 = \mathcal{C}'_F$ be the full subcategory of $\mathcal{C}$ generated by $\{F(A) \mid A \in \mathcal{C}'\}$ together with $S^0$. Examples of such a framework appear in the later sections. The resulting category of spectra will be denoted $\text{Spectra}(\mathcal{C}, F)$. This seems to be the only general framework available for $\mathcal{G}$ denoting any one of the three choices above.

(ii) One may also adopt the following alternate definition of $\mathcal{C}_0 = \mathcal{C}'$, when $\mathcal{G}$ denotes either a finite or profinite group. Let $\mathcal{T} = \mathbb{P}^1 = \mathbb{P}_k^1$. Let $K$ denote a normal subgroup of $\mathcal{G}$ with finite index. Let $\mathcal{T}_K$ denote the $\land$ of $\mathcal{G}/K$ copies of $\mathcal{T}$ with $\mathcal{G}$ acting by permuting the various factors above. We let $\mathcal{C}'$ denote the full sub-category of $\mathcal{C}$ generated by these objects under finite applications of $\land$ as $K$ is allowed to vary subject to the above constraints. This sub-category of $[\mathcal{C}', \mathcal{C}]$ will be denoted $[\mathcal{C}', \mathcal{C}]_{\mathcal{P}^1}$ and the corresponding category of spectra will be denoted $\text{Spectra}(\mathcal{C}, [\mathcal{P}^1])$. If we fix the normal subgroup $K$ of $\mathcal{G}$, the resulting category of $[\mathcal{P}^1]$-spectra will be denoted $\text{Spectra}(\mathcal{C}, \mathcal{G}/K, \mathcal{P}^1)$; here the subcategory $\mathcal{C}'$ will denote the full sub-category of $\mathcal{C}$ generated by these objects under finite applications of $\land$.

More generally, given any object $P \in \mathcal{C}$ together with an action by some finite quotient group $\mathcal{G}/K$ of $\mathcal{G}$, one may define the categories $[\mathcal{C}', \mathcal{C}]_P$ and the category, $\text{Spectra}(\mathcal{C}, P)$ of $P$-spectra similarly by replacing $\mathcal{P}^1$ above by $P$. (For example, $P$ could be $F(\mathcal{P}^1)$ for a functor $F$ as in (i).) In case $\text{char}(k_0) = 0$, the regular representation
of any finite group breaks up into the sum of irreducible representations and contains among the summands all such irreducible representations. Therefore, in case \( \text{char}(k_0) = 0 \), there is no loss of generality in adopting this framework to that of (i) when the group is finite. In arbitrary characteristics, and also for general groups, the framework of (i) is clearly more general.

(iii) Under the same hypotheses as in (ii), we may also define \( C' \) as follows. Let \( T = S^n \) (for some fixed positive integer \( n \)) denote the usual simplicial \( n \)-sphere. Let \( T_K \) denote the \( K \) \& \( G/K \) copies of \( T \) with \( G/K \) acting by permuting the factors. In case \( K = G \) and \( n = 1 \), we obtain a spectrum in the usual sense and indexed by the non-negative integers. Such spectra will be called ordinary spectra; when the constituent simplicial presheaves are all simplicial abelian presheaves, such spectra will be called ordinary abelian group spectra.

3.1.3. Spectra indexed by the non-negative integers. One may construct spectra indexed by the non-negative integers from \( \text{Spectra}(C) \) as follows. Let \( T_V \) denote a fixed element of \( C' \). One may now consider the full subcategory of \( C'_0 \) generated by \( T_V \) under iterated smash products along with the unit \( S^0 \). Any spectrum \( X \) in \( \text{Spectra}(C) \) may be restricted to this sub-category and provides a spectrum \( \overline{X} \) in the usual sense by defining \( \overline{X}_n = X(\wedge^n T_V) \). Given any motivic spectrum \( X \), one obtains this way, the (motivic) spectrum \( \overline{X} \) indexed the non-negative integers where \( \overline{X}_n = X(\wedge^n T_V) \).

Examples 3.4. Suspension spectra (i) Let \( C \) denote one of the categories as in Definition 3.11 and let \( A \in C \).

Then the motivic suspension spectrum associated to \( A \) is the spectrum \( \Sigma_{mot}(A) \) whose value on \( TV \in C'_0 \) is given by \( TV \wedge A \). The \( P^1 \)-motivic suspension spectrum associated to \( A \) is the spectrum whose value on \( TV \in C'_0 \) is given by \( TV \wedge A \). This will be denoted \( \Sigma_0^P(A) \). If the subgroup \( K \) of \( G \) is fixed, we will denote the corresponding suspension spectrum by \( \Sigma_{mot,K}(A) \). If we take \( A = S^0 \), we obtain the motivic sphere spectrum \( \Sigma_{mot} \) and the \( P^1 \)-motivic sphere spectrum \( \Sigma_{P^1} \). In case \( F : C \rightarrow C \) is a functor commuting with \( \Lambda \) and with colimits, then we obtain the motivic \( F \)-spectrum \( \Sigma_{mot,F} \) and the \( (P^1) \)-motivic sphere spectrum \( \Sigma_{F(P^1)} \).

(ii) If we take \( A = S^n \wedge B \), for some \( n \geq 1 \) and \( B \in C \) (as above), the resulting spectra are the \( S^n \)-suspensions of the above suspension spectra associated to \( A \). This will be denoted \( \Sigma_{mot}^n B \). If we take \( A = TV \wedge S^n \wedge B \), \( TV \in C'_0 \), the resulting \( P^1 \)-motivic suspension spectrum associated to \( A \) will be denoted \( \Sigma_{mot}^{n,0} \wedge B \), where \( |TV| = \text{dim}(V) \).

If we take \( A = (P_1)^n \wedge B \) for some \( B \in C \), the resulting \( P^1 \)-motivic suspension spectrum associated to \( A \) will be the \( (P^1)^n \)-suspension of the \( P^1 \)-motivic suspension spectrum associated to \( B \). This will be denoted \( \Sigma_{mot}^{2n,n} \). In case \( A = S^n \wedge (P_1)^n \wedge B \) for some \( B \in C \), the resulting \( P^1 \)-motivic suspension spectrum will be denoted \( \Sigma_{mot}^{n+2m,m} \).

3.1.4. Suspending and de-suspending spectra. Let \( E \) denote a spectrum in any one of the above categories, with \( TV \) denoting elements of \( C'_0 \). Then we define the suspension \( \Sigma_2 TV \wedge E \) to be the spectrum defined by \( \Sigma_2 TV \wedge E(T_V) = E(T_V \wedge TV) \). The de-suspension \( \Sigma_2^{-1} TV \wedge \overline{E} \) will be the spectrum defined by \( \Sigma_2^{-1} TV \wedge \overline{E} = \overline{E}(TV \wedge TV) \) and \( = 0 \) otherwise.

3.1.5. Equivariant vs. non-equivariant spectra. The equivariant spectra will denote spectra as in 3.3(i) and 3.3(ii). \( \Sigma_{P^1,G} \) will denote the \( G \)-equivariant sphere spectrum associated to \( P^1_k \). The non-equivariant spectra will denote the \( G \)-spectra when the group \( G \) is the trivial group. \( \Sigma_{P^1} \) will denote the (non-equivariant) sphere spectrum associated to \( P^1_k \). When we only consider equivariant spectra, \( \Sigma_{P^1,G} \) will also be used to denote \( \Sigma_{P^1,G} \).

It is important to observe that the equivariant spectrum \( \Sigma_{P^1,G} \) is a module spectrum over the non-equivariant spectrum \( \Sigma_{P^1} \).

3.1.6. Mapping spectra. Next we briefly discuss the various mapping spectra (i.e. \( \text{Hom} \)-spectra) that one needs to invoke. For the categories \( \text{PSH} \text{Sm}/k_0, \text{G} \), \( \text{PSH} \text{Sm}/k_0, \text{G} \) as well as the corresponding \( \Lambda^1 \)-localized model categories one has the external hom which will be denoted \( \text{Hom} \). One also has the internal Hom, which will be denoted \( \text{Hom} \) and in addition there is an enriched hom, enriched in simplicial sets, which will be denoted \( \text{Map} \). For \( P, Q \in \text{PSH} \text{Sm}/k_0, \text{G}, \text{Map}(P, Q) = \text{Hom}(P \wedge \Delta[n], Q) \). Observe also that \( \Gamma(U, \text{Hom}(P, Q)) = \text{Map}(P_U, Q_U) \) where \( P_U \) and \( Q_U \) denote the restriction of \( P \) and \( Q \) to the subcategory \( \text{Sm}/U \).

Using these, one obtains the corresponding \( \text{Hom} \)-functors for spectra: \( \text{Hom}_{S_P} \) will denote the external hom, while \( \text{Hom}_{S_P} \) will denote the internal hom and \( \text{Map}_{S_P} \) will denote the enriched hom.

One may observe that

\[
\text{Hom}_{S_P}(E, B)(TV) = \text{Eq}(\Pi_{TV} \text{Hom}(E(T_V), B(T_V \wedge TV)) \rightarrow \Pi_{TV, TV} \text{Hom}(TV \wedge E(T_V), B(T_V \wedge TV \wedge TV)))
\]
where $\mathcal{H}om$ denotes the internal hom in the appropriate category of simplicial presheaves and the two arrows denote the obvious maps.

3.1.7. Notational convention. When the context is clear, we will often omit the subscript $Sp$ in $\mathcal{H}om_{Sp}$. Moreover, we will also use $\mathcal{H}om$ or $\mathcal{H}om_{Sp}$ to denote $\Gamma(Spec k_0, \mathcal{H}om_{Sp}(\ , \ ))$.

3.1.8. Layer filtrations on spectra. Layer filtrations on spectra provide a convenient tool for inductive arguments: see [Sw, 8.7(ii)]. We extend this to our framework as follows. Let $K$ denote a spectrum in any one of the above categories and let $T_Y$ denote elements of $C'$

We define an ascending filtration, $\{\tilde{K}^n[n]\}$ on $K$ as follows: $\tilde{K}^n = *$ for $n \leq 0$ and $\tilde{K}^n = \cup\{F^V(e) | l(e) \leq n, T_Y \in C_0\}$. This is clearly an ascending filtration on $K$ and $K = \cup_{n \geq 1} \tilde{K}^n$.

Observe that $\tilde{K}^1 = \bigvee F_{TV_\alpha}(S^n)$ and $\tilde{K}^n/\tilde{K}^{n-1} = \bigvee F_{TV_\beta}(S^n)$ for some families of $\{TV_\alpha[\alpha]\}, \{TV_\beta[\beta]\} \subseteq C_0$.

For module-spectra $M$ over a ring spectrum $E$ considered below, one may also obtain a similar filtration with $\tilde{M}^1 = \bigvee\{F_{TV_\gamma}(S^n) \wedge E\}$ and $\tilde{M}^n/\tilde{M}^{n-1} = \bigvee\{F_{TV_\gamma}(S^n) \wedge E\}$ for some families of $\{TV_\gamma[\gamma]\}, \{TV_\beta[\beta]\} \subseteq C_0$. The main difference is that we will define $F^E_V(e) = F^V(e) \wedge E$ and use $F^V(e)$ in the place of $F^V(e)$, with the filtration defined similarly.

3.2. Smash products of spectra and ring spectra. First we recall the construction of smash products of enriched functors from [Dund1, 2.3]. Given $F, F \in [C_0', C]$, their smash product $F' \wedge F$ is defined as the Kan-extension along the monoidal product $\otimes : C_0' \times C_0' \rightarrow C_0'$ of the $C'$-enriched functor $F' \times F \in [C_0' \times C_0', C]$. Given spectra $X$ and $Y$ in $[Sp(C_0'), C]$, this also defines their smash product $X \wedge Y$. One also defines the derived smash product $X \wedge^L Y$ by $C(X) \wedge Y$, where $C(X)$ is a cofibrant replacement of $X$ in the stable model structure on $\textup{Spectra}(C)$. An algebra in $[C_0', C]$ is an enriched functor $X : C_0' \rightarrow C_0'$ providing with an associative and unital pairing $\mu : X \times X \rightarrow X$, i.e. for $TV, TW \in C_0'$, one is given a pairing, $X(TV) \wedge X(TW) \rightarrow X(TV \wedge TW)$ which is compatible as $TW$ and $TV$ vary and is also associative and unital.

A ring spectrum in $\textup{Spectra}(C)$ is an algebra in $[Sp(C_0'), C]$ for some choice of $C_0'$ satisfying the above hypotheses. A map of ring-spectra is defined as follows. If $\mathcal{X}$ is an algebra in $[Sp(C_0'), C]$ and $\mathcal{Y}$ is an algebra in $[Sp(D_0'), C]$ for some choice of subcategories $C_0'$ and $D_0'$, then a map $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ of ring-spectra is given by the following data: (i) an enriched covariant functor $\phi : C_0' \rightarrow D_0'$ compatible with $\wedge$ and (ii) a map of spectra $\mathcal{X} \rightarrow \phi_*(\mathcal{Y})$ compatible with the ring-structures. (Here $\phi_*(\mathcal{Y})$ is the spectrum in $\textup{Spectra}(C_0')$ defined by $\phi_*(\mathcal{Y}')(TV) = Y(\phi(TV))$.

Given a ring spectrum $A$, a left module spectrum $M$ over $A$ is a spectrum $M$ provided with a pairing $\mu : A \wedge M \rightarrow M$ which is associative and unital. One defines right module spectra over $A$ similarly. Given a left (right) module spectrum $M$ ($\Lambda \, N$ respectively) over $A$, one defines

\[
M \wedge N = \text{coequalizer}(M \wedge A \wedge N \rightarrow M \wedge N)
\]

where the coequalizer is taken in the category of spectra and the two maps correspond to the module structures on $M$ and $N$, respectively.

Let $\textup{Mod}(A)$ denote the category of left module spectra over $A$ with morphisms being maps of left module spectra over $A$. Then the underlying functor $U : \textup{Mod}(A) \rightarrow \textup{Spectra}(C)$ has a left-adjoint given by the functor $F_A(N) = A \wedge N$. The composition $T = F_A \circ U$ defines a triple and we let $TM = \textup{hocolim}\{T^n[M|n]\}$. Since a map $f : M' \rightarrow M$ in $\textup{Mod}(A)$ is a weak-equivalence of spectra if and only if $U(f)$ is, one may observe readily that $TM$ is weakly-equivalent to $M$. Therefore, one defines

\[
M \wedge_{\Lambda} N = TM \wedge_{\Lambda} N = \text{coequalizer}(TM \wedge A \wedge N \rightarrow M \wedge N)
\]

Examples 3.5. (i) Assume the situation of 3.3 (i). Let $F : C \rightarrow C$ denote a functor commuting with $\wedge$ and with colimits and provided with a natural augmentation $\epsilon : id \rightarrow F$ also compatible with $\wedge$. Then $\Sigma_{\text{mot,F}}$ is a ring spectrum in $\textup{Spectra}(C)$ and $\epsilon$ induces a map of ring spectra $\Sigma_{\text{mot,F}} \rightarrow \Sigma_{\text{mot,F}}$. In case $F$ is not provided with
the augmentation, then \( \Sigma_{mot,F} \) is a ring spectrum in \( \text{Spectra}(C,F) \); in fact it is the unit of this symmetric monoidal category.

Let \( \phi : F \to G \) denote a natural transformation between two functors \( C \to C \) that commute with the tensor structure \( \wedge \) and with all small colimits. Let \( C', C'' \) denote subcategories of \( C \) as before and let \( C'_F, C'_G \) denote the corresponding subcategories. Then \( \phi \) induces a functor \( C'_F \to C'_G \) compatible with \( \wedge \). Let \( \Sigma_{mot,F} \) and \( \Sigma_{mot,G} \) denote the motivic sphere spectra defined there. Then these are ring-spectra and \( \phi \) induces a map commuting with the ring structures.

(ii) Assume the situation of 3.3 (ii). Let \( \phi : F \to G \) be as above. Then for any \( P \in C, \Sigma_{F(P)} \) is a ring-spectrum in \( \text{Spectra}(C,F(P)) \) and \( \Sigma_{G(P)} \) is a ring-spectrum in \( \text{Spectra}(C,G(P)) \) with \( \phi \) inducing a map of these ring-spectra.

3.3. Model structures on \( \text{Spectra}(C) \). We will begin with the level-wise model structure on \( \text{Spectra}(C) \) which will be defined as follows.

**Definition 3.6.** (The level model structure on \( \text{Spectra}(C) \).) A map \( f : X' \to X \) in \( \text{Spectra}(C) \) is a level equivalence (level fibration, level trivial fibration, level cofibration, level trivial cofibration) if each \( Ev_{TV}(f) \) is a weak-equivalence (fibration, trivial fibration, cofibration, trivial cofibration, respectively) in \( C \). Such a map \( f \) is a projective cofibration if it has the left-lifting property with respect to every level trivial fibration.

Let \( I (J) \) denote the generating cofibrations (generating trivial cofibrations, respectively) of the model category \( C \). We define the generating cofibrations \( I_{Sp} \) to be \( \bigcup_{TV} \{ F_{TV}(i) \mid i \in I \} \) and the generating trivial cofibrations \( J_{Sp} \) to be \( \bigcup_{TV} \{ F_{TV}(j) \mid j \in J \} \).

**Proposition 3.7.** (i) If \( A \in C \) is small relative to the cofibrations (trivial cofibrations) in \( C \), then \( F_{TV}(A) \) is small relative \( I_{Sp} \).

(ii) A map \( f \) in \( \text{Spectra}(C) \) is level cofibration (level trivial cofibration) if and only if it has the left lifting property with respect to all maps of the form \( R_{TV}(g) \), where \( g \) is a trivial fibration (fibration, respectively) in \( C \).

(iii) Every map in \( I_{Sp} - cof \) is a level cofibration and every map in \( J_{Sp} - cof \) is a level trivial cofibration.

(iv) The domains of \( I_{Sp} \) (\( J_{Sp} \)) are small relative to \( I_{Sp} - cof \) (\( J_{Sp} - cof \), respectively).

**Proof.** (i) The main point here is that the functor \( Ev_{TV} \) right adjoint to \( F_{TV} \) commutes with all small colimits.

(ii) Since \( R_{TV} \) is right adjoint to \( Ev_{TV} \), \( f \) has the left lifting property with respect to \( R_{TV}(g) \) if and only if \( Ev_{TV}(f) \) has the left-lifting property with respect to \( g \). (ii) follows readily from this observation.

(iii) Recall every object of \( C'_0 \) is assumed to be cofibrant. Therefore, smashing with any \( TV \in C'_0 \) preserves cofibrations of \( C \). Therefore, every map in \( I_{Sp} \) is a level cofibration. By (ii) this means \( R_{TV}(g) \in I_{Sp} \) – inj for all trivial fibrations \( g \) in \( C \). Recall every map in \( I_{Sp} - cof \) has the left lifting property with respect to every map in \( I_{Sp} - inj \) and in particular with respect to every map \( R_{TV}(g) \), with \( g \) a trivial fibration in \( C \). Now the adjunction between \( Ev_{TV} \) and \( R_{TV} \) completes the proof for \( I_{Sp} - cof \). The proof for \( J_{Sp} - cof \) is similar.

(iv) follows readily in view of the adjunction between the free functor \( F_{TV} \) and \( Ev_{TV} \).

We now obtain the following corollary.

**Corollary 3.8.** The projective cofibrations, the level fibrations and level equivalences define a cofibrantly generated model category structure on \( \text{Spectra}(C) \) with the generating cofibrations (generating trivial cofibrations) being \( I_{Sp} \) (\( J_{Sp} \), respectively). This model structure (called the projective model structure on \( \text{Spectra}(C) \)) has the following properties:

(i) It is left-proper (right proper) if the corresponding model structure on \( C \) is left proper (right proper, respectively). It is cellular if the corresponding model structure on \( C \) is cellular.

(ii) The objects in \( \bigcup_{TV} \{ F_{TV}(C'_0) \} \) are all finitely presented. The category \( \text{Spectra}(C) \) is symmetric monoidal with the pairing \( X' \wedge X(T_V) = X'(T_V) \wedge X(T_V) \) and with the unit being the inclusion functor \( i : \bigcup_{TV} \{ F_{TV}(C'_0) \} \to \text{Spectra}(C) \).
(iii) The projective model structure on \( \text{Spectra}(\mathcal{C}) \) is weakly finitely generated when the given model structure on \( \mathcal{C} \) is weakly finitely generated.

(iv) The category \( \text{Spectra}(\mathcal{C}) \) is locally presentable if \( \mathcal{C} \) is.

Proof. The assertions in (ii) are clear. Those in (i) and (iii) may be proven making use of the last Proposition: see also [Hov-3, proof of Theorem 1.14].

One may characterize the projective cofibrations and projective trivial cofibrations as follows: a map \( f : X' \to X \) is a projective cofibration (projective trivial cofibration) if and only if (i) each \( Eval_{T_V}(f) \) is a cofibration (trivial cofibration, respectively) and (ii) the induced map \( X'(T_V \wedge T_W) \to T_V \wedge (X(T_W) \to X(T_V \wedge T_W)) \) is a cofibration (trivial cofibration, respectively) for all \( T_V, T_W \in \mathcal{C}_0' \). This may be proven as in [Hov-3, Proposition 1.15].

3.3.1. The stable model structure on \( \text{Spectra}(\mathcal{C}) \). We proceed to define the stable model structure on \( \text{Spectra}(\mathcal{C}) \) by applying a suitable Bousfield localization to the projective model structure on \( \text{Spectra}(\mathcal{C}) \). This follows the approach in [Hov-3, section 3]. (One may observe that the domains and co-domains of objects in \( \mathcal{I} \) are cofibrant, so that there is no need for a cofibrant replacement functor \( Q \) as in [Hov-3, section 3].)

**Definition 3.9.** (\( \Omega \)-spectra) A spectrum \( X \in \text{Spectra}(\mathcal{C}) \) is an \( \Omega \)-spectrum if it is level-fibrant and each of the natural maps \( X(T_V) \to \text{Hom}_C(T_W, X(T_V \wedge T_W)) \), \( T_V, T_W \in \mathcal{C}_0' \) is a weak-equivalence in \( \mathcal{C} \).

Observe that giving a map \( F_{T_V \wedge T_W}(T_W \wedge C) \to F_{T_V}(C) \) corresponds by adjunction to giving a map \( T_W \wedge C \to (F_{T_V}(C))(T_V \wedge C) \). Therefore we let \( \text{S} \) denote the morphisms \( \{ F_{T_V \wedge T_W}(T_W \wedge C) \to F_{T_V}(C) \mid C \in \text{Domains and Co-domains of } I, T_V, T_W \in \mathcal{C}_0' \} \) corresponding to the identity maps \( T_W \wedge C \to T_V \wedge C \) by adjunction.

The stable model structure on \( \text{Spectra}(\mathcal{C}) \) is obtained by localizing the projective model structure on \( \text{Spectra}(\mathcal{C}) \) with respect to the maps in \( \text{S} \). (The reason for considering such maps is as follows: let \( C \in \mathcal{C} \) be an object as above. Then \( \text{Map}(C, \text{Eval}_{T_V}(X)) \approx \text{Map}(F_{T_V}(C), X) \) and \( \text{Map}(C, \text{Hom}_C(T_W, \text{Eval}_{T_V \wedge T_W}(X))) \approx \text{Map}(F_{T_V \wedge T_W}(T_W \wedge C), X) \) provided \( X \) is fibrant in the projective model structure considered above. Therefore to convert \( X \) into an \( \Omega \)-spectrum, it suffices to invert the maps in \( \text{S} \).) The \( \text{S} \)-local weak-equivalences (\( \text{S} \)-local fibrations) will be referred to as the stable equivalences (stable fibrations, respectively). The cofibrations in the localized model structure are the level cofibrations in \( \text{Spectra}(\mathcal{C}) \).

**Proposition 3.10.** (i) The corresponding stable model structure on \( \text{Spectra}(\mathcal{C}) \) is cofibrantly generated, left proper and cellular when \( \mathcal{C} \) is assumed to have these properties.

(ii) The fibrant objects in the stable model structure on \( \text{Spectra}(\mathcal{C}) \) are the \( \Omega \)-spectra defined above.

(iii) The category \( \text{Spectra}(\mathcal{C}) \) is symmetric monoidal with the same structure as in the projective model structure.

(iv) The stable model structure on \( \text{Spectra}(\mathcal{C}) \) is weakly finitely generated when the given model structure on \( \mathcal{C} \) is. In general it is cellular and left proper assuming this holds for \( \mathcal{C} \). It is also locally presentable, assuming \( \mathcal{C} \) is locally presentable.

Proof. The proof of (i) is entirely similar to the proof of [Hov-3, Theorem 3.4] and is therefore skipped. If \( \mathcal{C} \) is left proper (cellular), so is the projective model structure on \( \text{Spectra}(\mathcal{C}) \) as proved above. It is shown in [Hirsh, ] that then the localization of the projective model structure is also left proper and cellular. The last two statements are obvious from the corresponding properties of the projective model structure.

**Definition 3.11.** Motivic and étale spectra, \( \mathbb{P}^1 \)-motivic and étale spectra. (i) If \( \mathcal{C} = \text{PSh}_{mot}^{C_0'} \) and \( \mathcal{C}_0' \) chosen as in Examples 3.3(i), the resulting category of spectra, \( \text{Spectra}(\mathcal{C}) \), with the stable model structure will be called *motivic spectra* and denoted \( \text{Spt}_{mot}(k_0, \mathcal{G}) \). If \( \mathcal{C} = \text{PSh}_{et}^{C_0'} \) with the étale topology and \( \mathcal{C}_0' \) chosen as in Examples 3.3(i), the resulting category of spectra, \( \text{Spectra}(\mathcal{C}) \) with the stable model structure will be called *étale spectra* and denoted \( \text{Spt}_{et}(k_0, \mathcal{G}) \). In case \( F : \mathcal{C} \to \mathcal{C} \) is a functor as in 3.3 (i), \( \text{Spt}_{mot,F}(k_0, \mathcal{G}) \) (\( \text{Spt}_{et,F}(k_0, \mathcal{G}) \) will denote the corresponding category.) Observe from Examples 3.3(i), that there are several possible choices for the sub-categories \( \mathcal{C}_0' \).
(ii) If $\mathcal{C} = \text{PSh}_{\text{mot}}^{\text{et}}$ and $\mathcal{C}'_0$ chosen as in Examples 3.3(ii), the resulting category of spectra, $\text{Spectra}(\mathcal{C})$, with the stable model structure will be called $\mathbb{P}^1$-motivic spectra and denoted $\text{Spt}_{\text{mot}}(k_0, G, [\mathbb{P}^1])$. If $\mathcal{C} = \text{PSh}_{\text{des}}^{\text{et}}$ with the étale topology and $\mathcal{C}'_0$ chosen as in Examples 3.3(ii), the resulting category of spectra, $\text{Spectra}(\mathcal{C})$ with the stable model structure will be called $\mathbb{P}^1$-étale spectra and denoted $\text{Spt}_{\text{et}}(k_0, G, [\mathbb{P}^1])$. In case $F : \mathcal{C} \to \mathcal{C}$ is a functor as in 3.3(i), $\text{Spt}_{\text{mot}}(k_0, G, F([\mathbb{P}^1]))$ (or $\text{Spt}_{\text{et}}(k_0, G, F([\mathbb{P}^1]))$ will denote the corresponding category.) Observe again from Examples 3.3(ii), that there are several possible choices for the sub-categories $\mathcal{C}'_0$.

If $S^n$ for some fixed positive integer $n$ is used in the place of $\mathbb{P}^1$ above, the resulting categories will be denoted $\text{Spt}_{\text{mot}}(k_0, G, S^n)$ and $\text{Spt}_{\text{et}}(k_0, G, S^n)$.

The group $G$ will be suppressed when we consider spectra with trivial action by $G$.

3.3.3. Diagrams of spectra vs. spectra of diagrams. If $\mathcal{C} = \text{PSh}_{\text{mot}}^{\text{et}}$ with $\mathcal{C}'_0$ chosen as in Examples 3.3(i) or (ii), then the resulting categories of spectra, $\text{Spectra}_{\mathcal{C}_0}(\mathcal{C})$ obtained above are in fact spectra with values in $\mathcal{C}$ which is a diagram category. One may instead start with $\mathcal{C} = \text{PSh}_{\text{mot}}$ and with $\mathcal{C}'_0$ chosen as in Examples 3.3(i) or (ii), then we also obtain the resulting categories of spectra, $\text{Spectra}_{\mathcal{C}_0}(\mathcal{C})$. The main observation we want to make here is that there is a natural isomorphism $\text{Spectra}_{\mathcal{C}_0}(\mathcal{C}) \cong \text{Spectra}_{\mathcal{C}_0}(\mathcal{C})$. The category on the left is a category of diagrams of type $\mathcal{C}_{\mathcal{C}_0}$ with values in the category of spectra, $\text{Spectra}_{\mathcal{C}_0}(\mathcal{C})$. Therefore, we will routinely identify objects of $\text{Spectra}_{\mathcal{C}_0}(\mathcal{C})$ with objects in the above diagram category of spectra. The same discussion applies also to the case where $\mathcal{C} = \text{PSh}_{\text{des}}^{\text{et}}$ with the étale topology and with $\mathcal{C}'_0$ chosen as in Examples 3.3(i) or (ii).

Notational convention: Given an object $X \in \text{Spectra}_{\mathcal{C}_0}(\mathcal{C})_{\mathcal{C}_0}$, we let $X^H$ denote the diagram $X$ evaluated at $G/H$.

Remark 3.12. Clearly it is possible to use any of the categories $\mathcal{C} = \text{PSh}_{\text{mot}}$, $\text{PSh}_{\text{mot, G}}$, $\text{PSh}_{\text{des, G}}$, or $\text{PSh}_{\text{des, et}}$ with $\mathcal{C}'_0$ chosen as in 3.3(i) or as in 3.3(ii). However, the simplicial presheaves forming the terms of these spectra are not sheaves unto homotopy. Hence the reason for adopting the above definitions. Clearly, for certain applications, it may be enough to consider such spectra.

Definition 3.13. (i) $\text{HSpt}_{\text{mot}}(k_0, G)$ ($\text{HSpt}_{\text{et}}(k_0, G)$) will denote the stable homotopy category of motivic (étale) spectra.

(ii) $\text{HSpt}_{\text{mot}}(k_0, G, [\mathbb{P}^1])$ ($\text{HSpt}_{\text{et}}(k_0, G, [\mathbb{P}^1])$) will denote the corresponding stable homotopy category of $\mathbb{P}^1$-motivic (étale) spectra.

The group $G$ will be suppressed when we consider spectra with trivial action by $G$.

3.3.4. An alternate construction of the category of spectra. In the construction of the stable category of spectra considered above, we start with unstable category of pointed simplicial presheaves, then localize this to invert $\mathbb{A}^1$-equivalences, then stabilize to obtain the category of spectra with the projective model structure which is then localized to obtain the category of spectra with the stable model structure. Instead of these steps, one may adopt the following alternate series of steps to construct the category of spectra with its stable model structure, where all the localization is carried out at the end.

One starts with the category of pointed simplicial presheaves $\mathcal{C} = \text{PSh}_{\text{mot}}^{\text{et}}$. Next we apply the stabilization construction as discussed above and obtain $\text{Spectra}(\mathcal{C})$ with its projective model structure considered above. This is left proper, cellular and weakly finitely generated. Assume first that $\mathcal{C} = \text{Nis}$. One defines a presheaf $P \in \text{PSh}(\text{Sm}/k_0)$ to be motivically fibrant if (i) $P$ is fibrant in $\text{Spectra}(\mathcal{C})$, (ii) $\Gamma(\phi, P)$ is contractible (where $\phi$ denotes the empty scheme), (iii) $\phi$ is an elementary distinguished squares in $\text{MV}$ to a homotopy cartesian square and (iv) the obvious pull-back $\Gamma(U, P) \to \Gamma(U, \mathbb{A}^1 \wedge \mathbb{A}^1, P)$ is a weak-equivalence. Then a map $f : A \to B$ in $\text{Spectra}(\mathcal{C})$ is a motivic weak-equivalence if the induced map $\text{Map}(f, P)$ is a weak-equivalence for every motivically fibrant object $P$, with $\text{Map}$ denoting the simplicial mapping space. Next we localize this by inverting maps that belong to any of the following classes: (i) $f$ is a motivic weak-equivalence and (ii) \{ $F_{T_Y \wedge T_W}(T_Y \wedge \mathbb{A}^1) \to F_{T_Y}(C)$ | $C \in \text{Domains and Co-domains of } I, T_Y, T_W \in \mathcal{C}_0 \}$ (Here $I$ is the set of generating cofibrations of $\mathcal{C}$) The cofibrations in the localized model structure will be the same as those of $\text{Spectra}(\mathcal{C})$. The fibrations are defined by the lifting property. By [Hirsch, Theorem 4.1.2], it follows that this is a left-proper cellular model category. The resulting stable model structure identifies with the stable model structures on motivic spectra and $\mathbb{P}^1$-motivic spectra obtained above.
One may in fact carry out this process in two stages. First one only inverts maps of the following form: \( \{F_{T_V \wedge T_W}(T_V \wedge C) \to F_{T_W}(C) \mid C \in \text{Domains and Co-domains of } I, T_V, T_W \in C'_0 \} \). (Here I is the set of generating cofibrations of C. This produces a stable model structure on Spectra(C) where the fibrant objects are \( \Omega \)-spectra. This stable model category will be denoted \( \text{Spectra}_{st}(C) \). Then one inverts the maps f which are motivic weak-equivalences to obtain the \( \mathcal{A}^1 \)-localized stable category of spectra.

It may be important to specify the generating trivial cofibrations for the localized category, which we proceed to do now: see [Dund1, Definition 2.14]. For any elementary distinguished square:

\[
Q = \begin{array}{c} \phi' \end{array} \begin{array}{c} \phi \end{array} \begin{array}{c} \phi'' \end{array} \begin{array}{c} \phi''' \end{array} \begin{array}{c} \phi'''' \end{array}
\]

we factor the induced map \( h_P \to h_Y \) as a cofibration (using the simplicial mapping cylinder) \( h_P \to C \) followed by a simplicial homotopy equivalence \( C \to h_Y \) and similarly factor the induced map \( sq = h_U \sqcup C \to h_X \) as a cofibration \( sq \to tq \) followed by a simplicial homotopy equivalence \( tq \to h_X \). Similarly we factor the obvious map \( h_U \times \mathbb{A}^1 \to h_U \) into a cofibration \( u : h_U \times \mathbb{A}^1 \to C_u \) followed by a simplicial homotopy equivalence \( C_u \to h_U \). We also similarly factor the above map \( F_{T_V \wedge T_W}(T_V \wedge C) \to F_{T_W}(C) \) into a cofibration \( w_{V,W} : F_{T_V \wedge T_W}(T_V \wedge C) \to D_{V,W} \) followed by a simplicial homotopy equivalence \( D_{V,W} \to F_{T_W}(C) \). Let

\[
(3.3.5) \quad \tilde{J}_0 = \{ * \to h_\phi \} \cup \{ u : h_U \times \mathbb{A}^1 \to C_u \mid U \in \text{Sm}/k_0 \} \cup \{ q : sq \to tq \mid q \text{ is an elementary distinguished square} \}
\]

\[
\tilde{J}_1 = \{ w_{V,W} : F_{T_V \wedge T_W}(T_V \wedge C) \to D_{V,W} \mid C \in \text{Domains and Co-domains of } I, T_V, T_W \in C'_0 \}
\]

\[
\tilde{J} = \{ F_{T_V}(\tilde{J}_0)|V \in C'_0 \} \cup \tilde{J}_1
\]

(3.3.6)

The set of generating trivial cofibrations, \( J'_p \), will then be the set \( \{ f \sqcup g \mid f \in \tilde{J}_0, g : \delta \Delta[n] \to \Delta[n], n \geq 0 \} \cup J'_p \), where \( f \sqcup g \) denotes the pushout-product and \( J'_p \) is the set of generating trivial cofibrations in the stable model structure on Spectra(C).

In order to obtain a set of generating trivial cofibrations for \( \text{Spectra}_{st}(C) \), one just lets \( \tilde{J}'_0 = \{ * \to h_\phi \} \) and \( \tilde{J}_1 \) and \( \tilde{J} \) as before.

Using the above information one may verify readily that this is also a symmetric monoidal model category which is also simplicial and that it is weakly finitely generated.

If one uses the framework of Examples 3.3 (i), one obtains the categories \( \text{Spt}_{p}(k_0, G) \), \( \text{Spt}_{tor,F}(k_0, G) \) and if one uses the framework of Examples 3.3(ii), one obtains the categories \( \text{Spt}_{tor}(k_0, G, \mathbb{P}^1) \), \( \text{Spt}_{tor}(k_0, G, F(\mathbb{P}^1)) \). One obtains a similar alternate construction of étale spectra as well.

3.4. Key properties of \( \text{Spt}_{tor,F}(k_0, G) \), \( \text{Spt}_{et,F}(k_0, G) \), \( \text{Spt}_{tor}(k_0, G, F(\mathbb{P}^1)) \) and \( \text{Spt}_{et}(k_0, G, F(\mathbb{P}^1)) \). We summarize the following properties which have already been established above. Similar properties also hold for the categories \( \text{Spt}_{tor}(k_0, G, S^0) \) and \( \text{Spt}_{et}(k_0, G, S^0) \) for any \( n \geq 1 \) though we do not state them explicitly.

(i) Weak-equivalences and fibration sequence A map \( f : A \to B \) in any one of the above categories of \( G \)-equivariant spectra is a weak-equivalence if and only if the induced map \( f^H : A^H \to B^H \) is a weak-equivalence of spectra for all subgroups \( H \leq \mathbb{W} \). A diagram \( F \to E \to B \) is a fibration sequence in any one of the above categories of \( \mathcal{H} \)-equivariant spectra if and only if the induced diagrams \( F^H \to E^H \to B^H \) are all fibration sequences of spectra for all subgroups \( H \leq \mathbb{W} \).

(ii) Stably fibrant objects. A spectrum \( E \) is fibrant in the above stable model structure if and only if each \( E(T_V) \) is fibrant in \( \mathcal{C} = \) the appropriate unstable category of simplicial presheaves and the induced map \( E(T_V) \to \mathcal{H} \text{ome}_{\mathcal{C}}(T_W, (E(T_V \wedge T_W))) \) is a weak-equivalence of \( \mathcal{G} \)-equivariant spaces for all \( T_V, T_W \in C'_0 \).

(iii) Cofiber sequences A diagram \( A \to B \to B/A \) is a cofiber sequence if and only if the homotopy fiber of the map \( B \to B/A \) is stably weakly-equivalent to \( A \).

(iv) Finite sums. Given a finite collection \( \{ E_\alpha | \alpha \} \) of objects, the finite sum \( V_\alpha E_\alpha \) identifies with the product \( V_\alpha \otimes E_\alpha \) up to stable weak-equivalence.
(v) **Additive structure.** The corresponding stable homotopy categories is an additive category.

(vi) **Shifts.** Each \( T_V \in \mathcal{C}_0 \) defines a shift-functor \( E \to E[T_V] \), where \( E[T_V] \circ (T_W) = E(T_V \land T_W) \). This is an automorphism of the corresponding stable homotopy category.

(vii) **Cellular left proper simplicial model category structure.** All of the above categories of spectra have the structure of cellular left proper simplicial model categories. The categories \( \text{Spt}_{\text{mot},E}(k_0, G) \), \( \text{Spt}_{\text{mot}}(k_0, G, F(\mathbb{P}^1)) \) and the corresponding categories of spectra on the \( \text{etale} \) site are weakly finitely generated. The above categories are locally presentable, so that the corresponding model categories are combinatorial.

(viii) **Symmetric monoidal structure.** There is a symmetric monoidal structure on all the above categories of spectra, where the product is denoted \( \land \). The sphere spectrum, (i.e. the inclusion of \( \mathcal{C}_0 \) into \( \mathcal{C} \)) is the unit in this symmetric monoidal structure. Given a fixed spectrum \( E \), the functor \( F \mapsto E \land F \) has a right adjoint which will be denoted \( \text{Hom} \) or often \( \text{Hom} \). This is the internal hom in the above categories of spectra. The derived functor of this \( \text{Hom} \) denoted \( \text{RHom} \) may be defined as follows. \( \text{RHom}(F, E) = \text{Hom}(C(F), Q(E)) \), where \( C(F) \) is a cofibrant replacement of \( F \) and \( Q(E) \) is a fibrant replacement of \( E \).

(ix) **Ring spectra.** Algebras in the above categories of spectra will be called these \( G \)-equivariant ring spectra (and **ring-spectra** for short.)

**Definition 3.14.** (Stable homotopy groups) Let \( E \) denote a fibrant spectrum in any of the above categories of spectra and let \( H \) denote any subgroup of \( G \) so that \( H \in \mathcal{W} \) and so that \( H \) acts trivially on the affine space \( V \) on which \( G \) acts linearly. Recall that since \( E \) is fibrant, the obvious map \( E(T_V) \to \Omega G(T_V \land T_W) = \text{Hom}_C(T_W, E(T_V \land T_W)) \) is a weak-equivalence. Therefore, the \( k \)-th iteration of the above map, \( E(T_V) \to \Omega^k G(E(T_V \land T_W^k)) \) is also a weak-equivalence. Therefore, we define

\[
\pi^H_{s+2|T_V|,T_V}(E(U)) = \lim_{n \to \infty} [T_V \land T_W^s \land S^s \land U, E^H_U], \quad U \in (\text{Sm}/k_0, G),
\]

Here \( [A, B] \) denotes \( \text{Hom} \) in the homotopy category associated to the corresponding category of \( G \)-equivariant spectra and where \( H \) also acts trivially on \( W \). Next assume that \( G \) denotes a profinite group and let \( H_G \) denote the largest normal subgroup of \( G \) contained in \( H \). (This is the core of \( H \) we considered earlier and has finite index in \( G \).)

\[
H^{s+2|T_V|,T_V}(U_+^-, E) = \text{Hom}_{\text{Spt},H_G}(\Sigma^{s+2|T_V|,T_V}U_+, E^H_U) = \\
\lim_{n \to \infty} \text{Hom}_{\text{Spt},H_G}(T_V \land T_W^s \land S^s \land U, E^H_U(T_W^n)), \quad U \in (\text{Sm}/k_0, G),
\]

where \( \text{Hom}_{\text{Spt},H_G} \) denotes the internal hom-functor defined in 3.16 for \( G/H_G \)-equivariant objects and \( ? \) denotes either the Nisnevich or the \( \text{etale} \) sites. We will use the notation \( H^{s+2|T_V|,T_V}(U_+^-, E) \) \((H^{s+2|T_V|,T_V}(U_+^-, E)) \) to denote the corresponding hypercohomology when \( E \in \text{Spt}_{\text{mot}}(k_0, G, T) \) \((E \in \text{Spt}_{\text{mot}}(k_0, G, T)) \), respectively.

Observe that if we restrict to \( \mathbb{P}^1 \)-motivic spectra, then a general \( T_W \in \mathcal{C}_0 \) is a finite iterated smash product of terms of the form \( G/K \), for some normal subgroup \( K \) of \( G \) with finite index. When the group actions are ignored (or trivial), a general \( T_W \in \mathcal{C}_0 \) is a finite iterated smash product of \( \mathbb{P}^1 \), so that in this case the above stable homotopy groups may be just indexed by two integers.

3.5. **Eilenberg-Maclane and Abelian group spectra.** It is shown in [Dund2, Example 3.4], how to interpret the motivic complexes, \( \mathbb{Z} = \oplus_{r \geq 0} \mathbb{Z}(r) \) and \( \mathbb{Z}/l = \oplus_{r \geq 0} \mathbb{Z}/l(r) \) as ring-spectra in the above stable motivic homotopy category. While this is discussed there only in the non-equivariant framework, the constructions there extend verbatim to the \( G \)-equivariant setting.

This spectrum will be denoted \( H^G(\mathbb{Z}) \). The generalized cohomology (and homology) with respect to it in the sense defined in [CJ1], will define \( G \)-equivariant motivic cohomology (homology, respectively). \( H^G(\mathbb{Z}/l) \) will denote the mod \(-l \) variant for each prime \( l \).

With \( \mathcal{C} \) denoting either \( \text{PSh}(\text{Sm}/k_0, G) \) or \( \text{PSh}^{G} \) (or one of their subcategories considered earlier), we let \( \mathcal{C}_{\text{Ab}} \) denote the Abelian group objects in \( \mathcal{C} \). (Observe that these are nothing but simplicial Abelian presheaves together with a continuous action by \( G \) in the first case and diagrams of simplicial Abelian presheaves indexed by the orbit category \( O^G_{\text{G}} \) in the second case.)

Let \( \mathcal{C}_0 \) denote the subcategory

\[ \{ T_V | V = \text{an affine space provided with an action by} \ G, \text{ so that there exists an} \ H \in \mathcal{W} \text{acting trivially on} \ V \}. \]
Then we define an Abelian group spectrum $A$ to be a functor $A : C^0_\et \to C_{Ab}$ so that for each $T_V, T_W \in C^0_\et$, one is provided with maps $Z/(T_V) \oplus A(T_W) \to A(T_V \wedge T_W)$ and these are compatible as $W$ and $V$ vary in $C^0_\et$. The category of **Abelian group spectra** in $C$ will be denoted $\mathsf{Spt}_{Ab}(k_0, G)$. If $C = \mathsf{PSh}_{mot}(\text{Sm}/k_0, G)$ ($C = \mathsf{Sh}_{dcs}(\text{Sm}/k_0, G)$) the corresponding category of abelian group spectra will be denoted $\mathsf{Spt}_{Ab, mot}(k_0, G)$ ($\mathsf{Spt}_{Ab, dcs}(k_0, G)$, respectively). The objects in this category will be called **motivic Abelian group spectra** (étale Abelian group spectra, respectively). One may now define the $\mathbb{P}^1$-suspension spectrum associated to the motive $\mathcal{M}_G(X)$ as an object in the above category of Abelian group spectra. This will be denoted $\Sigma_\mathbb{P}(\mathcal{M}_G(X))$.

Let $E \in \mathsf{Spt}_{Ab, mot}(k_0, G)$. Then one also observes the following universal property for any $X \in \mathsf{Spt}(\text{Sm}/k_0, G)$:

\begin{equation}
\Gamma(X, E) \cong \text{Hom}_{\mathsf{Spt}_{Ab, mot}(k_0, G)}(\Sigma_\mathbb{P}(\mathcal{M}_G(X)), E)
\end{equation}

3.5.2. **Spectra of $\mathbb{Z}/l$-vector spaces.**

(i) For a pointed simplicial presheaf $P \in \mathsf{PSh}(\text{Sm}/k_0)$, $Z/(l(P))$ will denote the presheaf of simplicial $\mathbb{Z}/l$-vector spaces defined in [B-K, Chapter 2.1]. Observe that the presheaves of homotopy groups of $Z/(l(P))$ identify with the reduced homology presheaves of $P$ with respect to the ring $\mathbb{Z}/l$. Hence these are all $l$-primary torsion. The functoriality of this construction shows that if $P$ has an action by the group $G$, then $Z/(l(P))$ inherits this action. Moreover, if the action by $G$ on $P$ is continuous, then so is the induced action on $Z/(l(P))$.

(ii) Next one extends the construction in the previous step to spectra. Let $C = \mathsf{Psh}^{C\text{\text{-}ac}}$ and let $E \in \mathsf{Spectra}(C)$. Then first one applies the functor $(\ ) \mapsto Z/(l(\ )$ to each $E(T_W), T_W \in C^0_\et$. Then there exist natural maps $Z/(l(T_V) \otimes Z/(l(E(T_W))) \to Z/(l(T_V \wedge E(T_W)))$. Therefore, one may compose the above maps with the obvious map $Z/(l(T_V \wedge E(T_W))) \to Z/(l(E(T_V \wedge T_W)))$ to define an object in $\mathsf{Spectra}_{Z/(l(\ )}$.\n
(iii) A pairing $M \wedge N \to P$ in $\mathsf{PSh}(\text{Sm}/k_0, G)$ induces a pairing $Z/(l(M) \otimes Z/(l(N)) \to Z/(l(M \wedge N)) \to Z/(l(P))$. Similarly a pairing $M \wedge N \to P$ in $\mathsf{Spectra}(C)$ induces a similar pairing $Z/(l(M) \otimes Z/(l(N)) \to Z/(l(P))$. (To see this, one needs to recall the construction of the smash-product of spectra from [Dund1, 2.3]: first one takes the point-wise smash-product $M \wedge N : \mathsf{Sph}(C^0_\et) \times \mathsf{Sph}(C^0_\et) \to C$ and the one applies a left-Kan extension of this along the $\wedge : \mathsf{Sph}(C^0_\et) \times \mathsf{Sph}(C^0_\et) \to \mathsf{Sph}(C^0_\et)$. The functor $Z/(l$ commutes with left-Kan extension.) This shows that if $R \in \mathsf{Spectra}(C)$ is a ring spectrum so is $Z/(l(R)$ and that if $M \in \mathsf{Spectra}(C)$ is a module over the ring spectrum $R$, $Z/(l(M)$ is a module object over the ring object $Z/(l(R)$.

(iv) If $\{f : A \to B\} = J_{Sp}$ is a set of generating trivial cofibrations for $\mathsf{Spectra}(C)$, then, $Z/(l(f) : Z/(l(A) \to Z/(l(B))|f \in J_{Sp})$ is a set of generating trivial cofibrations for $\mathsf{Spectra}_{Z/(l(\ )}$. The category of **spectra of $\mathbb{Z}/l$-vector spaces** in $C$ will be denoted $\mathsf{Spectra}_{Z/(l(\ )}$ of $C$. If $C = \mathsf{PSh}_{mot}(\text{Sm}/k_0, G)$ ($C = \mathsf{PSh}_{dcs}(\text{Sm}/k_0, G)$), respectively where $F$ is a functor $C \to C$ as in 3.3(i). These categories are also locally presentable and hence the model categories are combinatorial. Applying $Z/(l(\ )$ to the pairing $T_V \wedge E(T_W) \to E(T_V \wedge T_W)$, $T_V, T_W \in C^0_\et$ and pre-composing with the pairing $Z/(l(T_V) \otimes Z/(l(E)) \to Z/(l(T_V \wedge E))$, one observes that the functor $Z/(l(\ )$ induces a functor $Z/(l(\ ) : \mathsf{Spt}_{mot,F}(k_0, G) \to \mathsf{Spt}_{Z/(l, mot,F}(k_0, G)$ and $Z/(l(\ ) : \mathsf{Spt}_{et,F}(k_0, G) \to \mathsf{Spt}_{Z/(l, et,F}(k_0, G)$.

(One also obtains a similar functor for the category of $F(\mathbb{P}^1)$-spectra.)

3.6. **$\mathbb{A}^1$-localization in the étale setting.** Recall that $\mathsf{Spt}_{et}(k_0, G)$ denotes the category of spectra on the big étale site. We will let $\mathsf{Spt}_{et}(k_0)$ denote spectra with trivial action by the group $G$.

Observe that the étale homotopy type of affine-spaces over a separably closed field $k$, are trivial when completed away from $\text{char}(k)$.

**Proposition 3.15.** Assume the base field $k_0$ is separably closed and $\text{char}(k_0) = p$. Let $E \in \mathsf{Spt}_{et}(k_0)$ be a constant sheaf of spectra so that all the (sheaves of) homotopy groups $\pi_n(E)$ are $l$-primary torsion, for some $l \neq p$. Then $E$ is $\mathbb{A}^1$-local in $\mathsf{Spt}_{et}(k_0)$, i.e. the projection $P \wedge \mathbb{A}^1_{k_0,+} \to P$ induces a weak-equivalence: $\text{Map}(P, E) \cong \text{Map}(P \wedge \mathbb{A}^1_{k_0,+}, E)$, $P \in \mathsf{Spt}_{et}(k_0)$.

**Proof.** First let $P$ denote the suspension spectrum associated to some scheme $X \in S$. Then $\text{Map}(P, E) (\text{Map}(P \wedge \mathbb{A}^1_{k_0,+}, E))$ identifies with the spectrum defining the generalized étale cohomology of $X$ (of $X \times \mathbb{A}^1_{k_0},$ respectively) with respect to the spectrum $E$. There exist Atiyah-Hirzebruch spectral sequences that converge to these generalized.
étale cohomology groups with the $E^2_{s,t}$-terms being $H^s_{et}(X, \pi_{-t}(E))$ and $H^s_{et}(X \times \mathbb{A}_{k_0}^1, \pi_{-t}(E))$, respectively. Since the sheaves of homotopy groups $\pi_{-t}(E)$ are all $l$-torsion with $l \neq p$, and $k_0$ is assumed to be separably closed, $X$ and $X \times \mathbb{A}_{k_0}^1$ have finite $l$-cohomological dimension. Therefore these spectral sequences converge strongly and the conclusion of the proposition holds in this case. For a general simplicial presheaf $P$, one may find a simplicial resolution where each term is a disjoint union of schemes as above (indexed by a small set). Therefore, the conclusion of the proposition holds also for suspension spectra of all simplicial schemes and therefore for all spectra $P$.

**Corollary 3.16.** Assume the base field $k_0$ is separably closed and $\text{char}(k_0) = p$. Let $E \in \text{Spt}_{et}(k_0, G)$ be a constant sheaf of spectra so that all the (sheaves of) homotopy groups $\pi_n(E)$ are $l$-primary torsion, for some $l \neq p$. Then $E$ is $\mathbb{A}^1$-local in $\text{Spt}_{et}(k_0, G)$, i.e. the projection $P \wedge \mathbb{A}_{k_0}^1 \to P$ induces a weak-equivalence: $\text{Map}_G(P, E) \simeq \text{Map}_G(P \wedge \mathbb{A}_{k_0}^1, E)$ where $\text{Map}_G$ is part of the simplicial structure on $\text{Spt}_{et}(k_0, G)$.

**Proof.** The last proposition shows that the conclusion is true were it not for the group action. The functor $\Phi: P \mapsto \{P^H \mid H \in W\}$ sending

$$\text{Spt}_{et}(k_0, G) \to \Pi_{H \in W} \text{Spt}_{et}(k_0)$$

has a left-adjoint defined by sending $\{Q(G/H) \mid H \in W\}$ to $\bigvee Q(G/H) \wedge (G/H)_+$. This provides a triple, whereby one may find a simplicial resolution of a given $P \in \text{Spt}_{et}(k_0, G)$ by spectra of the form $Q \wedge G/H_+$ where $Q \in \text{Spt}_{et}(k_0)$. (To see this is a simplicial resolution, it suffices to show this after applying $\Phi$ in view of the fact that a map $f: A \to B$ in $\text{Spt}_{et}(k_0, G)$ is a weak-equivalence if and only if the induced maps $f^H$ are all weak-equivalences. On applying $\Phi$, the above augmented simplicial object will have an extra degeneracy.) Therefore, one reduces to considering $P$ which are of the form $Q \wedge G/H_+$. Then one obtains by adjunction, the following weak-equivalences: $\text{Map}_G(Q \wedge G/H_+, E) \simeq \text{Map}(Q, E^H)$ and $\text{Map}_G(Q \wedge \mathbb{A}_{k_0}^1 \wedge G/H_+, E) \simeq \text{Map}(Q \wedge \mathbb{A}_{k_0}^1, E^H)$ where $\text{Map}$ is part of the simplicial structure on $\text{Spt}_{et}(k_0)$. One obtains the weak-equivalence $\text{Map}(Q, E^H) \simeq \text{Map}(Q \wedge \mathbb{A}_{k_0}^1, E^H)$ by the last proposition. This completes the proof of the corollary.

4. Effect of $\mathbb{A}^1$-localization on ring and module structures and on mod-$l$ completions

We make strong use of ring and module structures on spectra in [CJ1] and sequels. Therefore, it is important to know that the process of $\mathbb{A}^1$-localization is (or can be made to be) compatible with these structures. Similarly completions, especially at a prime $l$, plays a key role in our work: therefore, it is again important to show that such completions may be carried out so as to be compatible with the process of $\mathbb{A}^1$-localization. Since the arguments needed to show both of these have several common features, we will present the common arguments in a unified manner.

4.0.1. Therefore, we will let $\text{Spt}(k_0, G)$ denote either $\text{Spectra}_{et}(\mathcal{C})$ as in 3.3.4 or $\text{Spt}_{Z/1,\text{mod}}(k_0, G)$. The first observation is that both categories are locally presentable. Therefore, there exists a set $\{G_\alpha \mid \alpha\}$ so that any object $P \in \text{Spt}_{Z/1,\text{mod}}(k_0, G)$ is the filtered colimit of a diagram involving the $G_\alpha$, i.e. $P = \lim G_\alpha$. We will let $\otimes (\oplus)$ denote the monoidal product (sum, respectively) in these categories.

**Lemma 4.1.** Let $f: A \to B$, $f': A' \to B'$ denote two maps in $\text{Spt}(k_0, G)$ that are trivial cofibrations and let $g: A \to Q$, $g': A' \to Q'$ be maps in $\text{Spt}(k_0, G)$ with $Q, Q'$ be among the $\{G_\alpha \mid \alpha\}$ considered above. Assume that $f \otimes \text{id}_Q$ and $f' \otimes \text{id}_{Q'}$ are both trivial cofibrations. Then (i) the induced map $Q \otimes A' \oplus A \otimes Q' \to Q \otimes B' \oplus A \otimes Q'$ is a trivial cofibration and (ii) so is the induced map $Q \otimes B' \oplus A \otimes Q' \to Q \otimes B' \oplus Q' \oplus A \otimes Q' \oplus A \otimes Q'$.

**Proof.** The proof of (i) follows from the commutative diagram where all the squares are co-cartesian:

$$\begin{array}{ccc}
A \otimes A' & \rightarrow & A \otimes Q' \\
Q \otimes A' & \rightarrow & Q \otimes A' \oplus A \otimes Q' \\
Q \otimes B' & \rightarrow & Q \otimes B' \oplus A \otimes Q'
\end{array}$$

$$\begin{array}{ccc}
A \otimes A' & \rightarrow & B \otimes Q' \\
Q \otimes A' \oplus A \otimes Q' & \rightarrow & Q \otimes B' \oplus B \otimes Q' \\
Q \otimes B' \oplus A \otimes Q' & \rightarrow & Q \otimes B' \oplus Q' \oplus A \otimes Q'
\end{array}$$
The left arrow and the top arrow in the bottom right-most square are trivial cofibrations and hence so are the other arrows in the same square. The required map in (i) is the composition of the left arrow and the bottom arrow in the same square, so that it is a trivial cofibration.

(ii) follows from the commutative diagram

\[
\begin{array}{ccc}
A \otimes A' & \rightarrow & B \otimes B' \\
\downarrow & & \downarrow \\
Q \otimes B' \oplus B \otimes Q' & \rightarrow & p \\
& & \downarrow \\
& & C'
\end{array}
\]

The first arrow in the top row is a cofibration, so that so is the first arrow in the bottom row. Therefore, it follows that \( C' \cong C \). Since the first arrow in the top row is also a trivial cofibration, \( C \) and hence \( C' \) is acyclic. Therefore, the first arrow in the bottom row is also a weak-equivalence, thereby proving (ii). \( \square \)

Let \( J' \) denote the set of generating trivial cofibrations chosen as in spectra.construct.2 for both \( \text{Spectra}_\mathbb{A}(C) \) and \( \text{Spt}_{\mathbb{A}}(k_0, G) \).

**Lemma 4.2.** Let \( A_i \rightarrow A_i, \ i = 1, 2 \) be trivial cofibrations in \( \text{Spt}(k_0, G) \) and let \( f_i : A_0^i \rightarrow A^i_i, \ i = 1, 2 \) be given. Assume that \( g : A_0^i \rightarrow A_3 \) is also a trivial cofibration in \( \text{Spt}(k_0, G) \). (i) Then the induced map \( A = A_1^i \oplus A_2^i \rightarrow B = A_1 \oplus A_2 \) and the induced map \( B = A_1^i \oplus A_2^i \rightarrow C = A_1 \oplus A_2 \oplus A_3 \) are trivial cofibrations.

(ii) If \( Q \in \text{Spt}(k_0, G) \) so that \( f_i \otimes id_Q \), for \( i = 1, 2 \) and \( g \otimes id_Q \) are trivial cofibrations, then the obvious induced maps \( A \otimes Q \rightarrow B \otimes Q \) and \( B \otimes Q \rightarrow C \otimes Q \) are trivial cofibrations.

(iii) If \( \text{Spt}(k_0, G) \) denotes \( \text{Spt}_{Z/l, \mathbb{A}}(k_0, G) \) and each \( f_i \) and \( g \) belong to \( Z/l(J'_\text{Sp}) \) then the hypothesis of (ii) holds.

**Proof.** The proof of (i) is identical to the proof of the corresponding statement in the previous lemma. (ii) follows from (i) by replacing the \( A_i \) and \( A'_i \) in (i) with \( A_i \otimes Q \) and \( A'_i \otimes Q \). Then the hypothesis in (ii) ensures that the hypothesis in (i) holds for \( A_i \otimes Q \).

For (iii), it suffices to prove the following: if \( f : A \rightarrow B \) is a trivial cofibration with \( A \) and \( B \) cofibrant, \( Z/l(f) \otimes id_Q \) is a weak-equivalence. Let \( Q^c \rightarrow Q \) denote a cofibrant replacement. Then one obtains the commutative square:

\[
\begin{array}{ccc}
Z/l(A) \otimes Q^c & \rightarrow & Z/l(B) \otimes Q^c \\
\downarrow & & \downarrow \\
Z/l(A) \otimes Q & \rightarrow & Z/l(B) \otimes Q
\end{array}
\]

The top row is a weak-equivalence, since \( Q^c \) is cofibrant. The vertical maps are weak-equivalences because both \( A \) and \( B \) and hence \( Z/l(A) \) and \( Z/l(B) \) are cofibrant. It follows the bottom row is also a weak-equivalence. \( \square \)

In order to show that \( A^1 \)–localization can be carried out to be compatible with ring and module structures on spectra and also with mod-\( l \) completions, it is necessary to enlarge the set \( J'_\text{Sp} \) as follows. We define a sequence \( J'_\text{Sp}(i), \ i \geq 0 \) of increasing sets of trivial cofibrations as follows. When \( \text{Spt}(k_0, G) = \text{Spt}_{Z/l, \mathbb{A}}(k_0, G) \) we let \( J'_\text{Sp}(0) = Z/l(J'_\text{Sp}) \). When \( \text{Spt}(k_0, G) = \text{Spectra}_\mathbb{A}(C) \), we let \( J'_\text{Sp}(0) = J'_\text{Sp} \) chosen as in spectra.construct.2.

Having defined \( J'_\text{Sp}(i) \), for \( 0 \leq i \leq n \), we define

\[
J'_\text{Sp}(n + 1) = \{ Q \otimes B' \oplus A_0 \otimes A', B \otimes Q' \rightarrow (Q \otimes B' \oplus A_0 \otimes A') \oplus A_2 \otimes A', B \otimes B'|A \rightarrow B, A' \rightarrow B' \in J'_\text{Sp}(n), g : A \rightarrow Q, g' : A' \rightarrow Q' \in \text{Spt}(k_0, G), Q, Q' \in \{ G_\alpha | \alpha \} \}
\]

\[
J'_\text{Sp}(n + 1) = J'_\text{Sp}(n) \cup J'_\text{Sp}(n + 1) \text{ together with all finite sums of objects in } J'_\text{Sp}(n + 1).
\]

One may now prove using ascending induction on \( n \) that the domain and co-domain of the maps in \( J'_\text{Sp}(n) \) are finitely presented. We will replace \( J'_\text{Sp} \) with \( \cup_{n \geq 0} J'_\text{Sp}(n) \) and continue to denote this larger set by \( J'_\text{Sp} \).
The following lemma is straightforward to prove and is therefore left to the reader.

**Lemma 4.3.** (i) The domain and co-domain of the maps in each $J_{Sp}^I(n)$ are finitely presented.

(ii) The maps in each $J_{Sp}^I(n)$ are among the trivial cofibrations generated by starting with $J_{Sp}^I$ defined as in 3.3.4 and by iterating the following operations finitely many times: finite coproducts, co-base-change by arbitrary maps and pushout products of the form $f \sqcup g$, where $f$ is already a trivial cofibration generated by starting with $J_{Sp}^I$, as in 3.3.4 and $g = \text{id}_Q$, where $Q$ is among the small generators of $\text{Spt}(k_0, G)$.

4.0.2. The basic construction.

(i) For each $Y \in \text{Spt}(k_0, G)$, we let $S(Y)$ denote the set of all commutative squares

\[
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow_{f} & & \downarrow \\
B & \longrightarrow & 0
\end{array}
\]

where $f \in J_{Sp}^I$. We let $\mathcal{P}(S(Y))_f$ denote the set of all finite subsets of $S(Y)$.

(ii) For each finite subset $T \subseteq S$, we let $P_T(Y) = (\oplus B) \oplus Y$ where the $\oplus B$ and $\oplus A$ are over the finite set $T$.

Then we let $P'(Y) = \lim_{\alpha \in J(Y)} P_T(Y) = (\oplus B) \oplus Y$ where the sum is over all elements of $S(Y)$.

(iii) Let $Y = \lim_{\alpha \in J(Y)} G_\alpha$, where $J(Y)$ is some filtered set. Then we let $P(Y) = \lim_{\alpha \in J(Y)} P'(G_\alpha) = \lim_{\alpha \in J(Y)} \lim_{\beta \in \mathcal{P}(S(G_\alpha))_f} P_T(G_\alpha)$. Observe that then, $P(Y) = (\oplus B) \oplus Y$ where the $\oplus B$ and $\oplus A$ are over the set $S(Y)$.

Now we start with an $E \in \text{Spt}_{Z/l, \text{mot}}(k_0, G)$ and will construct a factorization of the map $Z/l(E) \to *$ into $Z/l(E) \to F(E) \to *$, where the first map is a cofibration (that is an $A^1$-equivalence) followed by an $A^1$-local fibration all in the category $\text{Spt}_{Z/l, \text{mot}}(k_0, G)$. In case $E \in \text{Spectra}_{\text{mot}}(C)$, then we instead construct a factorization of the map $E \to *$ into $E \to F(E) \to *$, where the first map is a cofibration (that is an $A^1$-equivalence) followed by an $A^1$-local fibration all in the category $\text{Spectra}_{\text{mot}}(C)$. Let $F^0(E) = Z/l(E)$ in the first case and $E$ in the second case. The factorization is obtained as in the diagram

\[
F^0(E) \longrightarrow F^1(E) \longrightarrow F^2(E) \longrightarrow \cdots \longrightarrow F^\beta(E) \longrightarrow \cdots
\]

Assume we have constructed the above diagram upto $F^\beta(E)$ all in $\text{Spt}(k_0, G)$ beginning with $F^0(E)$. To continue the construction, we consider maps from families of maps $A \to B$ in $J_{Sp}^I$ to $(F^\beta(E)) \to * = Z/l(*)$. One then lets $F^{\beta+1}(E) = \text{be defined by the pushout (} \oplus B) \oplus F^\beta(E)$ in the category $\text{Spt}(k_0, G)$, i.e. $F^{\beta+1}(E) = P(F^\beta(E))$.

For a limit ordinal $\gamma$, $F^\gamma(E)$ is defined as $\lim_{\beta < \gamma} F^\beta(E)$ with the colimit taken in the same category $\text{Spt}(k_0, G)$. We obtain the required factorization by taking $F(E) = \lim_{\beta < \lambda} F^\beta(E)$ and with the obvious induced maps $Z/l(P) \to F(E)$ and $E \to *$. Here $\lambda$ is a sufficiently large enough ordinal: if the domains of $J_{Sp}^I$ are $k$-small for some ordinal $k$, then $\lambda$ is required to be $k$-filtered. This means $\lambda$ is a limit ordinal and that if $A \subseteq \lambda$ and $|A| \leq \kappa$, then $\sup A < \lambda$.

It is clear from the construction that $F(E) \in \text{Spt}(k_0, G)$, that it is $A^1$-local follows from the observation that the domain of any object of $J_{Sp}^I$ is finitely presented. Henceforth we will denote the above object $F(E)$ as $Z/l(E)$ in the first case and as $R_{A^1}(E)$ in the second case.

**Proposition 4.4.** The functor $E \to \tilde{Z}/l(E) : \text{Spt}_{\text{mot}}(k_0, G) \to \text{Spt}_{Z/l, \text{mot}}(k_0, G)$ is left-adjoint to the forgetful functor $U : \text{Spt}_{Z/l, \text{mot}}(k_0, G) \to \text{Spt}_{\text{mot}}(k_0, G)$.

**Proof.** We will start with a map $G \to U(E)$ in $\text{Spt}_{\text{mot}}(k_0, G)$, where $E \in \text{Spt}_{Z/l, \text{mot}}(k_0, G)$. This corresponds to map $Z/l(G) \to E$ of $Z/l$-module spectra. We proceed to show that this map induces a map $Z/l(G) \to E$ so
that pre-composing with the map $Z/l(G) \to \widetilde{E}$ is the map $Z/l(G) \to \widetilde{E}$. We will show inductively the above map $Z/l(G) \to \widetilde{E}$ extends to a map $F^\beta(G) \to \widetilde{E}$ for all $\beta < \lambda$. Assume that $\beta$ is such an ordinal for which we have extended the given map $Z/l(G) \to \widetilde{E}$ to a map $F^\beta(G) \to \widetilde{E}$. Let $\{A_\alpha \to B_\alpha | \alpha \}$ denote a family of trivial cofibrations belonging to $J_{Z/l,Spt}$ indexed by a small set so that one is provided with a map $\oplus_\alpha A_\alpha \to F^\beta(G)$. Since $\widetilde{E}$ is a fibrant object of $Spt_{Z/l,\text{mot}}(k_0,G)$, and each map $A_\alpha \to B_\alpha$ is a generating trivial cofibration in the same category, one obtains an extension $\oplus_\alpha B_\alpha \to \widetilde{E}$.

Next recall that $F^{\beta+1}(G) = (\oplus_\alpha B_\alpha) \oplus F^\beta(G)$. Therefore, this has an induced map of $Z/l$-module spectra to $\widetilde{E}$. Since $F^\gamma(G)$ for a limit ordinal $\gamma < \lambda$ is defined as $\lim_{\beta \to \gamma} F^\beta(G)$, it follows that we obtain the required extension of the given map to a map $\widetilde{Z/l}(G) \to \widetilde{E}$.

**Proposition 4.5.** Assume the above situation. Then $\widetilde{Z/l}(E)$ is $Z/l$-complete in the following sense. For every map $\phi : A \to B$ in $\text{Spectra}(C)$ with both $A$ and $B$ cofibrant and which induces an isomorphism $H_\ast(A,Z/l) \cong H_\ast(B,Z/l)$ of the homology presheaves with $Z/l$-coefficients, then the induced map $\phi^* : \text{Map}(B,U(\widetilde{Z/l}(E))) \to \text{Map}(A,U(\widetilde{Z/l}(E)))$ is a weak-equivalence of simplicial sets.

**Proof.** This follows readily from the following observations: (i) such a homology isomorphism induces a weak-equivalence $Z/l(A) \to Z/l(B)$ and (ii) $\text{Map}(B,U(\widetilde{Z/l}(E))) \cong \text{Map}(Z/l(B),Z/l(E))$ and $\text{Map}(A,U(\widetilde{Z/l}(E))) \cong \text{Map}(Z/l(A),Z/l(E))$ where the $\text{Map}$ on the left-side (right-side) is taken in the category $Spt_{\text{mot}}(k_0,G)$ ($Spt_{Z/l,\text{mot}}(k_0,G)$, respectively). One may observe that the functor $Z/l( \ )$ preserves cofibrant objects and since $\widetilde{Z/l}(E)$ is a fibrant object in $Spt_{Z/l,\text{mot}}(k_0,G)$, the weak-equivalence $Z/l(A) \to Z/l(B)$ induces the weak-equivalence $\text{Map}(Z/l(A),Z/l(E)) \cong \text{Map}(Z/l(A),\widetilde{Z/l}(E))$. 

4.0.4. Proposition 4.4 above shows that $(\widetilde{Z/l}_n(E)n \geq 1)$ provides a cosimplicial object in $Spt_{\text{mot}}(k_0,G)$ (with $\widetilde{Z/l}(E)$ in degree 0) which is group-like, i.e. each $\widetilde{Z/l}_n(E)$ belongs to $Spt_{Z/l,\text{mot}}(k_0,G)$ and that the structure maps of the cosimplicial object except for $d^n$ are $Z/l$-module maps. Such a cosimplicial object is fibrant in the Reedy model structure on the category of cosimplicial objects of $Spt_{Z/l,\text{mot}}(k_0,G)$: see for example, [B-K, Chapter 10, 4.9 proposition]. Therefore, one may take the homotopy inverse limit of this cosimplicial object just as in [B-K, Chapter X].

In view of the observation, it follows that if $\widetilde{Z/l}_{\leq n}(E)$ denotes the corresponding truncated cosimplicial object, truncated to degrees $\leq n$, one obtains a compatible collection of maps $\{\text{holim} \widetilde{Z/l}_n(E) \to \text{holim} \widetilde{Z/l}_{\leq n}(E)n \geq 1\}$. Moreover the fiber of the map $\text{holim} \widetilde{Z/l}_{\leq n}(E) \to \text{holim} \widetilde{Z/l}_{\leq n-1}(E)$ identifies with $\text{ker}(s^n) \cap \cdots \cap \text{ker}(s^0)$, where $s^i : \widetilde{Z/l}_n(E) \to \widetilde{Z/l}_{n-1}(E)$ is the i-th co-degeneracy.

**Definition 4.6.** ($Z/l$-completions.) First we will extend the set $J_{Sp}$ as above. We will then apply the construction above to define $\widetilde{Z/l}(P)$. One repeatedly applies the functor $\widetilde{Z/l}(\ )$ to an object $E \in \text{Spectra}(C)$, to define the tower $\{\widetilde{Z/l}_n(E)n \}$ in $\text{Spectra}(C)$. We let $\widetilde{Z/l}_\infty(E) = \text{holim}_n\{\widetilde{Z/l}_n(E)n \}$. In contrast, we let $\widetilde{Z/l}_{\infty}(E) = \text{holim}_n\{\widetilde{Z/l}_n(E)n \}$. 

**Proposition 4.7.** (i) Let $\mu : E \otimes E' \to E''$ denote a pairing of spectra in $Spt_{\text{mot}}(k_0,G)$. Then $\mu$ induces a pairing $\widetilde{Z/l}_n(E) \otimes \widetilde{Z/l}_n(E') \to \widetilde{Z/l}_n(E'')$ for all $n \geq 1$ and a pairing $\widetilde{Z/l}_\infty(E) \otimes \widetilde{Z/l}_\infty(E') \to \widetilde{Z/l}_\infty(E'')$.

(ii) Let $\mu : E \otimes E' \to E''$ denote a pairing in $\text{Spectra}_{\ast}(C)$. Then one obtains an induced pairing $R_{\lambda}\mu : R_{\lambda}E \otimes R_{\lambda}E' \to R_{\lambda}E''$.

The above pairings are coherently associative.

**Proof.** Since the proof of the second assertion is entirely similar to that of the first, we will only prove (i). Assume that $\widetilde{Z/l}(E)$, $\widetilde{Z/l}(E')$ and $\widetilde{Z/l}(E'')$ are defined by $\widetilde{Z/l}(E) = \lim_{\pi} F^\beta(E) \ | \ \beta$, $\widetilde{Z/l}(E') = \text{colim} F^\beta(E') \ | \ \beta$ and $\widetilde{Z/l}(E'') = \lim_{\pi} F^\beta(E'') \ | \ \beta$ and that we have already shown the existence of a pairing $F^\beta(E) \otimes F^\beta(E') \to F^\beta(E'')$.
compatible with the given pairing $Z/l(E) \otimes Z/l(E') \to Z/l(E'')$ for a non-limit ordinal $\beta$. Since every object in $\text{Spt}_{\text{mot}}(k_0, \mathbf{G})$ is a filtered colimit of a subset of the finitely presented objects denoted $\{G_\alpha \mid \alpha\}$ as in 4.0.1, we may replace $F^3(E)$ ($F^3(E')$, $F^3(E'')$) by $G_\alpha$ ($G_\beta$, $G_\gamma$, respectively) together with a pairing $G_\alpha \otimes G_\beta \to G_\gamma$.

Recall next that

$$F^{3+1}(G_\alpha) = P^*(G_\alpha) = \colim_{T \in \Psi(S(G_\alpha))} P_T(G_\alpha).$$

Observe that then, $P_T(G_\alpha) = (\oplus B) \oplus G_\alpha$

where the $\oplus B$ (and $\oplus A$) are over the finite subset $T \subseteq S(G_\alpha)$. Similarly

$$F^{3+1}(G_\beta) = P^*(G_\beta) = \colim_{T' \in \Psi(S(G_\beta))} P_{T'}(G_\beta).$$

Observe that then, $P_{T'}(G_\beta) = (\oplus B') \oplus G_\beta$

where the $\oplus B$ (and $\oplus A$) are over the finite subset $T' \subseteq S(G_\beta)$ and

$$F^{3+1}(G_\gamma) = P^*(G_\gamma) = \colim_{T'' \in \Psi(S(G_\gamma))} P_{T''}(G_\gamma).$$

Observe that then, $P_{T''}(G_\gamma) = (\oplus B'') \oplus G_\gamma$

where the $\oplus B$ (and $\oplus A$) are over the finite subset $T'' \subseteq S(G_\gamma)$. Therefore, in order to show that there is an induced pairing $F^{3+1}(E) \otimes F^{3+1}(E') \to F^{3+1}(E'')$, it suffices to show that for each finite subsets $T \subseteq S(G_\alpha)$ and $T' \subseteq S(G_\beta)$ there exists a finite subset $T'' \subseteq S(G_\gamma)$ together with a pairing $P_T(G_\alpha) \otimes P_{T'}(G_\beta) \to P_{T''}(G_\gamma)$.

One may now identify $P_T(G_\alpha) \otimes P_{T'}(G_\beta)$ with $((G_\alpha \otimes B' \oplus A \otimes A') B \otimes G_\beta) \oplus A \otimes A' B \otimes G_\beta$ is a trivial cofibration and (ii) so is the induced map $G_\alpha \otimes B' \oplus A \otimes A' B \otimes G_\beta$ is a trivial cofibration. Moreover these are in $J'_{Sp,Z/l}$. i.e. We obtain the following pushout:

\[
\begin{array}{ccc}
G_\alpha \otimes A' \oplus A \otimes A' & \to & G_\alpha \otimes G_\beta \\
(\alpha \otimes B') \oplus A \otimes A' & \otimes & G_\beta \oplus A \otimes A' B \otimes B' & \to & P_{T''}(G_\alpha \otimes G_\beta)
\end{array}
\]

One may see directly that, since $A \otimes A'$ maps naturally to $G_\alpha \otimes A' \oplus A \otimes A' A \otimes G_\beta$, that there is a natural map $P_T(G_\alpha) \otimes P_{T'}(G_\beta) \to P_{T''}(G_\alpha \otimes G_\beta)$ and the latter maps naturally to $P_{T''}(G_\gamma)$ via the map induced by the given map $G_\alpha \otimes G_\beta \to G_\gamma$. For a limit ordinal $\gamma$, one may take the colimit of the above pairings for $\beta < \gamma$.

We have therefore proven the existence of a pairing $Z/l(E) \otimes Z/l(E') \to Z/l(E) \otimes Z/l(E') \to Z/l(E'')$. We may now repeat the above constructions to obtain the pairing

$$Z/l_n(E) \otimes Z/l_n(E') \to Z/l_n(E) \otimes Z/l_n(E') \otimes Z/l_n(E'')$$

for all $n \geq 0$.

One may view this as a pairing $Z/l_n(E) \otimes Z/l_n(E') \to Z/l_n(E'')$ that is $Z/l$-bilinear for all $n \geq 0$. Next one takes the homotopy inverse limit in the category $\text{Spt}_{\text{mot}}(k_0, \mathbf{G})$: the homotopy inverse limit commutes with products, so that one obtains an induced pairing

$$Z/l_\infty(E) \otimes Z/l_\infty(E') \to Z/l_\infty(E'').$$

\[\square\]

Completions will be most often applied to pro-objects in $\text{Spectra}(\mathcal{C})$. Let $X = \{X_i \mid i \in I\} \in \text{pro} - \text{Spectra}(\mathcal{C})$ denote a pro-object indexed by a small category $I$. Let $E \in \text{Spectra}(\mathcal{C})$. Then we let

\[
\begin{align*}
\text{Hom}_{\text{Sp}}(X, Z/l_\infty(E)) &= \text{holim}_i \text{colim}_j \text{Hom}_{\text{Sp}}(X_i, (Z/l)_{l_i}(E)) \\
\text{Map}(X, Z/l_\infty(E)) &= \text{holim}_i \text{colim}_j \text{Map}(X_i, (Z/l)_{l_i}(E))
\end{align*}
\]

Here $\text{Hom}$ denotes the internal hom in the category $\text{Spectra}(\mathcal{C})$.

4.0.7. Properties of the completion functor. Here we list a sequence of key properties of the completion functor so as to serve as a reference.
(i) The $\mathbb{Z}/l$-completion, $\mathbb{Z}/l_\infty(E)$ is $\mathbb{Z}/l$-complete. i.e. (i) it is fibrant in $\text{Spt}_{\text{mot}}(k_0, G)$ and for every map $\phi : A \to B$ in $\text{Spt}_{\text{mot}}(k_0, G)$ between cofibrant objects which induces an isomorphism $H_n(A, \mathbb{Z}/l) \cong H_n(B, \mathbb{Z}/l)$ of the homology presheaves with $\mathbb{Z}/l$-coefficients, the induced map $\phi^* : \text{Map}(B, \mathbb{Z}/l_\infty(E)) \to \text{Map}(A, \mathbb{Z}/l_\infty(E))$ is a weak-equivalence of simplicial sets.

(ii) If $R \in \text{Spt}_{\text{mot}}(k_0, G)$ is a ring spectrum so are each $\mathbb{Z}/l_n(R)$ and $\mathbb{Z}/l_\infty(R)$. If $R \in \text{Spt}_{\text{mot}}(k_0, G)$ is a ring spectrum and $M \in \text{Spt}_{\text{mot}}(k_0, G)$ is a module spectrum over $R$, then each $\mathbb{Z}/l_n(M)$ is a $\mathbb{Z}/l_n(R)$-module spectrum and $\mathbb{Z}/l_\infty(M)$ is a $\mathbb{Z}/l_\infty(R)$-module spectrum.

(iii) If $R \in \text{Spt}_{\text{mot}}(k_0, G)$ is a ring spectrum, each $\mathbb{Z}/l_n(R)$ is a module spectrum over $\mathbb{Z}/l_\infty(R)$. (This property may be deduced from the discussion in the second paragraph of 4.0.4.)

(iv) For each $E \in \text{Spt}_{\text{mot}}(k_0, G)$, each $\mathbb{Z}/l_n(E)$ is $\mathbb{A}^1$-local and belongs to $\text{Spt}_{\mathbb{Z}/l_\infty}(k_0, G)$.

(v) For $E \in \text{Spt}_{\text{mot}}(k_0, G)$ and let $Z/l(E)$ denote the free $\mathbb{Z}/l$-vector space functor applied to $E$. Then $\mathbb{Z}/l_n(E) = Z/l(U(Z/l_{n-1}(E)))$ defines the $n$-th term of the tower defining the usual Bousfield-Kan $\mathbb{Z}/l$-completion. One obtains a natural map $\{Z/l_n(E) \to Z/l_{n-1}(E)|n\}$ of towers compatible with the induced multiplication if $E$ is a ring spectrum.

In this paper completions play a key role. This is to be expected since even the étale homotopy type of schemes has good properties only after completion away from the residue characteristics. We will adapt the Bousfield-Kan completion (see [B-K]) to our framework as follows.

First recall that the Bousfield-Kan completion is defined with respect to a commutative ring with 1: in our framework, this ring will be always $\mathbb{Z}/l$ for a fixed prime $l$ different from $\text{char}(k_0)$. Therefore, we will work with category $\text{Spt}_{\mathbb{Z}/l_\infty}(k_0, G)$. At our arguments and constructions work also on $\text{Spt}_{\mathbb{Z}/l_\infty}(k_0, G)$, but we do not discuss this explicitly.

The last propositions show that each $Z/l(E)$ is already $\mathbb{A}^1$-local in the étale setting, i.e. with $k_0$ separably closed and where the étale topology is used and $E$ is a constant sheaf. In the motivic framework, i.e. when the topology used is the Nisnevich topology, and/or when $E$ is not constant, one needs to modify the above definitions so that each $Z/l(E)$ is also $\mathbb{A}^1$-local. We proceed to do this presently, but we will digress to develop a bit of the technicalities.

**References**


