Grothendieck-Verdier duality in enriched symmetric monoidal t-categories

Roy Joshua

Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA.

E-mail address: joshua@math.ohio-state.edu

ABSTRACT. In this paper we extend the yoga of Grothendieck's six (derived) functors to as broad a setting as possible. The general frame-work we adopt for our work is that of enriched symmetric monoidal categories which is broad enough to include most or all of the applications. The theory has already found several applications: for example to the theory of character cycles for constructible sheaves with values in K-theory which is discussed in detail in Chapter V. In addition, other potential applications exist, for example, to the theory of derived schemes and motivic derived categories, some of which are surveyed in Chapter VI.

Table of contents

Introduction

One of the most global duality results in mathematics is the Grothendieck-Verdier duality. These are statements in the derived category of suitable sheaves on a topology and incorporate many other duality results: for example Poincaré duality for manifolds has a formulation in this setting using constructible sheaves. The yoga of the (six) derived functors is incorporated into this theory of duality. The original setting for these is either that of constructible sheaves on suitable Grothendieck topologies or coherent sheaves on schemes. However, progress in various fields has necessitated that the basic theory be extended to increasingly more general contexts: for example to the setting of D-modules, coherent sheaves on super-commutative varieties, algebraic geometry over DGAs, simplicial presheaves on general sites etc. The theory of sheaves of modules over differential graded algebras is finding important applications in present-day algebraic geometry: see for example [Kon], [CK1], [CK2] as well as [Voe-1], [K-M]. Moreover, the theory of simplicial presheaves, started in [B-G] over 25 years ago, has been finding ever increasing applications: see the various papers of Simpson, ([Simp-1] through [Simp-3]), Toen (see [Toe-1] and [Toe-2]), Morel and Voevodsky (see [M1] and [M-V]). Recent progress in the theory of motives has led to several conjectures on extending the machinery of Grothendieck-Verdier duality to the motivic setting as well. However, the mere fact that Grothendieck-Verdier duality is formulated in the derived category of abelian sheaves or coherent sheaves on schemes (or algebraic spaces), makes it rather restrictive: it does not apply to generalized cohomology theories, for example to K-theory.

In this monograph we establish a general version of Grothendieck-Verdier duality in a sufficiently broad setting so as to be readily applicable to the above situations as well as others. We discuss one particular application in detail in Chapter V, namely a direct construction of micro-local character cycles in response to a question of P. Schapira. Other applications are discussed briefly at the end of Chapter IV and in Chapter VI. To make our theory applicable to a wide variety of situations, (including that of presheaves of E^{∞}-differential graded modules over a sheaf of E^{∞}-differential graded algebras), we have adopted an axiomatic situation.

The frame-work adopted for our work is that of enriched symmetric monoidal t-categories. Such categories are triangulated categories (to be precise, what we call strongly triangulated categories) with the extra structure of a symmetric monoidal category and a strong t-structure. It is shown that, with minor modifications, this framework is broad enough to include all the above applications: it includes sites provided with sheaves of differential graded algebras, sheaves of differential graded algebras over an operad or presheaves E^{∞}-ring spectra (in the sense of algebraic topology).

Next we give an overview of our work by considering the problem of obtaining a good notion of (Grothendieck-Verdier) duality on ringed sites. Let $\mathfrak{S}$ denote a ringed site, i.e. a site provided with a sheaf, $\mathcal{R}$, of commutative rings with unit. Let $Sh_{\mathcal{R}}(\mathfrak{S})$ denote the category of sheaves of $\mathcal{R}$-modules on $\mathfrak{S}$. This is a symmetric monoidal category under the tensor product of sheaves of $\mathcal{R}$-modules and has $\mathcal{R}$ as a strict unit. In this context it is possible to obtain a left-derived functor: $-\underset{\mathcal{R}}{\overset{L}{\otimes}}- : D_b(Sh_{\mathcal{R}}(\mathfrak{S}))\times D_b(Sh_{\mathcal{R}}(\mathfrak{S})) \to D_b(Sh_{\mathcal{R}}(\mathfrak{S}))$ by finding a resolution of any sheaf of $\mathcal{R}$-modules M by a complex, each term of which is of the form $\underset{U\epsilon \mathcal{C}}{\oplus} j_{U!}j_U^*(\mathcal{R})$, where the sum is over U in the site $\mathfrak{S}$. (Here $D_b(Sh_{\mathcal{R}}(\mathfrak{S}))$ denotes the category of bounded complexes in $Sh_{\mathcal{R}}(\mathfrak{S})$. We may further assume that the site $\mathfrak{S}$ is small, for the time being.)

If we further assume that the site $\mathcal{C}$ has enough points, then we will show it is possible to define $\mathcal{R}\mathcal{H}om$ as the derived functor of the internal $\mathcal{H}om$ in the category $Sh_{\mathcal{R}}(\mathfrak{S})$.

Given any M, $N \varepsilon Sh_{\mathcal{R}}(\mathfrak{S})$, we choose a resolution $P(M)^{\bullet} \to M$ by a complex as before and let $\mathcal{R}\mathcal{H}om(M, N) = Tot\mathcal{H}om(P(M)^{\bullet}, G^{\bullet}N)$ where $G^{\bullet}N$ is the Godement resolution of N and Tot is a total complex. That this defines $\mathcal{R}\mathcal{H}om$ follows from the following observations. First, $\mathcal{H}om(j_{U!}j_U^*(\mathcal{R}), G^{\bullet}N) \cong j_{U*}G^{\bullet}(j_U^*(N))$, which shows that the bi-functor $\mathcal{R}\mathcal{H}om(\quad,\quad)$ preserves distinguished triangles and quasi-isomorphisms in the second argument. To see that the bi-functor $\mathcal{R}\mathcal{H}om(\quad,\quad)$ preserves distinguished triangles and quasi-isomorphisms in the first argument, one needs to use basic properties of the Godement resolution and the fact that $P(M)^{\bullet} \to M$ is a projective resolution of M at each stalk.

We observe that the above framework is also particularly suitable for obtaining a bi-duality theorem. A particularly simple form of this bi-duality is the observation that the obvious map $M \to \mathcal{R}\mathcal{H}om(\mathcal{R}\mathcal{H}om(M, \mathcal{R}), \mathcal{R})$ is a quasi-isomorphism if M is locally free and of finite rank.

One of the observations that started our project is the realization that, in the above example, the category $Sh_{\mathcal{R}}(\mathfrak{S})$ is symmetric monoidal with a strict unit $\mathcal{R}$ and that this fact plays a key role in being able to define $-\overset{L}{\underset{\mathcal{R}}{\otimes}}-$ as well as $\mathcal{R}\mathcal{H}om$. In a sense what we do in the paper is to replace the sheaf of rings $\mathcal{R}$ by a sheaf or presheaf of differential graded objects: a presheaf of ring spectra (or Γ-rings) is a generalization of a presheaf of differential graded algebras.

It has to be noted that there have been several attempts at obtaining a theory of Grothendieck-Verdier-duality. For example, in [Neem], it is shown that one can establish the existence of a functor $Rf^!$ (associated to a map of sites f) which is right adjoint to Rf_*. However, a bi-duality theorem and therefore the full theory of Grothendieck-Verdier duality does not seem to exist in this context. Any bi-duality theorem can hold only for objects that are *finite* in a suitable sense. The notions of being *perfect, pseudo-coherent and of finite tor-dimension* on a ringed site are all various forms of finiteness conditions. (See [SGA]6, Exposé I.) However, one may observe that if $(\mathfrak{S}, \mathcal{R})$ is a ringed site and $j_U : U \to \mathcal{S}$ is an object in the site, the sheaf $j_{U!}j_U^*(\mathcal{R})$ need not be *pseudo-coherent* but clearly is of finite tor dimension. On a general ringed site as above, not every bounded complex is pseudo-coherent, but one can find resolutions of any bounded complex by a complex whose terms are sums of sheaves of the form $j_{U!}j_U^*(\mathcal{R})$. The notions of pseudo-coherence and perfection seem useful only on ringed sites $(\mathfrak{S}, \mathcal{R})$ where every finitely presented sheaf of $\mathcal{R}$-modules has a resolution by a pseudo-coherent complex. Therefore, the appropriate notion of finiteness that one has on sheaves of modules on general ringed sites seems to be that of having finite tor dimension along with finite cohomological dimension and cohomology sheaves of finite presentation (or that are constructible). (It has to be pointed out that the notion of being constructible is limited to the case where $\mathcal{R}$ is a locally constant sheaf on $\mathfrak{S}$.) In case every finitely presented sheaf of $\mathcal{R}$-modules has a resolution by a pseudo-coherent complex, the notion of perfection seems to be the right notion of finiteness.

However, the property of having finite tor dimension is not necessarily preserved by taking sub-quotients and hence not preserved by spectral sequences. Therefore, we adopt the following mechanism for defining such a property in our setting. To simplify our discussion we consider a site $\mathfrak{S}$ provided with a presheaf of *differential graded algebras* $\mathcal{A}$. Moreover, we assume that there exists a canonical filtration on $\mathcal{A}$ whose associated graded terms $Gr(\mathcal{A})$ may be assumed to be a presheaf of graded rings. Therefore, we consider presheaves of modules $\mathcal{M}$ on the site $(\mathfrak{S}, \mathcal{A})$ provided with a filtration so that $Gr(\mathcal{M})$ is a presheaf of modules over the ringed site $(\mathfrak{S}, Gr(\mathcal{A}))$. Now we say $\mathcal{M}$ is of finite tor dimension (constructible) if $Gr(\mathcal{M})$ is of finite tor dimension (is constructible over $Gr(\mathcal{A})$, respectively). We show that this defines a good notion of finiteness. Those used to working with filtered derived categories, may find this approach quite familiar. Another issue that becomes important for us is to be able to work with ease in unbounded derived categories. The notion of homotopy colimits and limits provide adequate substitutes for the notion of total complexes in this setting.

The monograph is divided into six chapters and two appendices. In Chapter I we develop the basic axiomatic framework adopted throughout and in Chapter II we discuss several concrete realizations of this axiomatic set-up. In Chapter III, we establish several spectral sequences that form one of our key-techniques. Chapter IV is devoted to a thorough discussion of Grothendieck-Verdier style duality based on these techniques and in as broad a setting as possible. The results of Chapter IV, sections 1 and 2 hold in great generality: here we define the derived functors Rf_*, $Rf_!^{\#}$, Lf^* and $Rf_{\#}^!$. The stronger results on bi-duality and the remaining formalism of Grothendieck-Verdier duality hold on ringed sites $(\mathfrak{S}, \mathcal{R})$ only under the stronger hypothesis that the sheaf of rings $\mathcal{R}$ is locally constant or for perfect objects. (Perfect objects are defined in Chapter III, Definition (2.11).)

We discuss one application to micro-local character cycles for constructible sheaves in detail in Chapter V and survey some of the remaining applications in Chapter VI and at the end of Chapter IV. Each chapter has its own introduction and the reader may consult these now for a survey of our results. Appendix A shows that the categories of Γ-spaces and symmetric spectra satisfy the axioms of stable closed simplicial model categories while Appendix B discusses some rather well-known relations between simplicial objects, cosimplicial objects and chain complexes in an abelian category.

Acknowledgments. This has been a rather long project for us, especially so, since when we started on this project there was no well defined framework to work with, except that of [Rob]. (See [J-3] which is written in this set-up; in fact the application to micro-local character cycles was first worked out in this setting.) In the meanwhile, the theory of symmetric spectra and smash products for Γ-spaces, and the theory of sheaves of DGAs and modules over them were developed by several mathematicians, which necessitated a thorough revision. Rather than restrict to any of these special cases, we have chosen to work in a very general frame-work: the current and emerging applications seem to indicate that this decision has payed off well. Discussions with many mathematicians have contributed in several ways that may not be readily apparent. These include Spencer Bloch, Michel Brion, Patrick Brosnan, Jean-Luc Brylinski, Zig Fiedorowicz, Eric Friedlander, Mike Hopkins, Amnon Neeman, Pierre Schapira, J. P. Schneiders, Jeff Smith, the late Robert Thomason, Bertrand Toen, Burt Totaro and Rainer Vogt. Finally as pointed out earlier, various problems and conjectures from the theory of motives as well as the work on motivic cohomology by Morel and Voevodsky and the work on simplicial presheaves by Simpson (and his collaborators) have been a source of motivation for us. We also thank the Max Planck Institut and the IHES for generously supporting our work and for hospitality.

CHAPTER I

The basic framework

1. Introduction

The goal of this section is to formulate a framework for Grothendieck-Verdier duality as broad as possible. We begin by considering what are called strongly triangulated categories, which are stronger than triangulated categories. The typical example of this is the category of chain complexes in an *exact category* - see Example 2.6 for more details. The homotopy category and the derived category associated to such categories of chain complexes are both triangulated categories; however they are not closed under finite colimits and limits in general, and hence cannot be strongly triangulated. On the other hand the category of chain complexes in an exact category, though not triangulated, is strongly triangulated. In the rest of this chapter we consider unital monoidal structures and t-structures that are compatible with the strongly triangulated structure. We also need to consider homotopy colimits and limits of diagrams which may be thought of as derived functors of the colimits and limits respectively. We list the relevant axioms for these as well. A category with these structures is called *an enriched monoidal t-category.*

In summary *an enriched monoidal t-category* has three basic structures, namely (i) that of a strongly triangulated category (see below for the definition) which induces the structure of a triangulated category on the associated derived category, (ii) that of a monoidal category and (iii) a strong t-structure: these are required to be compatible in a certain sense. In addition, there are a few extra hypotheses needed to ensure the existence of the derived functors of the colimit and limit functors for small diagrams in such a category.

2. Axioms for strongly triangulated categories

Let $\mathcal{C}$ denote a pointed category. The distinguished *zero* object will be denoted $*$. We say $\mathcal{C}$ is *strongly triangulated* if it satisfies the axioms (STR0) through (STR7.3):

(STR0) $\mathcal{C}$ is closed under all *small* colimits and limits. The sums in the category $\mathcal{C}$ will be denoted $\sqcup$. We further require that $\mathcal{C}$ have a *small* family of generators.

(STR1) There exists an equivalence relation called homotopy on the Hom-sets in the category $\mathcal{C}$. If K, $L \varepsilon \mathcal{C}$, we will let $Hom_{H\mathcal{C}}(K, L)$ denote the set of these equivalence classes of morphisms in $\mathcal{C}$ from K to L. We require that this defines a category called the *homotopy category* and denoted $H\mathcal{C}$. (Observe that we are not requiring this category to be additive.) A map $f : K \to L$ is a *homotopy equivalence*, if there exists a map $g : K \to L$ so that $g \circ f$ and $f \circ g$ are homotopic to the identity. *We will assume that any map that is a homotopy equivalence is a quasi-isomorphism (defined in (STR3) below).* We say, a diagram commutes upto homotopy, if the appropriate compositions of the maps get identified under the above equivalence relation.

(STR2) $\mathcal{C}$ is provided with a collection of diagrams $A' \xrightarrow{i} A \xrightarrow{j} A'' \rightarrow TA'$ called *strong triangles* (often called triangles) and a translation functor $T : \mathcal{C} \rightarrow \mathcal{C}$ satisfying the properties (STR3) through (STR5):

(STR3) There exists a (covariant) cohomology functor $\{\mathcal{H}^n|n\} : \mathcal{C} \rightarrow$(an abelian tensor category $\mathbf{A}$) that sends triangles to long exact sequences. We will say a map $f : X \rightarrow Y$ in $\mathcal{C}$ (or $H\mathcal{C}$) is a *quasi-isomorphism* if $\mathcal{H}^n(f)$ is an isomorphism for all n. Then the class of maps in $H\mathcal{C}$ that are quasi-isomorphisms *admits a calculus of left fractions* and *a calculus of right fractions.* $D(\mathcal{C})$ will denote the localization of $H\mathcal{C}$ by inverting maps that are quasi-isomorphisms. We will also require that $\mathbf{A}$ is closed under all small limits and colimits, that filtered colimits in $\mathbf{A}$ are exact, and that each $\mathcal{H}^n$ commute with filtered colimits, with finite sums and products.

(STR4) $D(\mathcal{C})$ is a triangulated category. Let $F : \mathcal{C} \rightarrow D(\mathcal{C})$ denote the functor that is the identity on objects and sends a map f to its class in $D(\mathcal{C})$. Then the distinguished triangles in $D(\mathcal{C})$ are precisely the images of the triangles by F and the functor T in $\mathcal{C}$ is sent to the translation functor in $D(\mathcal{C})$. Moreover F has the following universal property:

(STR5) if $F' : \mathcal{C} \rightarrow \mathcal{D}$ is any functor to a triangulated category sending the triangles to the distinguished triangles, the functor T to the translation functor of $\mathcal{D}$, and quasi-isomorphisms to isomorphisms, there exists a unique functor $F'' : D(\mathcal{C}) \rightarrow \mathcal{D}$ of triangulated categories so that $F' = F'' \circ F$.

We will also require the following :

(STR6) There is given a collection of mono-morphisms in $\mathcal{C}$ called *admissible monomorphisms* which are stable under co-base extension, compositions and retracts so that if $\alpha : X \rightarrow Y$ is an admissible monomorphism in $\mathcal{C}$, $Cone(\alpha)$ (defined below) is quasi-isomorphic to $Coker(\alpha)$. We further require that admissible monomorphisms are stable under all (small) inverse limits, filtered colimits and homotopy colimits. (See 4.1.1 for their definition.) There is given also a collection of epi-morphisms in $\mathcal{C}$ called *admissible epimorphisms* that are stable under base extension, compositions, all (small) colimits and all homotopy inverse limits. If $\beta : Y \rightarrow Z$ is an admissible epimorphism, then $T(ker(\beta))$ is quasi-isomorphic to $Cone(\beta)$. Moreover the obvious map $ker(\beta) \rightarrow Y$ is an admissible monomorphism. All isomorphisms are both admissible mono-morphisms and admissible epi-morphisms. Objects X for which the obvious map $* \rightarrow X$ ($X \rightarrow *$) is an admissible monomorphism (admissible epimorphism, respectively) will be called *mono-objects* (*epi-objects*, respectively). We assume there exist functors $M : \mathcal{C} \rightarrow \mathcal{C}$ ($E : \mathcal{C} \rightarrow \mathcal{C}$) so that for each object X, there is given a natural quasi-isomorphism $M(X) \rightarrow X$ ($X \rightarrow E(X)$) with $M(X)$ a mono-object ($E(X)$ an epi-object, respectively). In addition we require that if $f : X \rightarrow Y$ is given with$X$ mono (Y epi), the map f factors as $X \rightarrow M(Y) \rightarrow Y$ ($X \rightarrow E(X) \rightarrow Y$, respectively).

Remark. Observe as a consequence of the axioms (STR7.1) (see below) and (STR6), that, if $X \rightarrow^{\alpha} Y$ is an admissible monomorphism with both X and Y mono-objects, $X \xrightarrow{\alpha} Y \rightarrow Coker(\alpha) \rightarrow TX$ is a strong triangle. Similarly the axiom (STR7.2)(see below) and (STR6), imply that if $Y \xrightarrow{\beta} Z$ is an admissible epimorphism with both of them epi-objects, $ker(\beta) \rightarrow Y \xrightarrow{\beta} Z \rightarrow T(ker(\beta))$ is a strong triangle. If one considers the category of complexes of presheaves in any abelian category, both the functors e and m may be taken to be the identity. (See, for example, Chapter II, section 3.) These functors become non-trivial, however, when $\mathcal{C} =$ a category of presheaves that has the structure of a closed model category - see Chapter II, section 4. See also the remark 2.5, below.

(STR7.1) *Existence of canonical cylinder objects.* Let $A \varepsilon \mathcal{C}$ and let $\nabla : A \sqcup A \to A$ be the obvious map. A *cylinder object* for A is an object $A \times I \varepsilon \mathcal{C}$ provided with an admissible mono-morphism $d_0 \oplus d_1 : A \sqcup A \to A \times I$ and a map $s : A \times I \to A$ such that the composition $s \circ (d_0 \oplus d_1) = \nabla$. We require s to be a quasi-isomorphism and that $A \mapsto A \times I$ is natural in A, preserves admissible monomorphisms and commutes with all small limits as well as filtered colimits. Furthermore we require the following conditions on a cylinder object.

(i) Let $f : A \to B$ denote a map in $\mathcal{C}$. Then let $Cyl(f) = A \times I \underset{A}{\sqcup} B$ where the map $A \to A \times I$ is d_0 and $A \to B$ is the given map f. Let $r : Cyl(f) \to B$ denote the map defined by s on $A \times I$, by f on A and by the identity on B. Then r is a homotopy-equivalence with inverse given by the obvious map $i : B \to Cyl(f)$. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"] \\ B \arrow[r, "g'"] & R \end{tikzcd}$$

in $H\mathcal{C}$, there exists a cylinder object $A \times I$ in $\mathcal{C}$ so that if $P = Cyl(f) \underset{A}{\sqcup} C$ (with the map $A \to Cyl(f)$ induced by $d_1 : A \to A \times I$ and the map $A \to C$ the given map g), there exists a unique map $P \to R$ in $H\mathcal{C}$ making the diagram

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[d, "f"'] & C \arrow[d] \arrow[ddr] & \\ B \arrow[r] \arrow[drr] & P \arrow[dr, "f'"] & \\ & & R \end{tikzcd}$$

commute in $H\mathcal{C}$. We call P the *homotopy pushout* of the two maps f and g.

(ii) It follows from the axioms in (STR6) that the map $d_1 : M(A) \to Cyl(M(f))$ is now an admissible mono-morphism. We let $Cone(M(f)) = Coker(d_1 : M(A) \to Cyl(M(f)))$. Now we also require that there exist a map $Cone(M(f)) \to TM(A)$, natural in f so that $M(A) \xrightarrow{d_1} Cyl(M(f)) \to Cone(f) \to TM(A)$ is a triangle. (Observe that this triangle corresponds to the distinguished triangle $A \to B \to Cone(f) \to TA$ in the derived category $D(\mathcal{C})$.)

(iii) We also require that if

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is a commutative square with $A \to A'$ and $B \to B'$ admissible monomorphisms, the induced map $Cyl(f) \to Cyl(f')$ is also an admissible monomorphism.

REMARK 2.1. Since $A \mapsto A \times I$ is natural in A, one may observe that $f \mapsto Cyl(f)$ is natural in f.

(STR7.2) *Existence of co-cylinder objects.* If $A\varepsilon\mathcal{C}$, let $\Delta : A \to A\times A$ denote the diagonal map. A co-cylinder object for A is an object $A^I\varepsilon\mathcal{C}$ with an admissible epi-morphisms $d^0 \times d^1 : A^I \to A \times A$, and a map $s : A \to A^I$ so that the composition $(d^0 \times d^1) \circ s = \Delta$. The map s is required to be a quasi-isomorphism. We also require that $A \to A^I$ is natural, commutes with filtered colimits and small limits while preserving admissible monomorphisms and epimorphisms. Furthermore we require the following conditions on a co-cylinder object.

(i) Let $f : A \to B$ denote a map in $\mathcal{C}$. Let $Cocyl(f) = B^I \underset{B}{\times} A$ where the map $A^I \to A$ is d_0 and $A \to B$ is the given map f. Let $r : A \to Cocyl(f)$ denote the map defined by $s \underset{f}{\times} id_A$. Then r is a homotopy-equivalence with inverse given by the obvious map $p : Cocyl(f) \to A$. Finally given a commutative diagram

$$\begin{tikzcd} R \arrow[r, "f'"'] \arrow[d, "g'"'] & C \arrow[d, "g"'] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in $H\mathcal{C}$, there exists a cocylinder object B^I in $\mathcal{C}$ so that if $P = Cocyl(f)\underset{B}{\times}C$ (with the map $Cocyl(f) \to B$ induced by $d_1 : B^I \to B$ and the map $C \to B$ the given map g) there exists a unique map $R \to P$ making the diagram

$$\begin{tikzcd} R \arrow[dr] \arrow[drr] \arrow[ddr, "g'"'] & & \\ & P \arrow[r, "f'"'] \arrow[d] & C \arrow[d, "g"'] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

commute in $H\mathcal{C}$. We call P the *homotopy pull-back* of the two maps f and g.

(ii) It follows once again from the axioms in (STR6) that the map $d^1 : Cocyl(E(f)) \to E(B)$ is now an admissible epi-morphism. We let $fib_h(E(f)) = ker(d^1 : Cocyl(E(f)) \to E(B))$ and call it the homotopy fiber of f. Now we also require that there exist a map $E(B) \to Tfib_h(f)$, natural in f so that $fib_h(E(f)) \to Cocyl(E(f)) \to E(B) \to Tfib_h(E(f))$ is a strong triangle.

(iii) Finally we require that if

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is a commutative square with $A \to A'$ and $B \to B'$ admissible monomorphisms (epimorphisms), the induced map $Cocy(f) \to Cocyl(f')$ is also an admissible monomorphism (epimorphism, respectively).

REMARK 2.2. Observe that $f \mapsto Cocyl(f)$ is also natural in f.

(STR7.3) Let $f : A \to B$ denote a map in $\mathcal{C}$. Let $i : fib_h(EM(f)) = d_1^{-1}(*) \to E(M(A))$ denote the composition of the obvious map $fib_h(EM(f)) \to Cocyl(EM(f))$ and

$p : Cocyl(EM(f)) \to EM(A)$. Then there exists a map $Cone(i) \to EM(B)$ natural in f which is a quasi-isomorphism.

DEFINITION 2.3. Let $X\varepsilon\mathcal{C}$ and let $p : EM(X) \to *$ ($i : * \to EM(X)$) denote the obvious maps to (from, respectively) the zero object $*$ of $\mathcal{C}$. Then we define $\Sigma X = Cone(p)$, $\Omega X = fib_h(i)$.

PROPOSITION 2.4. *Let $X\varepsilon\mathcal{C}$. Then there exists a natural quasi-isomorphism $X \simeq \Omega\Sigma X \simeq \Sigma\Omega X$. If $\{X_i | i\varepsilon I\}$ is a finite collection objects of $\mathcal{C}$, the natural map $\sqcup_i X_i \to \Pi_i X_i$ is a quasi-isomorphism.*

PROOF. Take f in (STR7.3) to be the map $i' : * \to \Sigma X$. Then (STR7.3) implies that there exists a natural quasi-isomorphism $Cone(\Omega\Sigma X \to *) \xrightarrow{\simeq} \Sigma X$. It follows that one obtains a long-exact sequence:

$$\cdots \longrightarrow \mathcal{H}^n(\Omega\Sigma X) \longrightarrow \mathcal{H}^n(*) \longrightarrow \mathcal{H}^n(\Sigma X) \longrightarrow \cdots$$

Since $X \to * \to \Sigma X \to *$ is a strong triangle, one also gets a similar long exact sequence involving the cohomology of X, $*$ and ΣX. A comparison of these two long exact sequences shows that X and $\Omega\Sigma X$ are naturally quasi-isomorphic. The quasi-isomorphism $\Sigma\Omega X \simeq X$ is obtained similarly. The last assertion follows from the hypothesis in (STR3) that the functor $\mathcal{H}^*$ commute with finite sums and products. □

2.0.1. **Convention**. *Apart from this section, we will routinely omit the functors m and e in forming the cylinder or cocylinder objects; we hope this will keep our notations simpler throughout.*

Axioms on cofibrant and fibrant objects

Now we will further assume the existence of full sub-categories of $\mathcal{C}$ called *the sub-category of cofibrant objects* (denoted $\mathcal{C}_{cf}$) and *the sub-category of fibrant objects* (denoted $\mathcal{C}_f$) with the following properties:

(STR8.1) there is given a functor $Q : \mathcal{C} \to \mathcal{C}_f$, along with a natural transformation $id \to Q$ so that the map $X \to Q(X)$ is a quasi-isomorphism for all $X\varepsilon\mathcal{C}$. Moreover, we require that the sub-category $\mathcal{C}_f$ be stable by the functor Q and that the functor Q preserves admissible monomorphisms and filtered colimits.

(STR8.2) For every object $X\varepsilon\mathcal{C}$, there exists a map $C(X) \xrightarrow{\simeq} X$ that is a quasi-isomorphism, with $C(X)\varepsilon\mathcal{C}_{cf}$

(STR8.3) For each $P\varepsilon\mathcal{C}_{cf}$ and $K\varepsilon\mathcal{C}$, the natural map $Hom_{HC}(P, QK) \to Hom_{DC}(P, K)$ is an isomorphism.

(STR8.4) If X is cofibrant (fibrant) the obvious map $* \to X$ is an admissible monomorphism ($X \to *$ is an admissible epimorphism, respectively).

REMARK 2.5. Observe as a consequence, that the condition $* \to X$ being an admissible monomorphism is assumed to be weaker than X being cofibrant. If $\mathcal{C} = Presh =$ a category of presheaves on a site that forms a stable simplicial model category as in Chapter II, Theorem 4.10, the condition $* \to X$ is an admissible monomorphism corresponds to requiring the stalks of X be cofibrant whereas X being cofibrant corresponds to X being cofibrant in the given model structure of presheaves. Similarly the condition that $X \to *$ is an admissible epimorphism corresponds to requiring the stalks of X to be fibrant, whereas X being fibrant corresponds to X being fibrant in the given model structure of presheaves. The last could be much stronger than the first.

EXAMPLES 2.6. (i) Let $\mathcal{C}$ denote the category of all complexes in an *abelian* category. This satisfies the axioms (STR0) through (STR5) with $T = [1]$. The axiom (STR1) ((STR3)) is satisfied with the homotopy being chain-homotopy (the cohomology functor being the usual one sending a complex to its cohomology objects, respectively). The triangles are the diagrams $A' \to A \to A'' \to A'[1]$ which are isomorphic in the homotopy category to mapping cone sequences which are defined as usual. One may take the admissible monomorphisms (epimorphisms) in (STR6) to be the maps of complexes that are degree-wise split monomorphisms (epimorphisms, respectively). Moreover the axioms (STR7.1) through (STR7.3) are satisfied where the cylinder objects and co-cylinder objects may be defined as in ([Iver] p. 24 or [T-T] (1.1.2). (For the sake of completeness we will presently recall these definitions. Let $g : A \to G$ denote a map. Then $Cyl(g)$ is the complex defined by $Cyl(g)^n = A^n \oplus A^{n+1} \oplus G^n$ with the differential defined by $d(a^n, a^{n+1}, x^n) = (d(a^n) + a^{n+1}, -d(a^{n+1}), d(x^n) - g(a^{n+1}))$. $Cocyl(g)$ is the complex defined by $Cocyl(g)^n = A^n \oplus A^{n-1} \oplus G^n$ with the differential defined by $d(a^n, a^{n-1}, y^n) = (d(a^n), -d(a^{n-1}) + a^n - g(y^n), d(y^n))$.) The remaining axioms need not be satisfied in general.

(ii) Let $\mathcal{C}$ denote the category of (bounded below) chain-complexes in an *exact category* $\mathcal{E}$ that is also *closed under finite limits and colimits.* Assume further that, for each morphism $f : K \to L$ in $\mathcal{C}$, the sequence $0 \to ker(f) \to K \to Coim(f) \to 0$ is exact. (Observe that, the existence of finite colimits and limits show that both $ker(f)$ and $Coim(f)$ exist in $\mathcal{C}$.) A typical example of this is the category of all filtered objects in an abelian category provided with an ascending filtration. Let $T = [1]$. Let the triangles denote the collection of diagrams $A \to B \to C \to TA$ in $\mathcal{C}$ that are isomorphic to mapping-cone-sequences in the homotopy category, which may be defined as in (i). (i.e. diagrams of the form : $A \xrightarrow{u} B \to Cone(u) \to A[1]$). Let a map $u : K \to L$ be called an admissible monomorphism if each of the maps $u^n : K^n \to L^n$ is an admissible mono-morphism in the exact category $\mathcal{E}$; admissible epimorphisms may be defined similarly.

PROPOSITION 2.7. *Assume the situation in 2.6(ii). Then $\mathcal{C}$ satisfies all the axioms (STR0) through (STR7.3) except possibly for the existence of arbitrary small colimits and limits.*

PROOF. Clearly the homotopy category is additive and a triangulated category. Let $h : \mathcal{E} \to \mathbf{A}$ denote a fully-faithful imbedding of the exact category into an abelian category. (See [Qu] section 2.) Then one defines a complex K to be *acyclic* if $h(K)$ is acyclic as a complex in the abelian category $\mathbf{A}$. It is shown in ([Lau] p. 158) that this is equivalent to the map $d^{n-1} : K^{n-1} \to ker(d^n)$ being an admissible epimorphism for all n. Now one may define a map $f : K \to L$ to be a quasi-isomorphism if $Cone(f)$ is acyclic. It is shown in ([Lau] p. 159) that the class of complexes that are acyclic form a null system in the sense of ([K-S] p. 43) and hence that the class of maps that are quasi-isomorphisms admits a calculus of left and right fractions. One may define the natural t-structure on $\mathcal{C}$ as in [Hu] p. 11 or [Lau] p. 160. This defines a cohomology functor $\mathcal{H}$ on $\mathcal{C}$ taking values in the heart of the derived category.

$$\mathcal{H}^n(K) = (... \to 0 \to Coim(d^{n-1}) \to ker(d^n) \to 0 \to ...)$$

Then $\mathcal{H}^n(K) = 0$ if and only if the above map $Coim(d^{n-1}) \to ker(d^n)$ is an isomorphism. The admissible epimorphism $K^{n-1} \to Coim(d^{n-1})$ implies $\mathcal{H}^n(K) = 0$ if and only if K is acyclic in degree n. In particular, it follows that K is acyclic if and only if $\mathcal{H}^n(K) = 0$ for all n. It follows that we obtain the axioms (STR0) through (STR5). The mapping cylinders and co-cylinders may be defined as in the case of complexes in an abelian category considered in (2.5)(i). One may readily verify that these (as well as the mapping cones and the homotopy fibers as in (STR7.1) and (STR7.2)) commute with the imbedding h. The map $h(Cone(i)) \rightarrow h(B)$ exists and is a quasi-isomorphism in the setting of (STR7.3); therefore one obtains the map $Cone(i) \rightarrow B$ which is also a quasi-isomorphism. To verify (STR6) observe that if $L' \overset{u}{\rightarrow} L$ is an admissible mono-morphism, then $h(Cone(u))$ is quasi-isomorphic to $h(L/L')$. One may similarly verify the hypothesis on admissible epimorphisms. □

2.1. To obtain another example of a different flavor, let $\mathcal{C}$ denote the category of all (simplicial) spectra as in [B-F]. (See Appendix A, section 2 for details.) Let T denote the suspension functor. Given a map $u : A \rightarrow B$, let $Cone(u)$ denote the mapping cone of u. One has a well-defined unstable (or strict) homotopy category defined in the usual manner - see [Qu-1]. We will denote this $H\mathcal{C}$. (Observe that this category is not additive, since we are considering the unstable situation.) Then we let the triangles be the diagrams $A \overset{u}{\rightarrow} B \overset{v}{\rightarrow} C \rightarrow TA$ that are isomorphic in $H\mathcal{C}$ to mapping cone-sequences. The fibrant (co-fibrant) objects in this category are the strictly fibrant spectra (the strictly cofibrant spectra, respectively) in the sense of [B-F]. Let $Q^{st} : \mathcal{C} \rightarrow \mathcal{C}$ denote a functor that converts a spectrum into a fibrant spectrum as in Appendix A. We define the stable homotopy groups of a spectrum K, by $\pi_n(K) = Hom_{H\mathcal{C}}(\Sigma^n \mathcal{S}, Q^{st}K)$, where $\mathcal{S}$ denotes the sphere spectrum, $\Sigma^n \mathcal{S}$ is its n-fold suspension. A map $f : K \rightarrow L$ of spectra is a quasi-isomorphism if it induces an isomorphism on all the stable homotopy groups. (Thus the cohomology functor $\mathcal{H}^n$ is given by the stable homotopy group π_{-n}.) Then one defines the derived category associated to $\mathcal{C}$ (denoted $D(\mathcal{C})$) to be the localization of $H\mathcal{C}$ by inverting maps that are quasi-isomorphisms. This is an additive category and is commonly called the stable homotopy category. Moreover, it is a triangulated category when one defines the distinguished triangles to be the ones that are isomorphic in $D(\mathcal{C})$ to mapping cone sequences. One thus obtains all the axioms through (STR5). One defines the cylinder and co-cylinder objects the usual manner: this readily shows (STR7.1) through (STR7.2) are satisfied. The axiom (STR7.3) is satisfied since we are working in the *stable* homotopy category. In (STR6) one takes the admissible mono-morphisms to be *strict cofibrations* and admissible epi-morphisms to be *strict fibrations* in the sense [B-F]. Moreover the axioms (STR8.1) through (STR8.4) are also satisfied with Q in (STR8.1) identified with the functor Q^{st}.

2.2. One may consider in a similar manner the category of all Γ-spaces, or the category of symmetric spectra. (We skip the details here. One may consult Appendix A for more details in this direction.)

3. Axioms on the monoidal structure

(M0) Next we assume $\mathcal{C}$ also has a *unital* monoidal structure, the operation being denoted $\otimes$, which we will assume, *commutes with all colimits in both arguments.* An object $M \varepsilon \mathcal{C}$ is *flat* if $M \otimes K$ is acyclic for all acyclic objects $K \varepsilon \mathcal{C}$. (An object K in $\mathcal{C}$ is *acyclic* if it is quasi-isomorphic to the zero-object $*$.) Moreover, we require that every cofibrant object is flat. In addition, we require that $* \otimes M = * = M \otimes *$ for any $M \varepsilon \mathcal{C}$.

We will further assume there exists a *small* full sub-category $\mathfrak{F}$ of flat objects such that the following hold:

(M1) for every object $M \varepsilon \mathcal{C}$, there exists an object $P(M) \varepsilon \mathfrak{F}$ and a quasi-isomorphism $P(M) \overset{\simeq}{\rightarrow} M$.

REMARK 3.1. The hypothesis that every cofibrant object is flat shows (STR8.2) implies (M1).

Observe as a consequence of the hypotheses in (ST0) on the existence of a small family of generators and the Special Adjoint Functor theorem (see [Mac] p.125) the following: let m, $K \varepsilon \mathcal{C}$ be fixed. Now the functor $M \to M \otimes -$ has a *right adjoint* which will be denoted $\mathcal{H}om(-, K)$. If $K \overset{\simeq}{\to} K'$ ($M' \overset{\simeq}{\to} M$) is a quasi-isomorphism between fibrant objects (objects that are mono, respectively), $\mathcal{H}om(M, K) \simeq \mathcal{H}om(M', K')$. If M is an object that is mono, $\mathcal{H}om(M, \quad)$ preserves triangles between objects that are fibrant; if K is a fibrant object, $\mathcal{H}om(\quad, K)$ preserves triangles between objects that are mono.

(M2) If $F \varepsilon \mathfrak{F}$, the functors $F \otimes -$ and $- \otimes F$ send triangles in $\mathcal{C}$ to triangles and preserve admissible monomorphisms.

Let $\mathcal{S} \varepsilon \mathcal{C}$ denote the unit for the operation $\otimes$. We require the following additional hypotheses on $\mathcal{S}$.

(M3) $\mathcal{S}$ is a *cofibrant* object in $\mathcal{C}$.

(M4.0)There exists a bi-functor $- \otimes - :$ (pointed simplicial sets) $\times \mathcal{C} \to \mathcal{C}$ *commuting with colimits in the second argument and satisfying the following* properties.

(M4.1) Let K denote a fixed pointed simplicial set. Now the functor $Y \to K \otimes Y$, $\mathcal{C} \to \mathcal{C}$ has a right adjoint which will be denoted Y^K. The functor $K \to K \otimes Y$, (pointed simplicial sets) $\to \mathcal{C}$ has a right adjoint, which will be denoted $Map(Y, .)_K$. (Observe that $Map(Y, .)_{\Delta[n]_+} = Map(Y, .)_n$.)

(M4.2) If K is a pointed simplicial set and X, $Y \varepsilon \mathcal{C}$, there exist an isomorphism $X \otimes (K \otimes Y) \cong K \otimes (X \otimes Y)$ natural in K, X and Y. (The naturality implies that if $\alpha : K \to L$ is a map of pointed simplicial sets, then $id_X \otimes (\alpha \otimes id_Y) \cong \alpha \otimes (id_X \otimes id_Y)$.)

(M4.3) If $K' \to K \to K/K' \to \Sigma K'$ is a cofibration sequence of pointed simplicial sets (i.e. the map $K' \to K$ is a mono-morphism) and $X \varepsilon \mathcal{C}$ is such that $* \to X$ ($X \to *$) is an admissible monomorphism (epimorphism, respectively), the induced diagram $K' \otimes X \to K \otimes X \to K/K' \otimes X \to \Sigma K' \otimes X$ ($X^{\Sigma K'} \to X^{K/K'} \to X^K \to X^{K'}$) is a strong triangle in $\mathcal{C}$. Moreover, the induced map $K' \otimes X \to K \otimes X$ ($X^K \to X^{K'}$) is an admissible monomorphism (epimorphism, respectively). If $Y' \overset{\simeq}{\to} Y$ ($Z' \overset{\simeq}{\to} Z$) is a quasi-isomorphism between cofibrant objects (fibrant objects, respectively), the induced map $Map(Y, Z') \to Map(Y', Z)$ is a weak-equivalence of pointed simplicial sets. If Y is a cofibrant object, $Map(Y, \quad)$ sends triangles between fibrant objects to fibration sequences of simplicial sets. If Z is a fibrant object, $Map(\quad, Z)$ sends triangles between cofibrant objects to fibration sequences of simplicial sets.

(M4.4) If K is a pointed simplicial set and $X \overset{f}{\to} Y$ is a *quasi-isomorphism* in $\mathcal{C}$, then the induced maps $id_K \otimes f : K \otimes X \to K \otimes Y$ and $f^{id} : X^K \to Y^K$ are also quasi-isomorphisms.

(M4.5) If $X' \to X \to X'' \to TX$ is a strong triangle in $\mathcal{C}$ and K is a pointed simplicial set, then the induced diagrams $K \otimes X' \to K \otimes X \to K \otimes X'' \to K \otimes TX$ and ${X'}^K \to X^K \to {X''}^K \to TX^K$ are also triangles in $\mathcal{C}$.

(M4.6) We also require that the functors $X \to X \times I$ and $X \to X^I$ are compatible with the given tensor structure in the following manner: there exists natural isomorphisms $X \otimes (Y \times I) \to (X \otimes Y) \times I$ and $(X \times I) \otimes Y \to (X \otimes Y) \times I$ and similarly $\mathcal{H}om(Y, X^I) \cong \mathcal{H}om(Y \times I, X) \cong \mathcal{H}om(Y, X)^I$.

(M5) *Compatibility of the functors, m, e and Q with the tensor structure.* Given objects $K_i \varepsilon \mathcal{C}$, $i = 1, ..., n$, there exists a map $\otimes_{i=1}^{i=n} Q(K_i) \rightarrow Q(\otimes_{i=1}^{i=n} K_i)$, *natural* in K_i. The same holds with the functor Q replaced by m (e, respectively).

REMARK 3.2. The hypotheses in (M4.0) through (M4.6) imply the axioms in (STR7.1) and (STR7.2) provided for every object $X \varepsilon \mathcal{C}$, the map $* \rightarrow X$ ($X \rightarrow *$) is an admissible monomorphism (epimorphism, respectively). To see this, observe that now one may define the canonical cylinder (cocylinder) object $A \times I$ to be $\Delta[1]_+ \otimes A$ ($A^I = A^{\Delta[1]_+}$, respectively).

PROPOSITION 3.3. *Assume that $\mathcal{C}$ is a category satisfying the axioms (STR0) through (STR8.4) provided with a bi-functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ satisfying the hypotheses (M1) and (M2). Now the bi-functor $\otimes$ induces a derived functor $\overset{L}{\otimes} : D(\mathcal{C}) \times D(\mathcal{C}) \rightarrow D(\mathcal{C})$ that sends distinguished triangles in either argument to distinguished triangles. $D(\mathcal{C})$ is a monoidal category with respect to $\overset{L}{\otimes}$. $\mathcal{S}$ is a unit for this monoidal structure on $D(\mathcal{C})$.*

PROOF. Let M, $N \varepsilon \mathcal{C}$ and let $P(M) \underset{\simeq}{\overset{\epsilon_M}{\rightarrow}} M$, $P(N) \underset{\simeq}{\overset{\epsilon_N}{\rightarrow}} N$, $P(N)' \underset{\simeq}{\overset{\epsilon'_N}{\rightarrow}} N$ denote flat objects in $\mathcal{C}$ chosen as in (M1). Now the induced map $d^1 : Cocyl(\epsilon'_N) \rightarrow N$ is an admissible epimorphism. Let $Q = Cocyl(\epsilon'_N) \underset{N}{\times} P(N)$; since $fib_h(\epsilon'_N) = ker(d^1)$ is acyclic, it follows that the induced maps $Q \rightarrow Cocyl(\epsilon'_N)$ and $Q \rightarrow P(N)$ are quasi-isomorphisms. Now apply (M1) to find a $P(N)'' \overset{\simeq}{\rightarrow} Q$ with $P(N)'' \varepsilon \mathfrak{F}$. It follows that we may assume without loss of generality that there exists a map $P(N)' \rightarrow^{\alpha} P(N)$ in $\mathcal{C}$ making the square

$$\begin{array}{ccc} P(N)' & \xrightarrow{\simeq} & N \\ \downarrow & & \downarrow{\scriptstyle id} \\ P(N) & \xrightarrow{\simeq} & N \end{array}$$

commute. Now we will show that the natural maps $P(M) \otimes P(N)' \rightarrow M \otimes P(N)'$ and $P(M) \otimes P(N) \rightarrow M \otimes P(N)$ are quasi-isomorphisms. To see this let $\beta : P(M) \rightarrow M$ denote the given map and let $Cone(\beta)$ be its cone. Since $P(N)'$ and $P(N)$ are flat, the diagrams:

$P(M) \otimes P(N)' \overset{\beta \otimes id}{\rightarrow} M \otimes P(N)' \rightarrow Cone(\beta) \otimes P(N)'$ and

$P(M) \otimes P(N) \overset{\beta \otimes id}{\rightarrow} M \otimes P(N) \rightarrow Cone(\beta) \otimes P(N)$

are triangles. Since $Cone(\beta)$ is acyclic and $P(N)'$, $P(N)$ are flat, the last terms are also acyclic showing the first maps are quasi-isomorphisms.

Now we will show that the induced map $P(M) \otimes P(N)' \overset{id \otimes \alpha}{\rightarrow} P(M) \otimes P(N)$ is also a quasi-isomorphism. To see this let $Cone(\alpha)$ denote the cone of α. Since $P(M)$ is flat, by (M2), the diagram $P(M) \otimes P(N)' \overset{id \otimes \alpha}{\rightarrow} P(M) \otimes P(N) \rightarrow P(M) \otimes Cone(\alpha) \rightarrow P(M) \otimes TP(N)'$ is a strong triangle. Since $Cone(\alpha)$ is acyclic and $P(M)$ is flat, it follows $P(M) \otimes Cone(\alpha)$ is also acyclic. It follows that the map $P(M) \otimes P(N)' \rightarrow^{id \otimes \alpha} P(M) \otimes P(N)$ is a quasi-isomorphism.

The arguments above show that we may choose a flat object $P(N) \overset{\simeq}{\rightarrow} N$ as in (M2) and consider the functor $- \otimes P(N) : H(\mathcal{C}) \rightarrow H(\mathcal{C})$. The arguments above show that the above functor preserves quasi-isomorphisms and induces a functor at the level of derived categories. (Moreover the same arguments show that the corresponding functor is independent of the choice of $P(N) \overset{\simeq}{\rightarrow} N$.) A similar argument works with N in the first argument. □

REMARK 3.4. A monoidal category will always mean one which satisfies all of the axioms (M0) through (M5) above. For emphasizing the existence of a unit, we will, however refer to such categories as *unital* monoidal. For the applications in Chapter IV, we will also need to assume the monoidal structure is symmetric.

4. Axioms on the strong t-structure

(ST1): For each integer n, there exists a functor $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}$ along with a natural transformation $\tau_{\leq n} Q(K) \to Q(K)$ which is an *admissible mono-morphism* for each $K \varepsilon \mathcal{C}$. The functors $\tau_{\leq n}$ preserve homotopies and quasi-isomorphisms; the induced functors at the level of the derived categories are idempotent.

(ST2): Moreover we require that $\mathcal{H}^i(\tau_{\leq n} QX) \cong \mathcal{H}^i(X)$ if $i \leq n$ and $\cong 0$ otherwise.

The functors $\tau_{\leq n}$ define a filtration on each $Q(K)$ by $F_n Q(K) = \tau_{\leq n} Q(K)$. We call this *the canonical Cartan filtration* . Clearly this is a non-decreasing filtration. Since we have assumed $\mathcal{C}$ is closed under small colimits, it follows that $Gr_{\mathcal{C}}(Q(K)) = \underset{n}{\sqcup} F_n(Q(K))/F_{n-1}(Q(K))$ also belongs to $\mathcal{C}$.

(ST3): Let $D(\mathcal{C})^{\leq n \geq}$ denote the heart of $D(\mathcal{C})$ shifted by n i.e. $D(\mathcal{C})^{\leq n \geq} = \{X \varepsilon D(\mathcal{C}) | \mathcal{H}^i(X) = 0 \text{ if } i \neq n\}$ Let $\mathbf{A}$ denote the abelian category in (STR3). We will assume that $\mathbf{A}$ is provided with a unital symmetric monoidal structure which we denote by $\otimes$. Furthermore, we will assume that the functor $\mathcal{H}^n : D(\mathcal{C})^{\leq n \geq} \to \mathbf{A}$ *is an equivalence of categories.* Moreover, there exists a sub-category, $\mathcal{C}_f^{\leq n \geq}$ of $\mathcal{C}_f$ so that the obvious functor $\mathcal{C} \to D(\mathcal{C})$ induces an equivalence of $\mathcal{C}_f^{\leq n \geq}$ with $D(\mathcal{C})^{\leq n \geq}$.

(ST4) Let $EM_n : \mathbf{A} \to \mathcal{C}_f^{\leq n \geq}$ denote an inverse to the composition $\mathcal{C}_f^{\leq n \geq} \to D(\mathcal{C})^{\leq n \geq} \to \mathbf{A}$. Each EM_n sends short-exact sequences in $\mathbf{A}$ to triangles in $\mathcal{C}$.

(ST5) We require that there exist a natural map $Gr_{\mathcal{C}}(Q(X)) \to GEM(\mathcal{H}^*(X)) = \underset{n}{\Pi} EM_n(\mathcal{H}^n(X))$ natural in $X \varepsilon \mathcal{C}$.

(ST6) Given π, π' in $\mathbf{A}$, there exists an induced pairing $EM_n(\pi) \otimes EM_m(\pi') \to EM_{n+m}(\pi \otimes \pi')$ *natural* in π and π' where the $\otimes$ on the left-hand-side (right-hand-side) denotes the given tensor structure $\otimes$ in $\mathcal{C}$ (the tensor product in $\mathbf{A}$, respectively). Moreover we require that this pairing makes the functor GEM (defined in (ST5)) into a monoidal functor sending the tensor product on $\mathbf{A}$ to the functor $\otimes$ on $\mathcal{C}$.

(ST7) We require that the tensor structure is compatible with the t-structure. i.e. If $X_i \varepsilon \mathcal{C}$, $i = 1, ..., n$ are provided with a pairing $\otimes_{i=1}^{i=n} X_i \to Z$, there exists an induced pairing $\otimes_{i=1}^{i=n} Gr_{\mathcal{C}}(Q(X_i)) \to Gr_{\mathcal{C}}(Q(Z))$.

(ST8) Finally we require that the maps in (ST5) and (ST7) are compatible. i.e. if $Gr_{\mathcal{C}}(Q(X)) \to GEM(\pi)$, $Gr_{\mathcal{C}}(Q(Y)) \to GEM(\pi')$ and $Gr_{\mathcal{C}}(Q(Z)) = GEM(\pi'')$, then the pairings $Gr_{\mathcal{C}}(Q(X)) \otimes Gr_{\mathcal{C}}(Q(Y)) \to Gr_{\mathcal{C}}(Q(Z))$ and $GEM(\pi) \otimes GEM(\pi') \to GEM(\pi'')$ are compatible.

COROLLARY 4.1. *Assume the above situation. Now $Gr_{\mathcal{C}}(Q(\mathcal{A}))$ and $GEM(\mathcal{H}^*(\mathcal{A}))$ are both algebras in $\mathcal{C}$ and the map in (ST5) is a quasi-isomorphism of algebras.*

PROOF. The proof is clear. □

DEFINITION 4.2. We say $\mathcal{C}$ is a *strong t-category* if $\mathcal{C}$ is a category satisfying the hypotheses (STR0) through (STR8.4) along with a strong t-structure satisfying the axioms (ST1) through (ST5)

4.1. We may also generalize the above situation as follows - see (6.1) for an example. A *pairing of strong-t-sub-categories* is the following data:

two strong t-sub-categories $\mathcal{C}'$, $\mathcal{C}''$ of $\mathcal{C}$ along with a bi-functor $\otimes : \mathcal{C}' \otimes \mathcal{C}'' \rightarrow \mathcal{C}$ so that the following conditions are satisfied:

(i)there exists a small sub-category $\mathfrak{F}'$ ($\mathfrak{F}''$) of $\mathcal{C}'$ ($\mathcal{C}''$, respectively) so that if $F' \varepsilon \mathfrak{F}'$ ($F'' \varepsilon \mathfrak{F}''$) then the functor $F' \otimes - : \mathcal{C}'' \rightarrow \mathcal{C}$ (the functor $- \otimes F'' : \mathcal{C}' \rightarrow \mathcal{C}$, respectively) sends an acyclic object to an acyclic object,

(ii) for each object $M' \varepsilon \mathcal{C}'$ ($M'' \varepsilon \mathcal{C}''$) there exists a quasi-isomorphism $P(M') \rightarrow M'$ with $P(M') \varepsilon \mathfrak{F}'$ ($P(M'') \rightarrow M''$ with $P(M'') \varepsilon \mathfrak{F}''$, respectively) and

(iii) *Compatibility of the t-structures with the pairing.* There exist strong t-sub-categories, $\mathcal{C}'_{gr}$, $\mathcal{C}''_{gr}$ of $\mathcal{C}$ so that the functor $Gr_{\mathcal{C}}$ sending an object in $\mathcal{C}$ to its associated graded object with respect to the Cartan filtration sends $\mathcal{C}'$ ($\mathcal{C}''$) to $\mathcal{C}'_{gr}$ ($\mathcal{C}''_{gr}$, respectively) and there exists a bi-functor $\otimes_{gr} : \mathcal{C}'_{gr} \times \mathcal{C}''_{gr} \rightarrow \mathcal{C}_{gr}$ so that if $M \varepsilon \mathcal{C}'$, $N \varepsilon \mathcal{C}''$ and $M \otimes N \varepsilon \mathcal{C}$ one obtains a natural map: $Gr_{\mathcal{C}}(M) \otimes_{gr} Gr_{\mathcal{C}}(N) \rightarrow Gr_{\mathcal{C}}(M \otimes N)$.

PROPOSITION 4.3. *Assume the above situation. Now the bi-functor $\otimes$ induces a derived functor*

$$\overset{L}{\otimes} : D(\mathcal{C}') \times D(\mathcal{C}'') \rightarrow D(\mathcal{C}).$$

PROOF. This is similar to that of (3.4) and is therefore skipped. □

In addition to these we will also need to define the analogue of the homotopy colimits and limits. For these we require the axioms denoted (HCl) and (Hl) below.

Let I denote a small category and let $\mathcal{C}^{I^{op}}$ denote the category of contravariant functors from I to $\mathcal{C}$. Let $n \rightarrow S(n)$ denote an object in $\mathcal{C}^{I^{op}}$ i.e. a functor $I^{op} \rightarrow \mathcal{C}$. Now we consider the functor:

$$T(\{S(n)|n\}) : I \times I^{op} \rightarrow \mathcal{C}, \text{ defined by } (n, m) \rightarrow I/m \otimes S(n)$$

4.1.1. We define the *homotopy colimit* $\underset{I}{\operatorname{hocolim}}\{S(n)|n\}$ to be the *co-end* of this functor in the sense of [Mac] p. 222. Now we require the axiom:

(HCl) A map $f : S' \rightarrow S$ of objects in $\mathcal{C}^{I^{op}}$ is called a quasi-isomorphism if the maps $f(n) : S'(n) \rightarrow S(n)$ (in $\mathcal{C}$) are all quasi-isomorphisms. A diagram $S' \rightarrow S \rightarrow S'' \rightarrow TS'$ in $(\mathcal{C})^{I^{op}}$ is a strong triangle if the corresponding diagrams $S'(n) \rightarrow S(n) \rightarrow S''(n) \rightarrow TS'(n)$ are strong-triangles in $\mathcal{C}$ for all n. Then the functor $\underset{I}{\operatorname{hocolim}}$ preserves triangles and quasi-isomorphisms. Moreover, in case $I = \Delta$, there exists a spectral sequence:

$$E^2_{s,t} = H^{-s}(\{H^{-t}(S_n)|n\}) \Rightarrow H^{-s-t}(\underset{\Delta}{\operatorname{hocolim}} S)$$

The $E^2_{s,t}$-term is the $-s$-th co-homology group of the simplicial abelian group $\{H^{-t}(S_n)|n\}$.

Let $n \rightarrow C(n)$ denote an object in $\mathcal{C}^I$ i.e. a covariant functor $I \rightarrow \mathcal{C}$. Now we consider the functor:

$$T(\{C(n)|n\}) : I \times I^{op} \rightarrow \mathcal{C}, \text{ defined by } (n, m) \rightarrow C(n)^{I \backslash m}$$

4.1.2. We define the *homotopy limit* $\underset{I}{\operatorname{holim}}\{C(n)|n\}$ to be the *end* of this functor in the sense of [Mac] p. 218. Now we require the axiom:

(Hl) A map $f : C' \to C$ of objects in $\mathcal{C}^I$ is called a quasi-isomorphism if the maps $f(n) : C'(n) \to C(n)$ (in $\mathcal{C}$) are all quasi-isomorphisms. A diagram $C' \to C \to C'' \to TC'$ in $(\mathcal{C})^I$ is a strong triangle if the corresponding diagrams $C'(n) \to C(n) \to C''(n) \to TC'(n)$ are strong-triangles in $\mathcal{C}$ for all n. Then the functor $\underset{\Delta}{\operatorname{holim}} \circ Q$ preserves triangles and quasi-isomorphisms. Moreover, when $I = \Delta$, there exists a spectral sequence:

$$E_2^{s,t} = H^s(\{H^t(C^n)|n\}) \Rightarrow H^{s+t}(\underset{\Delta}{\operatorname{holim}}\, C)$$

if each C^n is a *fibrant object* in $\mathcal{C}$. The $E_2^{s,t}$-term is the s-th (co-)homology of the cosimplicial abelian group $\{H^t(C^n)|n\}$.

In addition we will require the following axiom that enables one to compare two homotopy inverse limits or colimits.

Let I denote a small category and let $f : I \to J$ denote a covariant functor. We say f is *left-cofinal* if for every object $j \varepsilon J$, the nerve of the obvious comma-category f/j is contractible. Let $F : J \to \mathcal{C}$ be a functor.

Cofinality. We require that the induced map $\underset{J}{\operatorname{holim}}\, F \to \underset{I}{\operatorname{holim}}\, F \circ f$ is a quasi-isomorphism if the functor f is left-cofinal. We also require a parallel axiom on the cofinality of homotopy colimits.

PROPOSITION 4.4. *Assume the above situation. Let $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$ denote the category of $\mathbb{Z}$-graded objects in $\mathbf{A}$ and let $C_0(\underset{\mathbb{Z}}{\oplus}\mathbf{A})$ denote the category of co-chain complexes in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$ that are trivial in negative degrees. Then one may define a functor $Sp_0 : C_0(\underset{\mathbb{Z}}{\oplus}\mathbf{A}) \to \mathcal{C}$ that sends distinguished triangles (quasi-isomorphisms) of chain complexes to triangles (quasi-isomorphisms, respectively) in the category $\mathcal{C}$.*

PROOF. Let $M = \underset{n\varepsilon\mathbb{Z}}{\Pi} M(n)$ be a graded object of $\mathbf{A}$. Now recall $GEM(M) = \underset{n}{\Pi} EM_n(M(n))$ in $\mathcal{C}$. Next let $K = K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} K^{n-1} \xrightarrow{d^n} K^n \to \dots$ denote a co-chain complex in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$ that is trivial in negative degrees. $DN(K)$ denotes a co-simplicial object in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$. We apply the functor GEM degree-wise to $DN(K)$ to obtain a cosimplicial object of fibrant objects in $\mathcal{C}$. Finally we take the homotopy inverse limit of this cosimplicial object to define $Sp_0(K)$. Now the proposition follows readily from the hypothesis that the functor GEM preserves sends distinguished triangles to triangles and from the standard properties of the homotopy inverse limit functor. □

REMARKS 4.5. (i) If $M \varepsilon \underset{\mathbb{Z}}{\oplus}\mathbf{A}$ and $M[0]$ is the associated complex concentrated in degree 0, $Sp_0(M[0]) \simeq GEM(M)$. This follows from the degeneration of the spectral sequence for the homotopy limit considered in (Hl).

(ii) We may extend the functor Sp_0 to the whole category of bounded below co-chain complexes in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$ as follows. Let N denote an integer and let $C_N(\underset{\mathbb{Z}}{\oplus}\mathbf{A})$ denote the category of all co-chain complexes of $\mathbb{Z}$-graded objects in $\mathbf{A}$ that are trivial in degrees less than N. Let M denote such a co-chain complex and let $M[-N]$ denote the same complex shifted to the right N-times. Then $M[-N]$ is trivial in negative degrees. We let

$$Sp_N(M) = \Sigma^N Sp_0(M[-N]) \tag{4.1.3}$$

4.2. We proceed to show that there are natural maps $Sp_N(M) \to Sp_{N+1}(M)$ that are quasi-isomorphisms for any N so that the complex M is trivial in degrees less than N. To see this, recall first of all that the functor GEM sends short-exact sequences of complexes to triangles and preserves quasi-isomorphisms; the definition of the functor Sp_0 as above shows, it inherits the same property. One may deduce from this property that there exists a natural quasi-isomorphism $Sp_0(M[-N-1]) \simeq \Omega Sp_0(M[-N])$. This proves the required assertion.

DEFINITION 4.6. We let $Sp(M) = \lim\limits_{N \to \infty} Sp_N(M)$.

It follows that the functor Sp defines an extension of the functor Sp_0 to the category of all bounded below co-chain complexes in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$.

(ST9) *Standing hypothesis.* We will make the following hypothesis from now on. Let $\bar{M} = \underset{i}{\Pi}\bar{M}(i)$ ($\bar{N}$) denote a co-chain complex in $\underset{\mathbb{Z}}{\oplus}\mathbf{A}$ (in $\mathbf{A}$) trivial in negative degrees and bounded above by m. Then there exists a quasi-isomorphism:

$$Sp(\bar{M}) \simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(GEM(\bar{M}[m_h]))$$

$$(\underset{\Delta}{\mathrm{holim}}\, DN(EM_n(\bar{N})) \simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(EM_n(\bar{N}[m_h])), \text{ respectively})$$

natural in $\bar{M}$ (in $\bar{N}$, respectively), where $\bar{M}[m_h]$ ($\bar{N}[m_h]$) is the chain-complex defined by $(\bar{M}[m_h])_i = \bar{M}^{m-i}$ ($(\bar{N}[m_h])^i = \bar{N}^{m-i}$, respectively) and DN denotes the denormalization functor as in Appendix B that produces a simplicial object from a chain complex.

Remark. The purpose of this condition is to be able to pass between homotopy limits and colimits with ease. One knows that, in general, they are quite different; but for bounded simplicial and cosimplicial objects in abelian categories, they are both equivalent to a (in fact, the same) total complex construction. By making use of the given t-structure, one is able to reduce the general homotopy limits and colimits we consider to ones taking place in Abelian categories.

DEFINITIONS 4.7. (i) A category $\mathcal{C}$ is called a *strongly triangulated category* if it satisfies the axioms (STR0) through (STR8.4) and the axioms on the homotopy colimits and limits.

(ii) $\mathcal{C}$ is a *strongly triangulated monoidal category* if it is strongly triangulated and satisfies the axioms (M0) through (M5).

(iii) $\mathcal{C}$ is *an enriched monoidal t-category* if it is a strongly triangulated monoidal category satisfying the axioms (ST1) through (ST9) on the strong t-structure as well.

The next chapter is devoted to a thorough examination of various examples of such categories.

CHAPTER II

The basic examples of the framework

1. Sites

In this chapter, we consider in detail, various concrete examples of the axiomatic framework introduced in the first chapter. After discussing the general frame-work, we consider in detail three distinct contexts for the rest of our work: these are discussed in sections two, three and four respectively. Throughout this chapter $\mathfrak{S}$ will denote a site satisfying the following hypotheses.

1.0.1. In the language of [SGA]4 Exposé IV, there exists a conservative family of points on $\mathfrak{S}$. Recall this means the following. Let $(sets)$ denote the category of sets. Then there exists a set $\bar{\mathfrak{S}}$ with a map $p : (sets)^{\bar{\mathfrak{S}}} \to \mathfrak{S}$ of sites so that the map $F \to p_* \circ U \circ a \circ p^*(F)$ is injective for all Abelian sheaves F on $\mathfrak{S}$. (Equivalently, if $i_{\bar{s}} : (sets) \to \mathfrak{S}$ denotes the map of sites corresponding to a point $\bar{s}$ of $\bar{\mathfrak{S}}$, an Abelian sheaf F on $\mathfrak{S}$ is trivial if and only if $i_{\bar{s}}^* F = 0$ for all $\bar{s}\varepsilon\bar{\mathfrak{S}}$.) Here $(sets)^{\bar{\mathfrak{S}}}$ denotes the product of the category $(sets)$ indexed by $\bar{\mathfrak{S}}$. a is the functor sending a presheaf to the associated sheaf and U is the forgetful functor sending a sheaf to the underlying presheaf. We will also assume that the corresponding functor $p^{-1} : \mathfrak{S} \to (sets)^{\bar{\mathfrak{S}}}$ commutes with finite fibered products.

1.0.2. If X is an object in the site $\mathfrak{S}$, we will let $\mathfrak{S}/X$ denote the category whose objects are maps $u : U \to X$ in $\mathfrak{S}$ and where a morphism $\alpha : u \to v$ (with $v : V \to X$ in $\mathfrak{S}$) is a commutative triangle

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "u"'] & & V \arrow[dl, "v"] \\ & X & \end{tikzcd}$$

We will further assume that the site $\mathfrak{S}$ has a terminal object which will be denoted X (i.e. $\mathfrak{S}/X = \mathfrak{S}$) and that the category $\mathfrak{S}$ is closed under finite inverse limits.

1.0.3. Let Y be an object in the site $\mathfrak{S}$ and $i_{\bar{s}} : \bar{s} \to \mathfrak{S}$ a point of the site $\mathfrak{S}$. We say $i_{\bar{s}}$ is a point of Y if the map $i_{\bar{s}} : \bar{s} \to \mathfrak{S}$ factors through $\mathfrak{S}/Y$. Given a point $i_{\bar{s}}$ of Y, a *neighborhood* of $i_{\bar{s}}$ in the site $\mathfrak{S}$ is an object U in the site with a map $u : U \to Y$ together with a lifting of $i_{\bar{s}}$ to U.

1.0.4. We will assume that the system of neighborhoods of any point has a *small cofinal* family.

1.0.5. We will also assume the sites we consider are *coherent and locally coherent* sites as in [SGA]4 Exposé VI (2.3). Let $\mathfrak{S}$ denote a site and let $U\varepsilon\mathfrak{S}$. Then U *is quasi-compact* if every cover of U has a finite sub-cover. An object $U\varepsilon\mathfrak{S}$ is *quasi-separated* if for any two maps $V \xrightarrow{v} U$ and $W \xrightarrow{w} U$ in $\mathfrak{S}$, the fibered product $V \underset{U}{\times} W$ is quasi-compact. An object $U\varepsilon\mathfrak{S}$ is *coherent* if it is both quasi-compact and quasi-separated. A site $\mathfrak{S}$ is *coherent* if the following hold:

1.0.6. (i) Every object quasi-separated in $\mathfrak{S}$ is quasi-separated over the terminal object, X, of the site $\mathfrak{S}$ (an object $U\varepsilon\mathfrak{S}$ is quasi-separated over the terminal object X if the induced map $\Delta : U \to U \underset{X}{\times} U$ is quasi-compact) and

(ii) The terminal object X of the site $\mathfrak{S}$ is *coherent*.

A site $\mathfrak{S}$ satisfying the condition (i) is called *algebraic*. A site $\mathfrak{S}$ is called *locally coherent* if there exists a covering $\{X_i|i\}$ of the terminal object X so that the sites $\mathfrak{S}/X_i$ are all coherent. Observe that if X is a scheme and the site $\mathfrak{S}$ is either the Zariski or the small étale site of X, then $\mathfrak{S}$ is coherent if and only if X is quasi-compact and separated as a scheme. (See [SGA]4 Exposé VI.)

(The main need for coherence is given by the following result in [SGA]4 Exposé VI, Théoreme (5.1):

Theorem. Let $f : \mathfrak{S} \to \mathfrak{S}'$ denote a map of sites.

(i) Suppose $\mathfrak{S}$ satisfies the first condition in (1.1) and $\mathfrak{S}'$ is locally coherent. Then the functors $R^n f_*$ commute with filtered direct limits of abelian sheaves for each n.

(ii) Suppose $\mathfrak{S}$ is coherent (and X is the terminal object of $\mathfrak{S}$). Then, for each n, the functor $F \to H^n(\mathfrak{S}, F) = R^n\Gamma(X, F)$,

(abelian sheaves on $\mathfrak{S}$) $\to$(abelian groups)

commutes with filtered direct limits. □)

Observe that the same conclusions as in (i) hold if the sites are the obvious sites associated to locally compact Hausdorff topological spaces and f is a continuous map of these topological spaces. Therefore, we will make either of the following assumptions throughout the rest of the paper:

1.0.7. The site is locally coherent *or*

1.0.8. The site is the obvious site associated to a locally compact Hausdorff topological space.

We will explicitly consider only the first case, leaving the corresponding statements in the second case to the reader.

1.0.9. *Godement resolutions.* Let $Presh = Presh(\mathfrak{S})$ denote a category of presheaves on a site $\mathfrak{S}$ so that it satisfies the axioms (STR0) through (STR8.4)) and the axioms (Hl), (HCl) on the existence of homotopy limits and colimits. We will further assume that the abelian category $\mathbf{A}$ is a category of abelian sheaves on the site $\mathfrak{S}$. Recall our site has a conservative family of points as in (1.0.1). We will assume further that $Presh(\bar{\mathfrak{S}}))$ denotes a category of presheaves on the discrete site $\bar{\mathfrak{S}}$ satisfying the same axioms and that p induces functors $p^* : Presh(\mathfrak{S}) \to Presh(\bar{\mathfrak{S}})$, $p_* : Presh(\bar{\mathfrak{S}}) \to Presh(\mathfrak{S})$. Given a presheaf $P \varepsilon Presh$, we let $G^\bullet P : P ... GP ... G^2P ... G^nP ...$ denote the obvious cosimplicial object in $Presh$, where $G = p_* \circ U \circ a \circ p^*$. We let $\mathcal{G}P = \underset{\Delta}{\text{holim}}\{G^nP|n\}$. From the properties of the homotopy limits as in (Hl), the following are now obvious:

The induced map

$$\Gamma(U, \mathcal{G}Q(P')) \to \Gamma(U, \mathcal{G}Q(P)) \tag{1.0.10}$$

is a quasi-isomorphism for each U in the site and for each quasi-isomorphism $P' \to P$ of presheaves.

The induced diagram

$$\Omega\Gamma(U, \mathcal{G}Q(P'')) \to \Gamma(U, \mathcal{G}Q(P')) \to \Gamma(U, \mathcal{G}Q(P)) \to \Gamma(U, \mathcal{G}Q(P'')) \tag{1.0.11}$$

is a triangle for each U in the site and each diagram $\Omega P'' \to P' \to P \to P''$ which is a triangle.

A map of presheaves that induces a quasi-isomorphism at each stalk, will be denoted $\simeq$.

1.1. Definition of derived functors via the Godement resolution. Let $Presh$, $Presh'$ denote two categories of presheaves on sites as above and let $\phi : Presh \to Presh'$ denote a functor so that the following two properties hold.

• Given a triangle $P' \to P \to P'' \to P'[1]$ in $Presh$, the induced diagram $\phi\mathcal{G}P' \to \phi\mathcal{G}P \to \phi\mathcal{G}P'' \to \phi\mathcal{G}P'[1]$ is a triangle in $Presh'$ and

• Given a quasi-isomorphism $P' \to P$ in $Presh$, the induced map $\phi\mathcal{G}P' \to \phi\mathcal{G}P$ is also a quasi-isomorphism.

In this case we may define the right derived functor, $R\phi$ of ϕ by $\phi \circ \mathcal{G}$. Then the spectral sequence for the homotopy inverse limit in Chapter I provides a spectral sequence:

$$E_2^{s,t} = H^s(\{\phi G^n \mathcal{H}^t(P)|n\}) \Rightarrow \mathcal{H}^{s+t}(R\phi P) \tag{1.1.1}$$

The $E_2^{s,t}$-term is the s-cohomology of the cosimplicial abelian sheaf $\{\phi G^n\mathcal{H}^t(P)|n\}$. We will consider various examples of this in this paper. For example, let $\mathfrak{S}'$ denote another site and let $\phi : \mathfrak{S} \to \mathfrak{S}'$ denote a map of sites. We may now define the right derived functor $R\phi$ to be $\phi \circ \mathcal{G}$. In particular, one may take $\mathfrak{S}'$ to be the punctual site pt. A map of sites $\mathfrak{S} \to pt$ may be identified with the global section functor (i.e. sections over X= the terminal object of the site $\mathfrak{S}$.) Then we let $R\Gamma(X, P) = \Gamma(X, \mathcal{G}P)$. We will also denote this by $\mathbb{H}(X, P)$ and call it the hypercohomology object associated to X and P.

1.2. Algebras and modules. Assume in addition that the category $Presh$ is symmetric monoidal with respect to a bi-functor $\otimes$ and that it satisfies the axioms (M0) through (M4.6) in Chapter I except possibly for the existence of a unit for the monoidal structure. (Often we require, in addition, that there exist a unit $\mathcal{S}$ for $\otimes$) . Let $\mathcal{A}$ be an algebra in $Presh$. i.e. $\mathcal{A}$ is an object in $Presh$ provided with a coherently associative pairing $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. (Moreover if $\mathcal{S}$ is a unit, we require that there is a unit map $i : \mathcal{S} \to \mathcal{A}$ so that the composition $\mathcal{A} \cong \mathcal{S} \otimes \mathcal{A} \xrightarrow{i \otimes id} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A}$ is the identity and that i is an admissible monomorphism.) Let $Mod_l(\mathfrak{S}, \mathcal{A})$ ($Mod_r(\mathfrak{S}, \mathcal{A})$) denote the category of left-modules (right-modules, respectively) over $\mathcal{A}$. A left-module M over $\mathcal{A}$ consists of an object $M \varepsilon Presh$ provided with a coherently associative pairing $\mathcal{A} \otimes M \to M$.) We will always require that the presheaf $U \to \mathcal{H}^*(\Gamma(U, \mathcal{A}))$ be a presheaf of Noetherian rings.

1.2.1. In what follows we need to consider two distinct situations: (i) where the functors m and e (as in Chapter I, (STR6) are the identity. (See for example, section 3.) and (ii) where these functors are not necessarily the identity. (See for example, sections 2 and 4). We will first consider the situation in (i).

We will next define a pairing (i.e. a bi-functor)

$$\underset{\mathcal{A}}{\otimes} : Mod_r(\mathfrak{S}; \mathcal{A}) \times Mod_l(\mathfrak{S}; \mathcal{A}) \to Presh \tag{1.2.2}$$

If $M \varepsilon Mod_r(\mathfrak{S}; \mathcal{A})$ and $N \varepsilon Mod_l(\mathfrak{S}; \mathcal{A})$, $M \underset{\mathcal{A}}{\otimes} N$ is defined as the co-equalizer:

$$\text{Coeq}(M \otimes \mathcal{A} \otimes N \underset{g}{\overset{f}{\rightrightarrows}} M \otimes N)$$

where $f : M \otimes \mathcal{A} \otimes N \to M \otimes N$ ($g : M \otimes \mathcal{A} \otimes N \to M \otimes N$) is the map $f = \lambda_M \otimes id_N$, with $\lambda_M : M \otimes \mathcal{A} \to M$ the module structure on M ($g = id_M \otimes \lambda_N$, with $\lambda_N : \mathcal{A} \otimes N \to N$ the module structure on N, respectively). If $M \varepsilon Mod_l(\mathfrak{S}; \mathcal{A})$ and $N \varepsilon Mod_l(\mathfrak{S}; \mathcal{A})$ we also

define:
(1.2.3)

$$Hom_{\mathcal{A}}(M,N) = Equalizer(\ Hom_{Mod_l(\mathfrak{S};\mathcal{A})}(\mathcal{A},N) \overset{f}{\underset{g}{\rightrightarrows}} Hom_{Mod_l(\mathfrak{S};\mathcal{A})}(\mathcal{A}\otimes M,N)\ )$$

where $f = Hom_{Mod_l(\mathfrak{S};\mathcal{A})}(\lambda_M, N)$ and $g = Hom_{Mod_l(\mathfrak{S};\mathcal{A})}(\mathcal{A}\otimes M, \lambda_N)$.

Let $\mathcal{H}om$ denote the internal Hom in the category $Presh$. This exists as a right adjoint to $\otimes$ since the category $Presh$ has a small generating set. Observe that if M, N and $P\varepsilon Presh$, then,

$$(1.2.4)\qquad Hom_{Presh}(M, \mathcal{H}om(P,N)) \cong Hom_{Presh}(M\otimes P, N)$$

One defines $\mathcal{H}om_{\mathcal{A},l} : Mod_l(\mathfrak{S};\mathcal{A})^{op} \times Mod_l(\mathfrak{S};\mathcal{A}) \rightarrow Presh$ as in (1.2.3) using the functor $\mathcal{H}om$ in the place of Hom. Similarly one may define $\mathcal{H}om_{\mathcal{A},r} : Mod_r(\mathfrak{S};\mathcal{A})^{op} \times Mod_r(\mathfrak{S};\mathcal{A}) \rightarrow Presh$. (*When there is no cause for confusion, we will omit the the subscript l, r in $\mathcal{H}om_{\mathcal{A},l}$ and $\mathcal{H}om_{\mathcal{A},r}$.*) Finally one may also define the structure of a simplicial category on $Mod_l(\mathfrak{S};\mathcal{A})$ by

$$(1.2.5)\qquad Map_{\mathcal{A}}(M,N)_n = Hom_{\mathcal{A}}(\Delta[n]_+\otimes M, N), M, N\varepsilon Mod_l(\mathfrak{S};\mathcal{A})$$

Recall that $\Delta[n]_+\otimes M$ is defined as part of the axiom (M4.0) in Chapter I. One may now observe, the isomorphisms:

$$Map_{\mathcal{A}}(M,N)_0 \cong Hom_{\mathcal{A}}(M,N), \quad Hom_{\mathcal{A}}(\mathcal{A}\otimes M,N) \cong Hom(M, For(N))$$

(1.2.6)

$$Map_{\mathcal{A}}(\mathcal{A}\otimes M,N) \cong Map(M,For(N)) \quad and \quad Hom_{Presh}(M\underset{\mathcal{A}}{\otimes}P,N) \cong Hom_{\mathcal{A}}(P,\mathcal{H}om(M,N))$$

where M, $N\varepsilon Mod_l(\mathfrak{S};\mathcal{A})$ in the first three terms above, M, $N\varepsilon Mod_r(\mathfrak{S};\mathcal{A})$, $P\varepsilon Mod_l(\mathfrak{S};\mathcal{A})$ in the last term above and $For : Mod_l(\mathfrak{S};\mathcal{A}) \rightarrow Presh$ is the obvious forgetful functor.

Next we consider the situation in (ii) where the functors m and e in Chapter I, (STR6) are non-trivial. Now will define a pairing (i.e. a bi-functor) for $P\varepsilon Mod_r(\mathfrak{S},\mathcal{A})$, m, $N\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$. We may assume without loss of generality that $\mathcal{A}$ is mono.

$$(1.2.7)\qquad P\underset{\mathcal{A}}{\otimes}M = hocolim(\ m(P)\otimes\mathcal{A}\otimes M \overset{f}{\underset{g}{\rightrightarrows}} m(P)\otimes M\ )$$

where the two maps f and g are as in 1.2.2 and *hocolim* denotes the homotopy colimit. Similarly,

$$(1.2.8)\qquad \mathcal{H}om_{\mathcal{A},l}(M,N) = holim(\ \mathcal{H}om(m(M),e(N)) \overset{m^*}{\underset{n_*}{\rightrightarrows}} Hom(\mathcal{A}\otimes m(M), e(N))\ )$$

where the maps m^* and n_* are again as in 1.2.3 and *holim* denotes the homotopy inverse limit. We will define a bi-functor $Map_{\mathcal{A}} : Mod_l(\mathfrak{S},\mathcal{A}) \times Mod_l(\mathfrak{S},\mathcal{A}) \rightarrow (simplicial\quad sets)$ by

$$Map_{\mathcal{A}}(M,N) = holim(\ Map(m(M),e(N)) \overset{m^*}{\underset{n_*}{\rightrightarrows}} Map(\mathcal{A}\otimes m(M), e(N))\ ).$$

($Map : Presh \times Presh \rightarrow (simplicial\quad sets)$ is the obvious functor. Observe that $\pi_0(Map_{\mathcal{A}}(M,N))$ denotes homotopy classes of maps $f : M \rightarrow N$ which belong to the category $Mod_l(\mathfrak{S},\mathcal{A})$.

2. When *Presh* is a strongly triangulated unital monoidal category

Throughout the rest of this section, we will assume that $\mathcal{S}$ is a unit for $\otimes$. Let $\mathcal{B}$, $\mathcal{A}$ denote two algebras in $Presh$. We will let $Mod_{bi}(\mathfrak{S};\mathcal{A},\mathcal{B})$ denote the category of objects in $Presh$ that have the structure of a presheaf of left-$\mathcal{B}$ and weak right $\mathcal{A}$ bi-modules so that the two commute. (This means if $\lambda : \mathcal{A}\otimes M \to M$ and $\mu : M\otimes\mathcal{B} \to M$ are the two module structures, then $\lambda\circ(id\otimes\mu) = \mu\circ(\lambda\otimes id)$.) Now one of the key technical results we need in the body of the paper is the following.

PROPOSITION 2.1. *(i) Assume the above situation. Now* $P\underset{\mathcal{A}}{\otimes}\mathcal{A} \simeq P$, $\mathcal{A}\underset{\mathcal{A}}{\otimes}N \simeq N$ *and* $\mathcal{H}om_{\mathcal{A}}(\mathcal{A},N) \simeq \mathcal{N}$, P, $N\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$.

(ii) Assume that $M\varepsilon Mod_{bi}(\mathfrak{S};\mathcal{A},\mathcal{A})$.

Now $\mathcal{H}om_{\mathcal{A}}(M,\quad)$ *and* $\quad\underset{\mathcal{A}}{\otimes}M$ *induce functors* $Mod_r(\mathfrak{S};\mathcal{A}) \to Mod_r(\mathfrak{S};\mathcal{A})$. *Moreover if* N, $P\varepsilon Mod_r(\mathfrak{S};\mathcal{A})$, *one obtains a natural weak-equivalence:*

$$Map_{\mathcal{A}}(P\underset{\mathcal{A}}{\otimes}M,N) \simeq Map_{\mathcal{A}}(P,\mathcal{H}om_{\mathcal{A}}(M,N))$$

(iii) Assume that $N\varepsilon Mod_{bi}(\mathfrak{S};\mathcal{B},\mathcal{A})$, $P\varepsilon Mod_r(\mathfrak{S},\mathcal{B})$ *and* $M\varepsilon Mod_r(\mathfrak{S};\mathcal{A})$. *In this case one also obtains*

$$\mathcal{H}om_{\mathcal{B}}(M\underset{\mathcal{A}}{\otimes}N,P) \simeq \mathcal{H}om_{\mathcal{A}}(M,\mathcal{H}om_{\mathcal{B}}(N,P)) \quad and \tag{2.0.9}$$

$$Map_{\mathcal{B}}(M\underset{\mathcal{A}}{\otimes}N,P) \simeq Map_{\mathcal{A}}(M,\mathcal{H}om_{\mathcal{B}}(N,P)) \tag{2.0.10}$$

Moreover, one may also replace $\simeq$ *everywhere by* $\cong$ *in the situation (ii) considered in 1.2.1.*

PROOF. The diagram

$\mathcal{A}\otimes\mathcal{A}\otimes N \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{A}\otimes N \longrightarrow N$ is a *split fork* in the sense of [Mac] p. 145, the splitting provided by the maps $N \cong \mathcal{S}\otimes N \to \mathcal{A}\otimes N$ and $\mathcal{A}\otimes N \cong \mathcal{S}\otimes\mathcal{A}\otimes N \to \mathcal{A}\otimes\mathcal{A}\otimes N$. By ([Mac] Lemma p. 145) N is in fact the co-equalizer of the above diagram i.e.

$$\mathcal{A}\underset{\mathcal{A}}{\otimes}N \cong N \tag{2.0.11}$$

The weak-equivalence $\mathcal{H}om_{\mathcal{A}}(\mathcal{A},N) \simeq \mathcal{N}$ is established similarly. This completes the proof of (i). Clearly one may replace $\simeq$ by $\cong$ everywhere in the situation (i) of 1.2.1.

The right-module structure on $\mathcal{H}om_{\mathcal{A}}(M,N)$ $(P\underset{\mathcal{A}}{\otimes}M)$ is induced by the left-module structure (the right-module structure) of M over $\mathcal{A}$. Let $p : P\otimes\mathcal{A} \to P$, $m : \mathcal{A}\otimes M \to M$, $m' : M\otimes\mathcal{A} \to M$ and $n : N\otimes\mathcal{A} \to N$ denote the given module structures. Then

$$Map_{\mathcal{A}}(P\underset{\mathcal{A}}{\otimes}M,N) = holim\ Map(P\underset{\mathcal{A}}{\otimes}M,e(N)) \overset{n^*}{\underset{m'^*}{\rightrightarrows}} Map(P\underset{\mathcal{A}}{\otimes}M\otimes\mathcal{A},e(N)) \tag{2.0.12}$$

Next recall $P\underset{\mathcal{A}}{\otimes}M = hocolim(\ m(P)\otimes\mathcal{A}\otimes m(M) \overset{p_*}{\underset{m_*}{\rightrightarrows}} m(P)\otimes m(M)\ )$. This homotopy colimit pulls out of the Map as a homotopy inverse limit; the two homotopy inverse limits commute. Using the adjunction between $\otimes$ and $\mathcal{H}om$, we see that the last term may be identified with

$$holim(\ Map(m(P), \mathcal{H}om_{\mathcal{A}}(M,N)) \overset{p^*}{\underset{m^*}{\rightrightarrows}} Map(m(P)\otimes\mathcal{A}, \mathcal{H}om_{\mathcal{A}}(M,N))\ ) \tag{2.0.13}$$
$$\simeq Map_{\mathcal{A}}(m(P), \mathcal{H}om_{\mathcal{A}}(M,N))$$

Therefore, this proves (ii) in the case the functors m and e are nontrivial. The other case may be established more directly. The first identity in (iii) may be established in the situation (i) of 1.2.1 as follows. $N\varepsilon Mod_{bi}(\mathcal{A},\mathcal{B})$ and $P\varepsilon Mod_r(\mathfrak{S};\mathcal{B})$. Now consider the functor

$$Mod_r(\mathfrak{S};\mathcal{A}) \to Mod_r(\mathfrak{S};\mathcal{B}), \quad M \to M\underset{\mathcal{A}}{\otimes}N \tag{2.0.14}$$

The observation that the composition $M\otimes\mathcal{S} \to M\otimes\mathcal{A} \to M$ is the identity for any $M\varepsilon Mod_r(\mathfrak{S};\mathcal{A})$, shows that if $\mathfrak{F}$ is a *small generating set* for $Presh$, $\{F\otimes\mathcal{A}|F\varepsilon\mathfrak{F}\}$ is a generating set for $Mod_r(\mathfrak{S};\mathcal{A})$. Clearly the functor in (2.0.14) preserves all colimits and the category on the left is co-complete. By the special adjoint functor theorem (see [Mac] p. 125) the above functor has a right adjoint which we may identify with $\mathcal{H}om_{\mathcal{B}}(N,P)$. i.e. we obtain the isomorphism

$$Hom_{\mathcal{B}}(M\underset{\mathcal{A}}{\otimes}N, P) \cong Hom_{\mathcal{A}}(M, \mathcal{H}om_{\mathcal{B}}(N,P)) \tag{2.0.15}$$

We skip the proofs of the remaining assertions. □

REMARK 2.2. Observe that, in the situation of (ii) of 1.2.1, one really needs to adopt the definition of $\underset{\mathcal{A}}{\otimes}$ in 1.2.7 and 1.2.8 to obtain the results of the last proposition.

$\mathcal{H}om_{\mathcal{A}}(\quad,\quad)$ to obtain the

DEFINITION 2.3. Let $M\varepsilon Mod_l(\mathfrak{S};\mathcal{A})$. M is *flat* if for every acyclic module $N\varepsilon Mod_r(\mathfrak{S},\mathcal{A})$, $M\underset{\mathcal{A}}{\otimes}N$ is also acyclic. M is *locally projective* if for every acyclic object $N\varepsilon Mod_l(\mathfrak{S};\mathcal{A})$, $\mathcal{H}om_{\mathcal{A}}(M,N)$ is acyclic as well. If $P\varepsilon Presh$, P is flat if for every $K\varepsilon Presh$ that is acyclic $P\otimes K$ is also acyclic. P is locally projective if $\mathcal{H}om(P,R)$ is acyclic for every acyclic $R\varepsilon Presh$. (Here $\mathcal{H}om$ is the internal Hom in the category $Presh$.) We let $\mathcal{P}$ denote the full sub-category of objects in $Mod_l(\mathfrak{S},\mathcal{A})$ that are both flat and locally projective.

2.0.16. *Locally projective and flat resolutions.* Let $Presh$ denote a category of presheaves on a site $\mathfrak{S}$ so that it satisfies the axioms (STR0) through (STR7.3) and the axiom (HCl) on the existence of homotopy colimits. Now we may define a bi-functor $Map : Presh \times Presh^{op} \to (simplicial \quad sets)$ by $Map(M,N) = Hom_{Presh}(\Delta[n]_+ \otimes M, N)$ where the functor $\otimes$ is defined as in Chapter I, (M4.0). We will further assume that the following hypothesis holds:

(2.1.1.*) the abelian category $\mathbf{A}$ in Chapter I, (STR3), admits an imbedding into the category of abelian presheaves on the site $\mathfrak{S}$ where the latter is provided with the obvious tensor structure. Let the latter category be denoted $Presh_{Ab}(\mathfrak{S})$ and let the given imbedding be denoted U. We assume that U is compatible with the tensor structures and that the the presheaf $P \to U\circ\mathcal{H}^n(P)$ may be identified with the presheaf $P \to \pi_{-n}(Map(j_U^*(\underline{S}), j_U^*Q(P)))$, $j_U : \mathfrak{S}/U \to \mathfrak{S}$ in the site.

REMARK 2.4. 1. We will often denote $U\circ\mathcal{H}^n$ by just $\mathcal{H}^n$.

2. Observe that the restriction functor $j_U^* : Presh(\mathfrak{S}) \to Presh(\mathfrak{S}/U)$ has always a left-adjoint (see [SGA4] Exposé IV) which we will denote by $j_{U!}^{\#}$.

LEMMA 2.5. *Let $M \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$. Then there exists a set of integers $\{n_s|s\}$, a set of objects $\{j_U : U \to \mathfrak{S}|U\}$ in the site $\mathfrak{S}$ and a map*

$$\phi : \underset{U}{\sqcup}\underset{s}{\sqcup} j^{\#}_{U!}j^{*}_{U}\Sigma^{n_s}(\mathcal{S}) \otimes \mathcal{A} \to M$$

(where Σ^{n_s} is the n_s-fold iterate of Σ if $n_s \geq 0$ and the $-n_s$-fold iterate of Ω if $n_s < 0$) inducing a surjection:

$$\mathcal{H}^{-k}(\phi)_p : \underset{U}{\sqcup}\underset{s}{\sqcup}\mathcal{H}^{-k}(j^{\#}_{U!}j^{*}_{U}\Sigma^{n_s}(\mathcal{S}) \otimes \mathcal{A}))_p \overset{\mathcal{H}^{-k}(\phi)_p}{\to} \mathcal{H}^{-k}(M)_p, \quad \textit{for all } k \textit{ and all points } p.$$

PROOF. Recall the functor Q is compatible with the tensor structure. Therefore, we may assume, without loss of generality that M has been replaced by $Q(M)$. For each integer n, each point p and each class $[\alpha] \in \mathcal{H}^{-n}(M)_p$, let $\alpha : j^{\#}_{U!}j^{*}_{U}(\Sigma^n\mathcal{S}) \to Q(M)$ denote a representative map. If $i : \mathcal{S} \to \mathcal{A}$ denotes the unit of the algebra $\mathcal{A}$, we obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{S} \otimes j^{\#}_{U!}j^{*}_{U}\Sigma^n\mathcal{S} \arrow[r, "i\otimes id"] \arrow[d, "id"] & \mathcal{A} \otimes j^{\#}_{U!}j^{*}_{U}\Sigma^n\mathcal{S} \arrow[r, "id\otimes\alpha"] & \mathcal{A} \otimes Q(M) \arrow[d, "\mu"] \\
\mathcal{S} \otimes j^{\#}_{U!}j^{*}_{U}\Sigma^n\mathcal{S} \simeq j^{\#}_{U!}j^{*}_{U}\Sigma^n\mathcal{S} \arrow[rr, "\alpha"] & & Q(M)
\end{tikzcd}$$

where $\mu : \mathcal{A} \otimes Q(M) \to Q(M)$ is part of the structure making $Q(M)$ an $\mathcal{A}$-module. It follows that one may let $S = \{\alpha : \Sigma^n\mathcal{S} \to M|\alpha, n\}$; that this is a set follows from the hypotheses that one has a conservative family of points and that the system of neighborhoods of any point has a small cofinal family. □

DEFINITION 2.6. (i)The modules of the form $\underset{s\epsilon S}{\sqcup}\Sigma^{n_s}\mathcal{A}$ will be called *free* $\mathcal{A}$−modules.

(ii). The modules M for which there exists a finite set S so that the map $\mathcal{H}^*(\phi)$ in Lemma (2.5) is *a surjective map of $\mathcal{H}^*(\mathcal{A})$-modules* will be called *constructible* $\mathcal{A}$−modules. (*Recall that we have assumed the presheaf of graded rings $\mathcal{H}^*(\mathcal{A})$ to be Noetherian, which justifies this terminology.* Nevertheless, the notion of constructibility is useful only when $\mathcal{H}^*(\mathcal{A})$ is locally constant.)

(iii) *The free functor.* We define a free functor $\mathcal{F} : Presh \to Mod_l(\mathfrak{S}, \mathcal{A})$ by $\mathcal{F}(N) = \mathcal{A} \otimes N$. One defines a free functor $\mathcal{F} : Presh \to Mod_r(\mathfrak{S}, \mathcal{A})$ by $\mathcal{F}(N) = N \otimes \mathcal{A}$.

PROPOSITION 2.7. *Assume the hypotheses as in 2.0.16. Let $M \varepsilon Presh$. (i) Then there exists an object $\tilde{M}$, which is flat and locally projective, and a map $\epsilon : \tilde{M} \to M$ which is a quasi-isomorphism at each stalk. Moreover, there exists a simplicial object $P_\bullet = \{P_k|k\}$, each $P_k \varepsilon \mathcal{P}$ with an augmentation $P_0 \to M$ so that the following hold: a) $\tilde{M} = \underset{\Delta}{hocolim} P_\bullet$ and ϵ is the obvious induced map. (We will call $P_\bullet$ a simplicial resolution of $\tilde{M}$.) b) $\{\mathcal{H}^*(P_n)\tilde{\ }|n\}$ is a resolution of the sheaf of $\mathcal{H}^*(\mathcal{A})\tilde{\ }$-modules $\mathcal{H}^*(M)\tilde{\ }$.*

(ii) The same conclusions hold if $M \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ (or $M \varepsilon Mod_r(\mathfrak{S}, \mathcal{A})$) with $\mathcal{A}$ an algebra in Presh

PROOF. Observe that by taking $\mathcal{A} = \mathcal{S}$, we see that (i) is a special case of (ii). Therefore we will only prove (ii) in detail. Let $For : Mod_l(\mathfrak{S}, \mathcal{A}) \to Presh$ denote the obvious forgetful functor.

Let $M \varepsilon Mod_l(\mathcal{A})$ and let p denote a point of the site $\mathfrak{S}$. Then $For(M)_p = \underset{p\varepsilon U}{\mathrm{colim}}\Gamma(U, For(M))$ and $\mathcal{H}^{-n}(For(M)_p) \cong \underset{p\varepsilon U}{\mathrm{colim}}\mathcal{H}^{-n}(\Gamma(U, For(M))) \cong \underset{p\varepsilon U}{\mathrm{colim}}\mathcal{H}^{-n}(Hom(j^{\#}_{U!}j^{*}_{U}\mathcal{S}, For(M)))$ where

$\mathcal{S}$ is the constant presheaf on $\mathfrak{S}$ with stalks isomorphic to $\mathcal{S}$. In view of Lemma (2.5), it follows that one may find a set S, a covering $\mathcal{U} = \{U_s | s\varepsilon S\}$ of X and a collection of maps $j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{S}) \overset{\epsilon_s}{\to} For(M)$ so that for each integer n, the induced map $\mathcal{H}^{-n}\tilde{}(\underset{s\varepsilon S}{\sqcup}\epsilon_s))$: $\mathcal{H}^{-n}\tilde{}(\underset{s\varepsilon S}{\sqcup} j^{\#}_{U_s!}j^*_{U_s}(\Sigma^{n_s}\mathcal{S}) \to \mathcal{H}^{-n}\tilde{}(For(M))$ is a *surjective map of Abelian sheaves.*

Next we consider $P'_0 = \mathcal{A} \otimes (\underset{s\varepsilon S}{\sqcup} j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{S})) = \mathcal{F}(\underset{s\varepsilon S}{\sqcup} j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{S}))$. Then the natural map $P'_0 \to \underset{s\varepsilon S}{\sqcup} j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{A})$ is an isomorphism. Let d_{-1} denote the composition:

$$id_{\mathcal{A}} \otimes \underset{s\varepsilon S}{\sqcup}\epsilon_s : \mathcal{A} \otimes (\underset{s\varepsilon S}{\sqcup} \otimes j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{S})) \to \mathcal{A} \otimes M \to M$$

Since the composition of the map

$$S \otimes (\underset{s\varepsilon S}{\sqcup} \otimes j^{\#}_{U!}j^*_U(\Sigma^{n_s}\mathcal{S})) \to \mathcal{A} \otimes (\underset{s\varepsilon S}{\sqcup} j^{\#}_{U!}j^*_U(T^{n_s}\mathcal{S}))$$

and the map d_{-1} induces the map $\mathcal{H}^{-n}(\underset{s\varepsilon S}{\sqcup}\epsilon_s))\tilde{}$ in cohomology, it follows that $\mathcal{H}^{-n}(d_{-1})\tilde{}$ is a surjection for each n.

2.0.17. Let $Q_0 = M$ and $u'_0 = d_{-1} : P'_0 \to M$ the above map. We consider $P_0 = Cocyl(u'_0)$ and $u_0 : P_0 \to M$ is the map denoted d^1 in Chapter I, (STR7.2). (As mentioned in Chapter I, Remark (2.0.1), throughout the rest of the proof we will suppress mentioning the functor E explicitly, though it needs to be applied first before taking the co-cylinders. We hope this makes our discussion simpler.) Let $n > 0$ be a fixed integer. Assume we have defined $P_i \varepsilon \mathcal{P}$, X_i, $u'_i : P'_i \to X_i$, and P_i for all $0 \leq i \leq n$ so that the following hold

$P_i = Cocyl(u'_i)$, $X_i = fib_h(P'_{i-1} \overset{u'_{i-1}}{\to} X_{i-1})$, $u_i : P_i \to X_i$ is the map induced by u'_i (as in Chapter I, (STR7.2))

$\mathcal{H}^{-k}(u_i)\tilde{}$ is surjective as a map of Abelian sheaves for all k and

$\mathcal{H}^*(P_i)_p$ is a free graded module over the graded ring $\mathcal{H}^*(\mathcal{A})_p$ for all points p

We let $X_{n+1} = fib_h(u'_n : P'_n \to X_n)$. By replacing $M\varepsilon Presh$ by X_{n+1} and applying the arguments in 2.0.17 one may find an object $P'_{n+1}\varepsilon Presh$ along with a map $u'_{n+1} : P'_{n+1} \to X_{n+1}$ so that the following hold:

P'_{n+1} is locally projective and flat

$\mathcal{H}^*(P'_{n+1})_p$ is a free graded module over $\mathcal{H}^*(\mathcal{A})_p$ for all points p and

$\mathcal{H}^{-k}(u'_{n+1})\tilde{}$ is surjective as a map of Abelian sheaves for all k.

Now we let $P_{n+1} = Cocyl(u'_{n+1})$ and $u_{n+1} : P_{n+1} \to X_{n+1}$ the obvious map induced by u'_{n+1}. Now observe that there exists a natural homotopy-equivalence between P_n and P'_n for all n. It follows that if $K\varepsilon Presh$, $K \otimes P_n$ is homotopy equivalent to $K \otimes P'_n$ for all n and hence acyclic if K is. Therefore P_n is flat. One may similarly prove that P_n is also locally projective. Now observe that $X_n = ker(u_{n-1} : P_{n-1} \to X_{n-1})$ for all $n > 0$. We will let the composite map $P_n \overset{u_n}{\to} X_n \to P_{n-1}$, $n > 0$, be denoted d_n. It follows that the diagram

$$\to ...P_n \overset{d_n}{\to} P_{n-1} \overset{d_{n-1}}{\to} ... \overset{d_0}{\to} P_0$$

is a *complex in* $Mod_l(\mathfrak{S}; \mathcal{A})$ i.e. the composition of the two maps $d_{n-1} \circ d_n = *$. Moreover the map $u_0 : P_0 \to X_0 = M$ is an augmentation. Next we apply the denormalization functor DN to the above complex to produce a simplicial object in $Mod_l(\mathfrak{S}; \mathcal{A})$; this will be denoted $DN(P_\bullet)$. We let $\tilde{M} = \underset{\Delta}{\text{hocolim}} DN(P_\bullet)$. Let $K \varepsilon Mod_r(\mathfrak{S}; \mathcal{A})$. Then $K \underset{\mathcal{A}}{\otimes} DN(P_\bullet)$ is a simplicial object in $Presh$. Clearly each $DN(P_\bullet)_k \varepsilon Mod_l(\mathfrak{S}; \mathcal{A})$ is flat and locally projective; now the spectral sequence in Chapter I, (HCl) for the homotopy colimit shows that $\underset{\Delta}{\text{hocolim}} DN(P_\bullet)$ is also flat. The corresponding spectral sequence for the homotopy inverse limit (and the observation that homotopy colimits in the first argument in $\mathcal{H}om$ comes out of the $\mathcal{H}om$ as homotopy limits) shows that it is also locally projective. □

2.0.18. *The homotopy categories and the derived categories.* We define

$$Hom_{HMod_l(\mathfrak{S},\mathcal{A})}(K, L) = \pi_0(Map(\mathcal{S}, \mathcal{H}om_{\mathcal{A}}(K, L))) \tag{2.0.19}$$

Now observe that we obtain the following isomorphisms:

$$Hom_{HMod_l(\mathfrak{S},\mathcal{A})}(K, L) = \pi_0(Map(\mathcal{S}, \mathcal{H}om_{\mathcal{A}}(K, L))$$

$$\cong \pi_0 Map_{\mathcal{A}}(\mathcal{S} \otimes K, L) \cong \pi_0 Map_{\mathcal{A}}(K, L)$$

The first isomorphism follows from (2.0.15) by taking $\mathcal{B} = \mathcal{A}$ and $\mathcal{A} = \mathcal{S}$ in (2.0.15). Finally recall that $\otimes$ may be identified with $\underset{\mathcal{S}}{\otimes}$: this provides the last isomorphism. It follows immediately that there exists a functor from the category $\mathcal{P}resh$ to the homotopy category sending a map $f : K \to L$ to its class in $\pi_0 Map_{\mathcal{A}}(K, L) \cong Hom_{HMod_l(\mathfrak{S},\mathcal{A})}(K, L)$. Observe that the derived category associated to $Mod_l(\mathfrak{S}, \mathcal{A})$ is obtained from the above homotopy category by inverting maps that are stalk-wise quasi-isomorphisms. (See Proposition 2.9 below.) This will be denoted $D(Mod_l(\mathfrak{S}, \mathcal{A}))$.

We proceed to obtain a *concrete realization* of the derived category. Let $P(K) \to K$ denote a quasi-isomorphism from a locally projective object as in Proposition 2.7 .

PROPOSITION 2.8. *Assume the above situation. Then there exists isomorphisms:*

$$Hom_{D(Mod_l(\mathfrak{S},\mathcal{A}))}(K, L) \underset{\cong}{\overset{\phi}{\to}} \pi_0(Map_{\mathcal{A}}(P(K), \mathcal{G}Q(L))) \cong \pi_0(Map(\mathcal{S}, \mathcal{H}om_{\mathcal{A}}(P(K), \mathcal{G}Q(L))))$$

$$\cong \pi_0(\underset{\Delta}{holim}\{Map(\mathcal{S}, \mathcal{H}om_{\mathcal{A}}(P(K), \mathcal{G}^n Q(L)))|n\})$$

PROOF. The identification of the last term on the right-hand-side with

$\pi_0(Map(\mathcal{S}, \mathcal{H}om_{\mathcal{A}}(P(K), \mathcal{G}Q(L))))$ follows since $\underset{\Delta}{\text{holim}}$ commutes with $Map(\mathcal{S}, -)$ and $\mathcal{H}om_{\mathcal{A}}(P(K), -)$; this in turn follows from the definition of $\underset{\Delta}{\text{holim}}$ in the category $Mod_l(\mathfrak{S}, \mathcal{A})$ as an *end* (see section 1). The second $\cong$ is clear. Clearly there is a natural map from $Hom_{H\mathcal{P}resh}(K, L)$ to first term on the right-hand-side. One may readily see that if $K' \to K$, $L \to L'$ are stalk-wise quasi-isomorphisms, then one obtains an isomorphism:

$$\pi_0(Map_{\mathcal{A}}(P(K), \mathcal{G}Q(L')))) \cong \pi_0(Map_{\mathcal{A}}(P(K), \mathcal{G}Q(L))) \cong \pi_0(Map_{\mathcal{A}}(P(K'), \mathcal{G}Q(L))).$$

This shows the above map ϕ factors through the derived category. If $\alpha : P(K) \to \mathcal{G}Q(L)$ is a map (representing a class on the right-hand-side), the diagram

$$K \overset{\simeq}{\leftarrow} P(K) \to \mathcal{G}Q(L)$$

defines a class in the left hand side. Moreover if $K \overset{\simeq}{\leftarrow} K' \to L$ is sent to the map $P(K') \to \mathcal{G}Q(L)$ which is null-homotopic, one can see readily that the map $K \overset{\simeq}{\leftarrow} K' \to L \to \mathcal{G}Q(L)$ in the derived category $D(Mod_l(\mathcal{A}))$ is itself identified with the trivial map. Therefore the given map $K \overset{\simeq}{\leftarrow} K' \to L$ also is identified with the trivial map in the derived category. This provides the required isomorphism. □

PROPOSITION 2.9. *Assume the above situation. Then $Mod_l(\mathfrak{S};\mathcal{A})$ is a strongly triangulated category satisfying the axioms (STR8.1) through (STR8.4) with the functor Q in Chapter I, (STR8.1) given by the composition $\mathcal{G}\circ Q_0$ where $\mathcal{G}$ = the Godement resolution, the functor Q_0 = the given functor Q in $Presh$ and the cofibrant objects being identified with the locally projective objects. In case the given monoidal structure is symmetric and $\mathcal{A}$ is a commutative algebra, $Mod_l(\mathfrak{S};\mathcal{A}) = Mod_r(\mathfrak{S};\mathcal{A})$ is symmetric monoidal with respect to $\underset{\mathcal{A}}{\otimes}$ and $\mathcal{A}$ is a cofibrant object in $Mod_l(\mathfrak{S};\mathcal{A})$. Moreover there exists a functor $\mathcal{F}: Presh \to Mod_l(\mathfrak{S};\mathcal{A})$ left adjoint to the forgetful functor $For: Mod_l(\mathfrak{S};\mathcal{A}) \to Presh$.*

PROOF. One begins with the observation that the category $Mod_l(\mathfrak{S};\mathcal{A})$ is closed under the formation of the cylinder and co-cylinder objects as well the canonical homotopy fiber and mapping cones. This follows from the axioms (M4.6) and (M5). This also shows that $HMod_l(\mathfrak{S},\mathcal{A})$ admits a calculus of left and right fractions. The abelian category $\mathbf{A}$ in Chapter I, (STR3) and the cohomology functor $\mathcal{H}^n$ are the same as the ones for the category $Presh$. One may now readily verify that the category obtained by localizing with respect to cohomology isomorphisms is triangulated. The admissible monomorphisms (epi-morphisms) and the functors m, e are defined to be the corresponding ones in the category $Presh$. Propositions 2.7 and Proposition 2.8 show that the axiom (STR8.2) is satisfied; i.e. one may take cofibrant objects to be the locally projective ones. We define fibrant objects to be those objects M so that the obvious map $\Gamma(U,M)\to\Gamma(U,\mathcal{G}QM)$ is a quasi-isomorphism for every U in the site $\mathfrak{S}$. Now the axioms (STR8.1) and (STR8.4) are clear. The axiom (STR8.3) follows from Proposition 2.8 above. The assertions on the symmetric monoidal structure of $Mod_l(\mathfrak{S};\mathcal{A})$ are clear. Recall the free functor $\mathcal{F}$ is defined by $\mathcal{F}(N) = \mathcal{A}\otimes N$. That this is left adjoint to the forgetful functor follows from the observation that the composition $\mathcal{S}\otimes N \to \mathcal{A}\otimes N \to N$ is the identity for any $N \varepsilon Mod_l(\mathfrak{S},\mathcal{A})$. $\square$

REMARK 2.10. It follows from the above result that the category of complexes of sheaves of modules over a ringed site $(\mathfrak{S},\mathcal{R})$, where $\mathcal{R}$ is a sheaf of commutative rings with unit, is strongly triangulated. We will assume here that for every object X, the obvious map $* \to X$ $(X\to *)$ is an admissible monomorphism (admissible epimorphism, respectively). One defines the tensor product $S\otimes K$ (between a pointed simplicial set S and an object $K\varepsilon C(Mod(\mathfrak{S},\mathcal{R})))$ by taking the $\underset{\Delta}{hocolim}$ of the obvious simplicial object $n \to \underset{S_n}{\oplus} K$; this defines a bi-functor $(pointed \quad simplicial \quad sets)\times Mod_l(\mathcal{C},\mathcal{A}) \to Mod_l(\mathcal{C},\mathcal{A})$. Moreover the remaining axioms on the monoidal and t-structure are satisfied so that $C(Mod(\mathfrak{S},\mathcal{R}))$ is an enriched unital symmetric monoidal t-category. If $\mathcal{A}$ is a sheaf of differential graded algebras over a site $\mathfrak{S}$, the category of sheaves of modules over $\mathcal{A}$ is a strongly triangulated category. It is neither monoidal nor has a strong t-structure in general. We proceed to establish that, similarly, if $\mathcal{A}$ is a sheaf of differential graded algebras over an E^∞-operad on a site $\mathfrak{S}$, the category of sheaves of modules over $\mathcal{A}$ is strongly triangulated. However, in general, the category of modules over such an E^∞ sheaf of DGAs is neither monoidal nor has a strong t-structure. *These observations make it necessary to consider this case separately in the next section. Observe that, since $C(Mod(\mathfrak{S},\mathcal{R}))$ is an enriched unital monoidal category, many of the techniques from the last sections carry over with minor modifications.*

3. Sheaves of algebras and modules over operads

Let $\mathfrak{S}$ denote a site and let $\mathcal{R}$ denote a sheaf of commutative Noetherian rings on $\mathfrak{S}$. For the purposes of this introduction to operads in $C(Mod(\mathfrak{S},\mathcal{R}))$ we will let $\otimes$ denote $\underset{\mathcal{R}}{\otimes}$. (See [K-M] for more details.)

Recall that an *(algebraic) operad* $\mathcal{O}$ in $Mod(\mathfrak{S};\,\mathcal{R})$ is given by a sequence $\{\mathcal{O}(k)|k\geq 0\}$ of differential graded objects in $Mod(\mathfrak{S};\,\mathcal{R})$ along with the following data:

for every integer $k\geq 1$ and every sequence $(j_1,...,j_k)$ of non-negative integers so that $\Sigma_s j_s = j$ there is given a map $\gamma_k: \mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes...\otimes\mathcal{O}(j_k)\to\mathcal{O}(j)$ so that the following associativity diagrams commute, where $\Sigma j_s = j$ and $\Sigma i_t = i$; we set $g_s = j_1 + ... + j_s$ and $h_s = i_{g_{s-1}+1} + ... + i_{g_s}$ for $1 \le s \le k$:

(3.0.20)
$$\begin{tikzcd}
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \arrow[r, "\gamma\otimes id"'] \arrow[dd, "shuffle"'] & \mathcal{O}(j) \otimes (\overset{j}{\underset{r=1}{\otimes}} \mathcal{O}(i_r)) \arrow[d, "\gamma"'] \\
 & \mathcal{O}(i) \\
\mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(j_s)) \otimes (\overset{j_s}{\underset{q=1}{\otimes}} \mathcal{O}(i_{g_{s-1}+q})) \arrow[r, "id\otimes(\otimes_s \gamma)"] & \mathcal{O}(k) \otimes (\overset{k}{\underset{s=1}{\otimes}} \mathcal{O}(h_s)) \arrow[u, "\gamma"']
\end{tikzcd}$$

In addition one is provided with a unit map $\eta : \mathcal{R} \to \mathcal{O}(1)$ so that the diagrams

$$\begin{tikzcd}
\mathcal{O}(k) \otimes (\mathcal{R}^{\overset{k}{\otimes}}) \arrow[r, "\cong"'] \arrow[d, "id\otimes\eta^k"'] & \mathcal{O}(k) \\
\mathcal{O}(k) \otimes \mathcal{O}(1)^{\overset{k}{\otimes}} \arrow[ur, "\gamma"'] &
\end{tikzcd}
\text{ and }
\begin{tikzcd}
\mathcal{R} \otimes \mathcal{O}(j) \arrow[r, "\cong"'] \arrow[d, "\eta\otimes id"'] & \mathcal{O}(j) \\
\mathcal{O}(1) \otimes \mathcal{O}(j) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

commute.

An A^∞-operad is an associative operad $\{\mathcal{O}(k)|k\}$ so that each $\mathcal{O}(k)$ is *acyclic.* A *symmetric operad* is an associative operad as above provided with an action by the symmetric group Σ_k on each $\mathcal{O}(k)$ so that the above diagrams are equivariant with respect to the actions by the appropriate symmetric groups. (See [K-M] p. 13.) An E^∞-operad is an A^∞-operad $\{\mathcal{O}(k)|k\}$ which is also symmetric so that, in addition, the given action of Σ_k on each $\mathcal{O}(k)$ is free.

REMARK 3.1. An operad of pointed simplicial sets, topological spaces, Gamma spaces or symmetric spectra is defined similarly with the following important changes: we replace $C(Mod(\mathfrak{S}, \mathcal{R}))$ with the category of pointed simplicial sets, Gamma spaces or symmetric spectra. Observe that these are all (unital) monoidal categories. The objects $\{\mathcal{O}(k)|k\}$ will be a sequence of objects in this category satisfying similar hypotheses. Now all of the discussion in this section applies with minor changes: for example the term differential graded object will need to be replaced by an object in one of the above categories. In particular such a discussion will define A^∞ and E^∞ objects in the category of Gamma spaces or symmetric spectra.

REMARK 3.2. Next observe that if $\mathcal{O}'$ is an operad of topological spaces (as above), by applying the singular functor followed by the free-abelian-group-functor one may convert it to an operad which will be a chain complex of abelian groups. One may now tensor the resulting complex with $\mathcal{R}$ to obtain an algebraic operad in the above sense.

An A^∞-*differential graded algebra* $\mathcal{A}$ over an A^∞-operad $\mathcal{O}$ is an object in $C(Mod_r(\mathfrak{S}; \mathcal{R}))$ provided with maps $\theta : \mathcal{O}(j) \otimes \mathcal{A}^j \to \mathcal{A}$ for all $j \ge 0$ that are associative and unital in the sense that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^j \arrow[r, "\gamma \otimes id"'] \arrow[dd, "shuffle"'] & \mathcal{O}(j) \otimes \mathcal{A}^j \arrow[d, "\theta"'] \\
 & A \\
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^{j_1} \otimes ... \otimes \mathcal{O}(j_k) \otimes \mathcal{A}^{j_k} \arrow[r, "id \otimes \theta^k"'] & \mathcal{O}(k) \otimes \mathcal{A}^k \arrow[u, "\theta"']
\end{tikzcd}$$

and
$$\begin{tikzcd}
\mathcal{R} \otimes \mathcal{A} \arrow[r, "\simeq"'] \arrow[d, "\eta \otimes id"'] & \mathcal{A} \\
\mathcal{O}(1) \otimes \mathcal{A} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

If $\mathcal{A}$ is an A^{∞} algebra over an operad $\mathcal{O}$ as above, one defines a left $\mathcal{A}$-module M to be an object in $C(Mod_r(\mathfrak{S}; \mathcal{R}))$ provided with maps $\lambda : \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \to M$ satisfying similar associativity and unital conditions. Right-modules are defined similarly.

An E^{∞} algebra over an E^{∞}-operad $\mathcal{O}$ is an A^{∞} algebra over the associative operad $\mathcal{O}$ so that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(j) \otimes \mathcal{A}^{\otimes^j} \arrow[rr, "\sigma \otimes \sigma^{-1}"'] \arrow[dr, "\theta"'] & & \mathcal{O}(j) \otimes \mathcal{A}^{\otimes^j} \arrow[dl, "\theta"] \\
 & \mathcal{A} &
\end{tikzcd}$$

Given an E^{∞}-algebra $\mathcal{A}$ over a commutative operad $\mathcal{O}$, an E^{∞} left-module M over $\mathcal{A}$ is an A^{∞} left-module M so that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \arrow[rr, "\sigma \otimes \sigma^{-1} \otimes id"'] \arrow[dr, "\lambda"'] & & \mathcal{O}(j) \otimes \mathcal{A}^{j-1} \otimes M \arrow[dl, "\lambda"] \\
 & M &
\end{tikzcd}$$

If $\mathcal{A}$ denotes either an A^{∞} or E^{∞}-algebra in $\mathcal{C}$, the category of all left modules (right modules) over $\mathcal{A}$ will be denoted $Mod_l(\mathfrak{S}; \mathcal{A})$ ($Mod_r(\mathfrak{S}; \mathcal{A})$, respectively).

One may now observe the following. For each integer, let $\mathcal{R}[\Sigma_n] = \underset{\Sigma_n}{\oplus} \mathcal{R}$ denote the sum of $\mathcal{R}$ indexed by the symmetric group Σ_n. Now one may define the structure of a monoid on $\mathcal{R}[\Sigma_n]$ as follows:

let $\mathcal{R}_g$ denote the copy of $\mathcal{R}$ indexed by $g \varepsilon \Sigma_n$. Now we map $\mathcal{R}_g \otimes \mathcal{R}_h$ to $\mathcal{R}_{g.h}$ by the given map $\mu : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$.

If $\mathcal{O}$ is a commutative operad in $Mod(\mathfrak{S}; \mathcal{R})$, one may now observe that each $\mathcal{O}(k)$ is a right-module over the monoid $\mathcal{R}[\Sigma_k]$. (Observe that $\mathcal{O}(k) \otimes \mathcal{R}[\Sigma_k] \cong \underset{g \varepsilon \Sigma_k}{\oplus} \mathcal{O}(k) \otimes \mathcal{R}_g$. We map $\mathcal{O}(k) \otimes \mathcal{R}_g$ to $\mathcal{O}(k) \otimes \mathcal{R}$ by the map $g \otimes id$. Now apply the given map $\mathcal{O}(k) \otimes \mathcal{R} \to \mathcal{O}(k)$.)

Finally observe that if $\mathcal{O}$ is an operad as above, the structure in (3.0.20) with $k = 1$ and $j = 1$ shows $\mathcal{O}(1)$ is a differential graded algebra. Moreover $\mathcal{O}(2) \varepsilon Mod_r(\mathfrak{S}; \mathcal{O}(1))$ as well by letting the second factor in $\mathcal{O}(1)^{\otimes^2}$ act trivially on $\mathcal{O}(2)$.

We may assume without loss of generality that the operad $\{\mathcal{O}(i)|i\}$ is obtained from the linear isometries operad as in [K-M] p. 130. In this case, we make the following additional observations.

3.1. (i) there exist maps $\sigma : \mathcal{O}(2) \rightarrow \mathcal{O}(1)$ and $\tau : \mathcal{O}(1) \rightarrow \mathcal{O}(2)$ in $Mod_r(\mathfrak{S}; \mathcal{O}(1))$

(ii) there exist homotopies $H : \Delta[1] \otimes \mathcal{O}(2) \rightarrow \mathcal{O}(2)$ in $Mod_r(\mathfrak{S}; \mathcal{O}(1))$ and $K : \Delta[1] \otimes \mathcal{O}(1) \rightarrow \mathcal{O}(1)$ in $Mod_r(\mathfrak{S}; \mathcal{O}(1))$ so that $H \circ d_0 = \tau \circ \sigma$, $H \circ d_1 = id_{\mathcal{O}(2)}$, $K \circ d_0 = \sigma \circ \tau$ and $K \circ d_1 = id_{\mathcal{O}(1)}$

(iii) there exist augmentations $\eta : \mathcal{R} \rightarrow \mathcal{O}(i)$ and $\nu : \mathcal{O}(i) \rightarrow \mathcal{R}$ so that the compositions $\nu \circ \eta$ and $\eta \circ \nu$ are chain homotopic to the identity. (In case $i = 1$, we assume these are maps of sheaves of differential graded algebras.)

Now we may define a (non-unital) symmetric-monoidal structure $\boxtimes$ (called the *operadic tensor product* on the category $Mod(\mathfrak{S}; \mathcal{O}(1))$ by :

$$M \boxtimes N = \mathcal{O}(2) \underset{\mathcal{O}(1)^2}{\otimes} M \underset{\mathcal{R}}{\otimes} N, \quad M, \quad N \varepsilon Mod(\mathcal{C}; \mathcal{O}(1)) \tag{3.1.1}$$

(Observe that $M \otimes N$ belongs to $Mod_l(\mathfrak{S}; \mathcal{O}(1))$ using the left-module structure of $\mathcal{O}(2)$ over $\mathcal{O}(1)$.) See [K-M] p. 101 for a proof that this defines a symmetric monoidal structure on $C(Mod(\mathfrak{S}, \mathcal{O}(1)))$. We will let $\mathcal{C}$ denote the monoidal category $Mod(\mathfrak{S}, \mathcal{O}(1))$ provided with the operadic tensor product.

REMARK 3.3. One may now define all the functors in (2.0.14) and (1.2.3) in this context if $\mathcal{A}$ is an algebra in *the monoidal category* $\mathcal{C}$. However, since $\mathcal{R}$ is *not* in general, a unit for the functor $\boxtimes$, one will *not* obtain the isomorphisms $M \underset{\mathcal{A}}{\boxtimes} \mathcal{A} \cong \mathcal{M}$, $M \varepsilon Mod_r(\mathfrak{S}; \mathcal{A})$, $N \cong \mathcal{A} \underset{\mathcal{A}}{\boxtimes} N$, $N \varepsilon Mod_l(\mathfrak{S}; \mathcal{A})$ and similarly $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, N) \cong N$. We correct this problem by defining the following functors.

REMARK 3.4. Observe that $\mathcal{O}(1)$ is a DGA provided with augmentations $\mathcal{R} \rightarrow \mathcal{O}(1)$ and $\mathcal{O}(1) \rightarrow \mathcal{R}$ whose composition is the identity. A sheaf of modules M over $\mathcal{O}(1)$ is a *unital* $\mathcal{O}(1)$-module if there is an augmentation $\mathcal{R} \rightarrow M$ compatible with the augmentation of $\mathcal{O}(1)$. It is shown in [K-M] pp. 112-113 that the category of unital $\mathcal{O}(1)$-modules may be provided with a bi-functor (which is a variant of the operadic tensor product) with respect to which it is symmetric monoidal. *A commutative monoid in this category now corresponds to an E^{∞}-algebra over the operad $\{\mathcal{O}(k)|k\}$.*

Let $\mathcal{A}$ denote an algebra in the category $\mathcal{C}$ as above. Recall that $\mathcal{R}[0]$ is not, in general, a unit for the bi-functor $\boxtimes$ provided on $\mathcal{C}$. Let $M \varepsilon \mathcal{C}$. Now we may define $\mathcal{A} \triangleleft M$ by the pushout:

$$\begin{array}{ccc} \mathcal{R} \boxtimes M & \xrightarrow{i \otimes id} & \mathcal{A} \boxtimes M \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \phi} \\ M & \longrightarrow & A \triangleleft M \end{array}$$

One may define $M \triangleright \mathcal{A}$ similarly by interchanging the $\mathcal{A}$ and M. Moreover the above definition applies to any algebra in $\mathcal{C}$. Therefore it applies in particular to the algebra $\mathcal{R}[0]$. It should be clear from the above definition that $\mathcal{R}[0] \triangleleft M \cong M \cong M \triangleright \mathcal{R}[0]$. We let $Mod_l(\mathcal{C}, \mathcal{A})$ ($Mod_r(\mathcal{C}, \mathcal{A})$) denote the full sub-category of $\mathcal{C} = Mod(\mathfrak{S}, \mathcal{O}(1))$ of left-modules (right-modules, respectively) over $\mathcal{A}$. By identifying chain-homotopy classes of maps we obtain the (additive) homotopy categories associated to $Mod_l(\mathcal{C}, \mathcal{A})$ and $Mod_r(\mathcal{C}, \mathcal{A})$. We use the same cohomology functors $\mathcal{H}^n$ to define quasi-isomorphisms in $Mod_l(\mathcal{C}, \mathcal{A})$ and $Mod_r(\mathcal{C}, \mathcal{A})$. (The Abelian category $\mathbf{A}$ in (STR3) is simply the category $Mod(\mathfrak{S}, \mathcal{R})$.) Observe that both $Mod_l(\mathcal{C}, \mathcal{A})$ and $Mod_r(\mathcal{C}, \mathcal{A})$ are closed under the formation of the mapping cylinder, mapping cone, the co-cylinders and the canonical homotopy fibers. Since quasi-isomorphisms are defined as above, one may readily see that the axioms (STR0) through (STR7.3) for a strongly triangulated category are satisfied. (One defines the tensor product $S \otimes K$ (between a pointed simplicial set S and an object $K \varepsilon C(Mod(\mathfrak{S}, \mathcal{R}))$) as in Remark 2.10; this defines a bi-functor $(pointed \quad simplicial \quad sets) \times Mod_l(\mathcal{C}, \mathcal{A}) \to Mod_l(\mathcal{C}, \mathcal{A})$. As in remark 3.2, Chapter I, one may use this observation to construct functorial cylinder and co-cylinder objects.) Therefore one may define the homotopy category associated to these categories in the obvious manner. The derived categories $D(Mod_l(\mathcal{C}, \mathcal{A}))$ and $D(Mod_r(\mathcal{C}, \mathcal{A}))$ are defined by inverting maps in the homotopy categories that are quasi-isomorphism. Next we may define the *free*-functor

$$F_{\mathcal{A},l} : \mathcal{C} \to Mod_l(\mathcal{C}, \mathcal{A}) \quad (F_{\mathcal{A},r} : \mathcal{C} \to Mod_r(\mathcal{C}, \mathcal{A})) \quad by \quad F_{\mathcal{A},l}(M) = \mathcal{A} \triangleleft M \\ (F_{\mathcal{A},r}(M) = M \triangleright \mathcal{A}, respectively) \tag{3.1.2}$$

DEFINITION 3.5. Let $\mathcal{C}$ be as before. We define an internal hom in $\mathcal{C}$ as an adjoint to $\boxtimes$ exactly as in 2.0.15. This will be denoted $\mathcal{H}om_{\mathcal{C}}$. We say an object F (P) in $\mathcal{C}$ is *flat* (*locally projective*) if for every acyclic object $K \varepsilon \mathcal{C}$, $F \boxtimes K$ ($\mathcal{H}om_{\mathcal{C}}(P, K)$, respectively) is acyclic.

PROPOSITION 3.6. *Let $F_{\mathcal{O}(1)} : C(Mod(\mathfrak{S}, \mathcal{R})) \to Mod(\mathfrak{S}, \mathcal{O}(1))$ be the functor defined by $F_{\mathcal{O}(1)}(M) = \mathcal{O}(1) \underset{\mathcal{R}}{\otimes} M$. Then $F_{\mathcal{O}(1)}$ is right adjoint to the forgetful functor $For : Mod(\mathfrak{S}, \mathcal{O}(1)) \to C(Mod(\mathfrak{S}, \mathcal{R}))$. Moreover the following conditions are satisfied*

(i) if $K \varepsilon \mathcal{C}$, the natural map $F_{\mathcal{O}(1)}(For(K)) \to K$ is an epimorphism

(ii)$\mathcal{O}(1)$ is flat with respect to the operadic tensor product $\boxtimes$

(iii) if $K \varepsilon \mathcal{C}$ and $M \varepsilon C(Mod(\mathfrak{S}, \mathcal{R}))$, $K \boxtimes F_{\mathcal{O}(1)}(M)$ $(F_{\mathcal{O}(1)}(M) \boxtimes K)$ is naturally chain-homotopy equivalent to $K \underset{\mathcal{R}}{\otimes} M$ $(M \underset{\mathcal{R}}{\otimes} K)$.

(iv)$\mathcal{H}om_{\mathcal{C}}(F_{\mathcal{O}(1)}(L), K)$ is homotopy equivalent to $\mathcal{H}om_{\mathcal{R}}(L, K)$, for every $K \varepsilon \mathcal{C}$ and $L \varepsilon C(Mod(\mathfrak{S}; \mathcal{R}))$, with the homotopy equivalence being natural in L and K

PROOF. The epimorphism in the first statement is induced by the epimorphism $\mathcal{O}(1) \to \mathcal{R}$ and is therefore obvious. If $K \varepsilon \mathcal{C}$, $K \boxtimes \mathcal{O}(1)$ is chain homotopy equivalent to K. (See 3.1.) Therefore, if K is acyclic, so is $K \boxtimes \mathcal{O}(1)$. This proves (ii). Similar observations prove (iii).

Let $K', K \varepsilon \mathcal{C}$ and let $L \varepsilon C(Mod(\mathfrak{S}, \mathcal{R}))$. Now $Hom_{\mathcal{C}}(K', \mathcal{H}om_{\mathcal{C}}(F_{\mathcal{O}(1)}(L), K)) \cong Hom_{\mathcal{C}}(K' \boxtimes \mathcal{O}(1) \underset{\mathcal{R}}{\otimes} L, K) \simeq Hom_{\mathcal{C}}(K' \underset{\mathcal{R}}{\otimes} L, K)$ where the last is a chain homotopy equivalence. Making use of the fact that $\mathcal{R}$ is commutative and that any map of $\mathcal{O}(1)$-modules is a map of $\mathcal{R}$-modules, one may show the last term is clearly isomorphic to $Hom_{\mathcal{C}}(K', \mathcal{H}om_{\mathcal{R}}(L, K))$. Since this holds for all $K' \varepsilon \mathcal{C}$, it follows from lemma (3.8) below that $\mathcal{H}om_{\mathcal{C}}(F_{\mathcal{O}(1)}(L), K)$ is chain homotopy equivalent to $\mathcal{H}om_{\mathcal{R}}(L, K)$. This proves (iv). □

PROPOSITION 3.7. *(i) $F_{\mathcal{A},l}$ $(F_{\mathcal{A},r})$ is left-adjoint to the forgetful functor $For : Mod_l(\mathcal{C}, \mathcal{A}) \to \mathcal{C}$
$(For : Mod_r(\mathcal{C}, \mathcal{A}) \to \mathcal{C})$ and the following conditions are satisfied:*

(ii) if $M \varepsilon Mod_l(\mathcal{C}, \mathcal{A})$ $(N \varepsilon Mod_r(\mathcal{C}, \mathcal{A}))$, the natural map $F_{\mathcal{A},l}(For(M)) \to M$

$(F_{\mathcal{A},r}(For(N)) \to N)$ is an epimorphism

(iii) if $M \varepsilon Mod_r(\mathcal{C}, \mathcal{A})$ and $K \varepsilon \mathcal{C}$, $M \underset{\mathcal{A}}{\otimes} F_{\mathcal{A},l}(K)$ is naturally isomorphic to $K \boxtimes M$ while $\mathcal{H}om_A(F_{\mathcal{A},l}(K), M)$ is naturally isomorphic to $\mathcal{H}om_{\mathcal{C}}(K, M)$.

(iv)$F_{\mathcal{A},l}(\mathcal{O}(1) \otimes K)$ is naturally isomorphic in the derived category to $\mathcal{A}\underset{\mathcal{R}}{\otimes}K$,

$K\varepsilon C(Mod(\mathfrak{S},\mathcal{R}))$.

(v) $F_{\mathcal{A}}(\mathcal{O}(1))$ and $\mathcal{A}$ are flat and locally projective in $Mod_l(\mathcal{C};\mathcal{A})$

(vi) $\mathcal{H}om_{\mathcal{A}}(F_{\mathcal{A},l}(F_{\mathcal{O}(1)}(K)), L)$ is chain homotopy equivalent to $\mathcal{H}om_{\mathcal{R}}(K, For(L))$,

$L\varepsilon Mod_l(\mathcal{C};\mathcal{A})$ and $K\varepsilon C(Mod(\mathfrak{S};\mathcal{R}))$ with the chain homotopy-equivalence being natural in L and K and preserving the obvious filtrations.

Proof. The first two statements follow from the observation that the composition $\mathcal{R}\triangleleft M \to \mathcal{A}\triangleleft M \to M$ ($N\triangleright\mathcal{R} \to N\triangleright\mathcal{A} \to N$) is the identity if $M\varepsilon Mod_l(\mathcal{C},\mathcal{A})$ ($N\varepsilon Mod_r(\mathcal{C},\mathcal{A})$, respectively).

One may obtain the first assertion in (iii) as follows. Take $P = F_{\mathcal{A},l}(K)$, $K\varepsilon\mathcal{C}$ in the last adjunction in (1.2.6) to obtain the isomorphism

$$Hom_{\mathcal{C}}(M\underset{\mathcal{A}}{\otimes}F_{\mathcal{A},l}(K), N) \cong Hom_{Mod_l(\mathfrak{S},\mathcal{A})}(F_{\mathcal{A},l}(K), \mathcal{H}om_{\mathcal{C}}(M,N)).$$

By (i) this is isomorphic to $Hom_{\mathcal{C}}(K, \mathcal{H}om_{\mathcal{C}}(M,N)) \cong Hom_{\mathcal{C}}(M \boxtimes K, N)$. Since this holds for all $N\varepsilon\mathcal{C}$, one obtains a natural isomorphism $M\underset{\mathcal{A}}{\otimes}F_{\mathcal{A},l}(K) \cong M \boxtimes K$. One obtains the second assertion in (iii) similarly.

Next we consider (iv) assuming (vi). By (vi) $\mathcal{H}om_{\mathcal{A}}(F_{\mathcal{A},l}(F_{\mathcal{O}(1)}(K)), N)$ is chain-homotopy equivalent to $\mathcal{H}om_{\mathcal{R}}(K, N)$. On the other hand, if $K = j_{U!}j_U^*(\mathcal{R})$ for an object U in the site $\mathfrak{S}$, we see that $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}\underset{\mathcal{R}}{\otimes}K, N) \simeq \mathcal{H}om_{\mathcal{R}}(K,N)$ by chain homotopy equivalences that are natural in K. In general, one may find a resolution of the given K by a complex each term of which is a sum of terms of the form $j_{U!}j_U^*(\mathcal{R})$, $U\varepsilon\mathfrak{S}$. Therefore (iv) follows.

Take $K = \mathcal{O}(1)$ in (iii) to see that $M\underset{\mathcal{A}}{\otimes}F_{\mathcal{A},l}(\mathcal{O}(1))$ is chain homotopy equivalent to $M \boxtimes \mathcal{O}(1)$; therefore, if $M\varepsilon Mod_r(\mathcal{C},\mathcal{A})$ is acyclic, so is $M\underset{\mathcal{A}}{\otimes}F_{\mathcal{A},l}(\mathcal{O}(1))$. This shows that $F_{\mathcal{A},l}(\mathcal{O}(1))$ is flat. Now one observes that $\mathcal{A}$ is chain homotopy equivalent to $\mathcal{A} \triangleleft \mathcal{O}(1) = F_{\mathcal{A},l}(\mathcal{O}(1))$. Therefore $\mathcal{A}$ is also flat. Finally observe that $\mathcal{H}om_{\mathcal{A}}(F_{\mathcal{A},l}(F_{\mathcal{O}(1)}(K)), L) \cong \mathcal{H}om_{\mathcal{C}}(F_{\mathcal{O}(1)}(K), L) \simeq \mathcal{H}om_{\mathcal{R}}(K, L)$ which proves (vi) and the assertion on the local projectivity in (v). □

Lemma 3.8. *Let $\mathbf{A}$ and $\mathbf{B}$ denote two categories of chain complexes of abelian sheaves on a site $\mathfrak{S}$. Let $F, F' : \mathbf{A} \to \mathbf{B}$ ($G, G' : \mathbf{B} \to \mathbf{A}$) denote two functors so that F (F') is left-adjoint to G (G', respectively). Let $\phi : F \to F'$ and $\psi : F' \to F$ denote two natural transformations so that the composition $\psi \circ \phi$ ($\phi \circ \psi$) is homotopy-equivalent to the identity natural transformation id_F ($id_{F'}$, respectively). Assume that $F(\Delta[1] \otimes K) \cong \Delta[1] \otimes F(K)$, and similarly $F'(\Delta[1] \otimes K) \cong \Delta[1] \otimes F'(K)$ for $K\varepsilon\mathbf{A}$.*

Let $\phi^* : G' \to G$ and $\psi^* : G \to G'$ denote the two induced natural transformations. Then there exists an induced homotopy equivalence $\psi^* \circ \phi^* \simeq id_{G'}$ ($\phi^* \circ \psi^* \simeq id_G$, respectively)

Proof. This is straightforward □

Corollary 3.9. *Assume the situation as above. Then the categories $Mod_l(\mathcal{C},\mathcal{A})$ and $Mod_r(\mathcal{C},\mathcal{A})$ are strongly triangulated categories.*

Proof. The proof is more or less clear from the above discussion. We begin with the observation that the category $C(Mod(\mathfrak{S},\mathcal{R}))$ is clearly a strongly triangulated category.

Now we obtain a free functor:

$$\mathcal{F} : C(Mod(\mathfrak{S}, \mathcal{R})) \to Mod_l(\mathcal{C}, \mathcal{A}) \tag{3.1.3}$$

by $\mathcal{F}(M) = F_{\mathcal{A},l}(F_{\mathcal{O}(1)}(M))$ and similarly $\mathcal{F} : C(Mod(\mathfrak{S}, \mathcal{R})) \to Mod_r(\mathcal{C}, \mathcal{A})$ by $\mathcal{F}(N) = F_{\mathcal{A},r}(F_{\mathcal{O}(1)}(N))$. These are now left adjoint to the obvious forgetful functors. Using these and the observation that $C(Mod(\mathfrak{S}, \mathcal{R}))$ has a small family of generators, one concludes that so do $Mod_l(\mathcal{C}, \mathcal{A})$ and $Mod_r(\mathcal{C}, \mathcal{A})$. For this observe that the compositions $\mathcal{R} \triangleleft M \to \mathcal{A} \triangleleft M \to M$ and $N \triangleright \mathcal{R} \to N \triangleright N \to N$ are the identity if $M \varepsilon Mod_l(\mathcal{C}, \mathcal{A})$ and $N \varepsilon Mod_r(\mathcal{C}, \mathcal{A})$: now the same argument as in (2.0.15) applies. Clearly $Mod_l(\mathcal{C}, \mathcal{A})$ and $Mod_r(\mathcal{C}, \mathcal{A})$ are closed under all small limits. As observed earlier, these categories satisfy all the axioms (STR0) through (STR7.3) for a strongly triangulated category.

In order to prove the remaining hypotheses (STR8.1) through (STR8.4) it suffices to show that *lemma 2.5 as well as Propositions 2.7 and 2.8 again hold in this setting once we replace $j^{\#}_{U!} j^*_U \Sigma^n(\mathcal{S}) \otimes \mathcal{A}$ with $\mathcal{F}(j^{\#}_{U!} j^*_U(\mathcal{R}))$. This is clear for Lemma (2.5) and Proposition 2.7 by the above propositions. Observe that the functor Q in this context is the identity.* Now the isomorphism
$Hom_{D(Mod_l(\mathcal{C},\mathcal{A}))}(K, L) \overset{\simeq}{\to} \pi_0(Map_{\mathcal{A}}(P(K), \mathcal{G}L))$ is clear by the same argument as in Proposition 2.8. The functor $Map_{\mathcal{A}} : Mod_l(\mathcal{C}, \mathcal{A})^{op} \times Mod_l(\mathcal{C}, \mathcal{A}) \to$ (*pointed simplicial sets*) is defined so that we obtain the isomorphism:

$$Hom_{pointed \quad simpl \quad sets}(S, Map_{\mathcal{A}}(M, N)) \cong Hom_{Mod_l(\mathcal{C},\mathcal{A})}(S \otimes M, N) \tag{3.1.4}$$

) $\square$

DEFINITION 3.10. In the above situation we will denote the category $Mod_l(\mathcal{C}, \mathcal{A})$ ($Mod_r(\mathcal{C}, \mathcal{A})$) by $Mod_l(\mathfrak{S}; \mathcal{A})$ ($Mod_r(\mathfrak{S}; \mathcal{A})$, respectively).

REMARK 3.11. Observe that the category $C(Mod(\mathfrak{S}, \mathcal{R}))$ satisfies all the axioms (ST1) through (ST8) on the strong t-structure with $\tau_{\leq n}$ denoting a familiar functor that kills the cohomology in degrees above n. One lets the functor EM_n in this context be defined by $EM_n(\bar{M}) = \bar{M}[-n]$. Observe also that if $\bar{A} = \prod_i \bar{A}(i)$ is a sheaf of graded modules in $Mod(\mathfrak{S}, \mathcal{R})$, one may define $GEM(\bar{A}) = \prod_i EM_i(\bar{A}(i)) = \prod_i \bar{A}(i)[i]$ and $GEM(\bar{M}) = \prod_i EM_i(\bar{M}(i))$. Now $GEM(\bar{A})$ is a sheaf of differential graded algebras in $C(Mod(\mathfrak{S}, \mathcal{R}))$ which one may view as a sheaf of algebras over an operad in a trivial manner. Moreover if $\bar{M} \varepsilon Mod_l(\bar{A})$, $GEM(\bar{M}) \varepsilon Mod_l(GEM(\bar{A}))$. One may now see readily that $C(Mod(\mathfrak{S}, \mathcal{R}))$ is an *enriched monoidal t-category.* However, the category $Mod(\mathfrak{S}, \mathcal{O}(1))$ is not unital though otherwise symmetric monoidal with the operadic tensor product and clearly the axioms on the strong t-structure do not hold here. Therefore $Mod(\mathfrak{S}, \mathcal{O}(1))$ is not an enriched monoidal t-category. Similarly $Mod_l(\mathfrak{S}, \mathcal{A})$ and $Mod_r(\mathfrak{S}, \mathcal{A})$ are also not enriched monoidal t-categories, if $\mathcal{A} \varepsilon C(Mod(\mathfrak{S}, \mathcal{R}))$ is an E^∞-algebra over an E^∞-operad. *Nevertheless the observation that the category $C(Mod(\mathfrak{S}, \mathcal{R}))$ is an enriched symmetric monoidal t-category enables us to consider the categories $Mod_l(\mathfrak{S}, \mathcal{A})$ and $Mod_r(\mathfrak{S}, \mathcal{A})$ without difficulty in the next chapter.*

4. Presheaves with values in a strongly triangulated symmetric monoidal category

As one of the last examples, we will establish the following theorem.

THEOREM 4.1. *Let $\mathcal{C}$ denote a strongly triangulated monoidal category and let $\mathfrak{S}$ denote a site as in section1. Let $\mathcal{S}$ denote the unit for the tensor structure on $\mathcal{C}$. Assume further that the hypothesis (2.1.1.*) is satisfied. Now the category $Presh_{\mathcal{C}}(\mathfrak{S})$ of presheaves on $\mathfrak{S}$ with values in $\mathcal{C}$ is also a strongly triangulated monoidal category. In case $\mathcal{C}$ is an enriched* *monoidal t-category, the category $Presh_{\mathcal{C}}(\mathfrak{S})$ is also an enriched monoidal t-category. In case $\mathcal{C}$ is symmetric monoidal, so is $Presh_{\mathcal{C}}(\mathfrak{S})$.*

REMARK 4.2. We will establish below that both the categories of symmetric spectra and Γ-spaces are enriched symmetric monoidal t-categories. It will follow as a consequence that the corresponding categories of presheaves on a site (as in section 1) are also enriched symmetric monoidal t-categories.

PROOF. Observe that if $\{P_\alpha|\alpha\}$ is a diagram of presheaves indexed by a small category I, the colimit $\lim\limits_{\overrightarrow{I}} P_\alpha$ (the limit $\lim\limits_{I} P_\alpha$) is the presheaf defined by $\Gamma(U, \lim\limits_{\overrightarrow{I}} P_\alpha) = \lim\limits_{\overrightarrow{I}} \Gamma(U, P_\alpha)$ ($\Gamma(U, \lim\limits_{I} P_\alpha) = \lim\limits_{I} \Gamma(U, P_\alpha)$, respectively). It follows that the category $Presh_{\mathcal{C}}(\mathfrak{S})$ is closed under all small colimits and limits. One defines a pairing $\otimes : Presh_{\mathcal{C}}(\mathfrak{S}) \times Presh_{\mathcal{C}}(\mathfrak{S}) \rightarrow Presh_{\mathcal{C}}(\mathfrak{S})$ by $\Gamma(U, P' \otimes P) = \Gamma(U, P') \otimes \Gamma(U, P)$. One may verify that this functor is symmetric monoidal with the constant presheaf $\underline{\mathcal{S}}$ associated to $\mathcal{S}$ as a unit. Next we define a pairing $(pointed \quad simplicial \quad sets) \times Presh_{\mathcal{C}}(\mathfrak{S}) \rightarrow Presh_{\mathcal{C}}(\mathfrak{S})$ by $\Gamma(U, K \otimes P) = K \otimes \Gamma(U, P)$.

We let $f \simeq g$ be the equivalence relation of homotopy defined on the morphisms of $Presh_{\mathcal{C}}(\mathfrak{S})$ and generated by the following: if $H : \Delta[1]_+ \otimes P \rightarrow P'$ is a map, then H defines a homotopy between $H \circ (d_0 \otimes id)$ and $H \circ (d_1 \otimes id)$ where $d_i : P \cong \Delta[0]_+ \otimes P \rightarrow \Delta[1]_+ \otimes P$ is the obvious face map. The resulting homotopy category is denoted $HPresh_{\mathcal{C}}(\mathfrak{S})$.

Given a map $f : P' \rightarrow P$, we may define $Cyl(f)$ ($Cone(f)$) by $\Gamma(U, Cyl(f)) = Cyl(\Gamma(U, f))$ ($\Gamma(U, Cone(f)) = Cone(\Gamma(U, f))$, respectively). One defines $Cocyl(f)$ and $fib_h(f)$ similarly. We define the functor $T(f)$ by $\Gamma(U, T(f)) = \Gamma(U, \Sigma(f))$ where Σ is defined as in Chapter I, Definition (2.3). A diagram $P' \rightarrow P \rightarrow P'' \rightarrow TP'$ is a triangle if it is isomorphic in the homotopy category to a diagram of the form: $P' \xrightarrow{f} P \rightarrow Cone(f) \rightarrow \Sigma(f)$.

Let $\{\mathcal{H}^n|n\}$ denote the cohomology functor on the category $\mathcal{C}$. Now we define a cohomology functor $\{\mathcal{H}^n|n\}$ on $Presh_{\mathcal{C}}(\mathfrak{S})$ by letting $\mathcal{H}^n$ denote the sheaf associated to the presheaf $P \rightarrow \mathcal{H}^n(\Gamma(U, P))$, U in the site $\mathfrak{S}$. We define a map $f : P' \rightarrow P$ to be a quasi-isomorphism if the induced maps $\mathcal{H}^n(f)$ are all isomorphisms. The following lemma shows that $HPresh_{\mathcal{C}}(\mathfrak{S})$ admits a calculus of left and right fractions.

LEMMA 4.3. *The class of maps in $HPresh_{\mathcal{C}}(\mathfrak{S})$ that are quasi-isomorphisms admits a calculus of left and right fractions*

PROOF. Let Qis denote the class of maps in $HPresh_{\mathcal{C}}(\mathfrak{S})$ that are quasi-isomorphisms. Recall that Qis admits a calculus of left fractions, if the following hold:

(i) Qis is closed under finite compositions and contains all the maps that are the identities

(ii) Given a diagram $X_2 \xleftarrow{q} X_1 \xrightarrow{f} X_3$ in $Presh_{\mathcal{C}}(\mathfrak{S})$ with q in Qis, there exists a diagram $X_2 \xrightarrow{g} X_4 \xleftarrow{q'} X_3$ in $Presh_{\mathcal{C}}(\mathfrak{S})$ with q' in Qis so that the square:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_3 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ X_2 & \xrightarrow{g} & X_4 \end{array}$$

commutes.

(iii) Given $X_1 \xrightarrow{q} X_2 \underset{g}{\overset{f}{\rightrightarrows}} X_3$ with q in Qis with $f \circ q = g \circ q$, there exists a map $X_3 \xrightarrow{q'} X_4$ so that $q' \circ f = q' \circ g$ and $q' \varepsilon Qis$.

Now (i) is clear. In order to prove (ii), one may first replace the map $q : X_1 \to X_2$ by the induced map $X_1 \to Cycl(q)$ and $f : X_1 \to X_3$ by the corresponding map $X_1 \to Cyl(f)$. Now take $X_4 = X_2 \underset{X_1}{\sqcup} X_3$. The natural map $X_2/X_1 \to X_4/X_3$ is now an isomorphism. Since $X_1 \to X_2 \to X_2/X_1 \to TX_1$ is a triangle and the homology functor $\mathcal{H}^*$ takes it long-exact sequences with $\mathcal{H}^*(X_1) \overset{\mathcal{H}^*(q)}{\to} \mathcal{H}^*(X_2)$ an isomorphism, it follows that $\mathcal{H}^*(X_4/X_3) \cong \mathcal{H}^*(X_2/X_1) \cong 0$. Now $X_3 \to X_4 \to X_4/X_3 \to \Sigma X_3$ is a strong triangle and therefore the map q' is a quasi-isomorphism.

In order to prove (iii), one may assume once again that q is also replaced by the corresponding map $X_1 \to Cyl(q)$. Let $H : \Delta[1]_+ \otimes X_1 \to X_3$ denote a homotopy between the two maps $f \circ q$ and $g \circ q$. Now let *Spool* denote the direct limit of the diagram

$$\begin{tikzcd}
\Delta[0] \otimes X_2| & & \Delta[1]_+ \otimes X_1 & & \Delta[0] \otimes X_2 \\
& \Delta[0] \otimes X_1 \arrow[ul, "id \otimes q"] \arrow[ur, "d_0 \otimes id"'] & & \Delta[0] \otimes X_1 \arrow[ul, "d_1 \otimes id"] \arrow[ur, "id \otimes q"'] &
\end{tikzcd}$$

and let Cyl denote $\Delta[1]_+ \otimes X_2$. Now one may observe readily that the obvious map $Spool \to Cyl$ is a quasi-isomorphism. (To see this: observe that $Spool \to Cyl \to \Sigma(X_2/X_1) \to \Sigma Spool$ is a strong-triangle and that the map $q : X_1 \to X_2$ is a quasi-isomorphism. It follows that $\mathcal{H}^*(\Sigma(X_2/X_1)) \cong 0$ which proves the map $Spool \to Cyl$ is also a quasi-isomorphism.) Now let X_4 be defined by the pushout square:

$$\begin{tikzcd}
Spool \arrow[r] \arrow[d] & X_3 \arrow[d] \\
Cyl \arrow[r] & X_4
\end{tikzcd}$$

The top row is defined by the two maps f, g and the homotopy H. Now the induced map $X_3 \to X_4$ is also a monomorphism and the natural map $Cyl/Spool \to X_4/X_3$ is an isomorphism. It follows that the induced map $X_3 \to X_4$ is also a quasi-isomorphism. These arguments prove that Qis admits a calculus of left fractions. The proof that it also admits a calculus of right fractions is similar using co-cylinders instead. $\square$

Observe that $\underline{S}$ is an algebra in $Presh_{\mathcal{C}}(\mathfrak{S})$. Taking $\mathcal{A} = \underline{S}$, in Proposition 2.7, one may produce cofibrant resolutions for presheaves. A presheaf $P \varepsilon Presh_{\mathcal{C}}(\mathfrak{S})$ will be called *cofibrant* if it is locally projective and flat. We define the functor $Q_{\mathcal{C}}$ on $Presh_{\mathcal{C}}(\mathfrak{S})$ by $\Gamma(U, Q_{\mathcal{C}}P) = Q(\Gamma(U, P))$ where the Q on the right is the functor as in (STR8.1) for $\mathcal{C}$.

A presheaf P (as above) is fibrant if the obvious map $\Gamma(U, P) \to \Gamma(U, \mathcal{G}Q_{\mathcal{C}}P)$ is a quasi-isomorphism for each U in the site $\mathfrak{S}$. (We also let the functor $Q = \mathcal{G} \circ Q_{\mathcal{C}}$.) Now the axiom Chapter I, (STR8.3) may be verified as in Proposition 2.7. One may also readily verify the axiom Chapter I, (M5). One defines admissible monomorphisms (epimorphisms) to be maps $F' \to F$ so that for each stalk, the induced map is an admissible monomorphism (epimorphism, respectively). In order to prove that the derived category is additive, observe in view of (STR8.3) that it suffices to show the homotopy category is additive. More specifically observe that if $f, g : \Sigma P(X) \to \Omega\Sigma Y$ are two maps, their *sum* in $Hom_{HPresh_{\mathcal{C}}(\mathfrak{S})}(P(X), Y)$ is given by the composition:

$$\Sigma P(X) \cong S^1 \otimes P(X) \overset{\nabla}{\to} (S^1 \sqcup S^1) \otimes P(X) \cong (\Sigma P(X)) \sqcup (\Sigma P(X)) \overset{f \sqcup g}{\to} \Omega\Sigma Y.$$

Observe $\Omega\Sigma Y$ is a homotopy associative monoid with the operation induced by the map $S^1 \to S^1 \sqcup S^1$. Now [Sp] p. 43 shows that the above sum is commutative. i.e. $Hom_{D(\mathcal{C})}(X, Y)$ is an *Abelian group* for all X and Y. It follows that the derived category $D(\mathcal{C})$ is additive.

We leave the verification that the triangles defined above in fact satisfy the axioms for distinguished triangles.

One may take the sub-category $\mathfrak{F}$ to be the same as cofibrant objects. The remaining axioms in Chapter I, (M0) through (M5) are easily verified. We have thereby shown that if $\mathcal{C}$ is a strongly triangulated symmetric monoidal category, so is $Presh_{\mathcal{C}}(\mathfrak{S})$.

Next assume that $\mathcal{C}$ satisfies the axioms on the strong t-structure as well. We define functors $\tau_{\leq n}$ on $Presh_{\mathcal{C}}(\mathfrak{S})$ by $\Gamma(U, \tau_{\leq n}P) = \tau_{\leq n}\Gamma(U, P)$. Now the axioms in Chapter I, (ST1) and (ST2) are clear. Observe that $D(Presh_{\mathcal{C}}(\mathfrak{S}))^{\leq n \geq} = \{X \varepsilon D(Presh_{\mathcal{C}}(\mathfrak{S})) | \mathcal{H}^i(X) = 0 \quad if \quad i \neq n\}$. Let $\mathbf{A}_{\mathcal{C}}$ denote the Abelian category which is equivalent to the heart of $D(\mathcal{C})$. We let $\mathbf{A}$ = the category of sheaves on $\mathfrak{S}$ with values in $\mathbf{A}_{\mathcal{C}}$. We define a functor $EM'_n : \mathbf{A} \rightarrow Presh_{\mathcal{C}}(\mathfrak{S})$ by $\Gamma(U, EM'_n(F)) = EM^{\mathcal{C}}_n(\Gamma(U, F))$ where $EM^{\mathcal{C}}_n$ is the functor in Chapter I, (ST3) associated to the category $\mathcal{C}$ as part of its strong t-structure. We let $EM_n = \mathcal{G} \circ EM'_n$. Now it is clear that the axioms in Chapter I on the strong t-structure are satisfied. □

4.0.5. *Presheaves with values in enriched stable model categories.* With a view to further possible applications, (see Chapter V and Appendix A) we show that the following set of axioms on a category $\mathcal{C}$ imply the category is strongly triangulated.

Definition 4.4. (Stable simplicial model categories) A category $\mathcal{C}$ is called a *stable simplicial model category* if it satisfies the following axioms (PM0) through (PM4), (M4), (SM0) through (SM3.4), (SM4) through (SM6), and the axioms (HCl), (Hl) along with (cofinality) on the homotopy limits and colimits.

We will assume that $\mathcal{C}$ has a zero object $*$ and that it is closed under all small colimits and limits. *Sums in the category $\mathcal{C}$ will be denoted $\sqcup$.* We will further assume that filtered colimits in $\mathcal{C}$ commute with finite limits.

(PM0) A *partial model structure on $\mathcal{C}$* is provided by three classes of maps called *weak-equivalences*, *cofibrations* and *fibrations* satisfying the following conditions:

(PM1) The class of fibrations is stable under compositions and base change; any isomorphism is a fibration. The class of cofibrations is stable under compositions and co-base change; any isomorphism is a cofibration. Moreover any retract of a fibration (a cofibration, a weak-equivalence) is a fibration (a cofibration, a weak-equivalence respectively).

(PM2) Any isomorphism is a weak-equivalence. If f and g are maps in $\mathcal{C}$ so that $g \circ f$ is defined and two of the maps f, g or $g \circ f$ are weak-equivalences, so is the third.

Any map that is both a fibration and a weak-equivalence (a cofibration and a weak-equivalence) will be called a trivial fibration (trivial cofibration, respectively).

(PM3) Any map f can be factored as $f = p \circ i$ with p a fibration, i a trivial cofibration and both depending functorially on f. Any map f can also be factored as $f = p \circ i$ with p a trivial fibration, i a cofibration and both depending functorially on f.

(PM4) Every cofibration in $\mathcal{C}$ is a monomorphism. (The converse is not assumed to be true. In particular, not every object in $\mathcal{C}$ need be cofibrant.)

A model category structure on $\mathcal{C}$ is a partial model category structure satisfying the axioms (PM1) through (PM3) and also satisfying the following *lifting axiom*:

(M4) For every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

in $\mathcal{C}$, there is a map $h : B \to X$ making the two triangles commute provided either

(a) i is a trivial cofibration and p is a fibration or

(b) i is a cofibration and p is a trivial fibration.

Such a model structure is *closed* if the fibrations (cofibrations) are characterized by the lifting property in (a) ((b), respectively) and a map is a weak-equivalence if and only if it can be factored as the composition of a trivial cofibration and a trivial fibration. (We will always assume this is the case and omit the adjective closed henceforth.) It is a *simplicial model structure* if one has a bi-functor $Map : \mathcal{C}^{op} \times \mathcal{C} \to (pointed \quad simplicial \quad sets)$ so that $Map_0 = Hom_{\mathcal{C}}$. Moreover we require the following. For each fixed $M \varepsilon \mathcal{C}$, the functor $N \mapsto Map(M, N)$, $\mathcal{C} \to (pointed \quad simplicial \quad sets)$ has a left adjoint which will be denoted $- \otimes M$.

A *stable simplicial model category structure* on $\mathcal{C}$ is provided by two structures:

(SM0) a *simplicial model structure* on $\mathcal{C}$ where the cofibrations (fibrations, weak-equivalences) are called *strict cofibrations* (*strict fibrations*, *strict weak-equivalences*, respectively) as well as another simplicial model structure (where the fibrations (cofibrations, weak-equivalences) are called *stable* fibrations (*stable* cofibrations, *stable* weak-equivalences, respectively)) so that the conditions (SM1) through (SM7) are satisfied:

(SM1) every strict weak-equivalence is a stable weak-equivalence

(SM2) every stable fibration (stable cofibration) is a strict fibration (strict cofibration, respectively)

(SM3.1) There exist two functors $Q : \mathcal{C} \to \mathcal{C}$ and $Q^{st} : \mathcal{C} \to \mathcal{C}$ along with natural transformations $id \to Q$, $Q \circ Q \to Q$ and $id \to Q^{st}$, $Q^{st} \circ Q^{st} \to Q^{st}$ so that if $X \varepsilon \mathcal{C}$, $Q(X)$ is strictly fibrant while $Q^{st}(X)$ is stably fibrant.

(SM3.2) The maps $X \to Q(X)$, $Q(Q(X)) \to Q(X)$ ($X \to Q^{st}(X)$, $Q^{st} \circ Q^{st}(X) \to Q^{st}(X)$) are required to be strict weak-equivalences (stable weak-equivalences respectively).

(SM3.3) The functor Q (Q^{st}) preserves strict fibrations (stable fibrations, respectively).

(SM3.4) We will also require that the two functors $\otimes : (pointed \quad simplicial \quad sets) \times \mathcal{C} \to \mathcal{C}$ defined as part of the simplicial model structure for the strict and stable model structures coincide. (Observe, as a consequence, that the two functors Map associated to the strict and stable simplicial model structures also coincide.) Moreover the following are assumed to hold: if K is a pointed simplicial set, $K \otimes -$ preserves strict and stable cofibrations as well as strict and stable weak-equivalences. If $M \varepsilon \mathcal{C}$ is a stably cofibrant object of $\mathcal{C}$, $- \otimes M$ sends cofibrations of simplicial sets to stable cofibrations and weak-equivalences to stable weak-equivalences. It is also required to commute with colimits in either argument.

Observe that the axiom (PM3) implies the existence of functorial cylinder and cocylinder objects for the partial model structure - see [Qu] chapter I. (Using the simplicial structure, it is possible to define them explicitly in the usual manner as well.)

4.0.6. Now we define cylinder and cocylinder objects using the strict and stable model structures as follows - see [Qu] chapter I. Let $X \sqcup X \overset{d_0+d_1}{\rightarrow} Cyl_{strict}(X) \overset{s}{\rightarrow} X$ denote a factorization of the obvious map $\nabla : X \sqcup X \to X$ as the composition of a strict cofibration followed by a strict weak-equivalence. We will call $Cyl_{strict}(X)$ an *strict cylinder object* for X. A cylinder object defined using the stable model structure will be denoted $Cyl_{st}(X)$. Let $\Delta[1]$ denote the obvious simplicial set; $\Delta[1]_+$ will denote this with an extra base point added. If $X \varepsilon \mathcal{C}$, we let $Cyl_{can}(X) = \Delta[1]_+ \otimes X$ with $d_i : X \cong \Delta_+[0] \otimes X \to \Delta[1]_+ \otimes X$, the obvious map for $i = 0, 1$ and $s : \Delta_+[1] \otimes X \to \Delta[0]_+ \otimes X \cong X$ the obvious maps. We call $Cyl_{can}(X)$ *the canonical cylinder object* for X. If the object X is strictly cofibrant, this is in fact a strict cylinder object for X, and it is a *stable cylinder object* for any X that is stably cofibrant. (These conclusions follow readily since the bi-functor $\otimes$ is considered part of the stable structure.)

4.0.7. Let $f : X \to Y$ denote a map in $\mathcal{C}$; now we let $Cyl_{strict}(f) = Cyl_{strict}(X)\underset{X}{\sqcup}Y$ and call it a *strict mapping cylinder of* f. $Cyl_{can}(f) = Cyl_{can}(X)\underset{X}{\sqcup}Y$ will be called the *canonical mapping cylinder of* f. Similarly we let $Cyl_{st}(f) = Cyl_{st}(X)\underset{X}{\sqcup}Y$ and call it the *stable mapping cylinder of* f. Observe that the canonical mapping cylinder will be a stable mapping cylinder if X and Y are stably cofibrant. We will often denote any one of the above mapping cylinders generically by $Cyl(f)$.

(1.1.3) Now the map induced by d_1, $X \to Cyl(f)$ is a strict cofibration (stable cofibration if X and Y are strictly cofibrant (stably cofibrant, respectively). (This map will be denoted d_1 henceforth.) The pushout

$$\begin{tikzcd} X \arrow[r, "d_1"] \arrow[d] & Cyl(f) \arrow[d] \\ * \arrow[r] & Cone(f) \end{tikzcd}$$

defines the mapping cone $Cone(f)$. This is strictly cofibrant (stably cofibrant) if X and Y are strictly cofibrant (if X and Y are stably cofibrant, respectively). The mapping cone defined using the canonical (strict, stable) mapping cylinder will be denoted $Cone_{can}(f)$ ($Cone_{strict}(f)$, $Cone_{st}(f)$, respectively). If both X and Y are strictly (stably) cofibrant, this coincides with $Cone_{strict}(f)$ ($Cone_{st}(f)$, respectively).

4.0.8. Next we consider the dual notion of a co-cylinder object. Let $X \to Cocyl^{strict}(X) \to X \times X$ denote the factorization of the diagonal as the composition of a strict weak-equivalence and a strict fibration. We call $Cocyl^{strict}(X)$ a strict co-cylinder of X. A co-cylinder object defined similarly using the stable model structure will be called a stable co-cylinder object of X and will be denoted $Cocyl^{st}(X)$. We will let $Cocyl_{can}(X) = X^{\Delta[1]_+}$ with the map $d^0 \times d^1 : X^{\Delta[1]_+} \to X^{\Delta[0]_+ \sqcup \Delta[0]_+} \cong X \times X$ and $s : X \cong X^{\Delta[0]_+} \to X^{\Delta[1]_+}$ the obvious maps. This will be a strict (stable) co-cylinder object for X if X is strictly fibrant (stably fibrant, respectively). Once again these conclusions follow readily from the assumption that the bi-functor Map is considered part of the stable structure.

4.0.9. Let $f : X \to Y$ denote a map in $\mathcal{C}$. We let $Cocyl^{strict}(f) = Cocyl^{strict}(Y)\underset{Y}{\times}X$ and call it a *strict mapping co-cylinder for* f. The corresponding functor defined using $Cocyl^{st}$ will be called a *stable mapping co-cylinder for* f. Finally $Cocyl^{can}(f) = Cocyl^{can}(Y)\underset{Y}{\times}X$ will be called the *canonical mapping co-cylinder for* f: this will be a strict (stable) mapping co-cylinder for f if X and Y are strictly fibrant (stably fibrant, respectively). We will denote any one of the above mapping co-cylinders generically by $Cocyl(f)$.

4.0.10. The map induced by d_1, $Cocyl(f) \rightarrow Y$ is a strict (stable) fibration if X and Y are strictly (stably, respectively) fibrant. (This map will be denoted d^1 henceforth.) The pull-back

$$\begin{tikzcd}
fib_h(f) \arrow[r] \arrow[d] & Cocyl(f) \arrow[d, "d_1"'] \\
* \arrow[r] & Y
\end{tikzcd}$$

defines the homotopy-fiber of f. This is strictly fibrant (stably fibrant) if X and Y are strictly fibrant (stably fibrant, respectively). The homotopy fiber defined using the canonical (strict, stable) co-cylinder for f will be denoted $fib_h^{can}(f)$ ($fib_h^{strict}(f)$, $fib_h^{st}(f)$, respectively).

Now we will require several axioms that will ensure that the derived category associated to the stable model structure is in fact an additive category. Let $P^{st} : \mathcal{C} \rightarrow \mathcal{C}$ denote the functor of stably cofibrant approximation as in (PM3); i.e. the obvious map $* \rightarrow X$ factors as $* \rightarrow P^{st}(X) \rightarrow X$ with $P^{st}(X)$ stably cofibrant and the map $P^{st}(X) \rightarrow X$ a stable weak-equivalence.

Let $f : X \rightarrow Y$ denote a map in $\mathcal{C}$. Let $i : fib_h^{can}(Qf) \rightarrow QX$ and $p : Y \rightarrow Cone_{can}(f)$ denote the obvious maps. Now

(SM4) $Cone_{can}(f)$ is stably equivalent to $Cone_{st}(P^{st}(f))$ always. If $f : X \rightarrow Y$ is a monomorphism, there exists a stable weak-equivalence $Cone_{can}(f) \simeq Coker(f)$ where $Coker(f)$ denotes the cokernel of f.

(SM5) $Cone_{can}(i)$ is naturally stably weakly equivalent to QY and hence Y as well

(SM5)' $fib_h^{can}(Qp)$ is naturally stably weakly equivalent to QX.

(SM6) $fib_h^{st}(Q^{st}(f))$ is naturally stably weakly equivalent to $Q^{st}(fib_h^{can}(Q(f)))$. (Observe that this axiom implies that the functor Q^{st} preserves stable fibration sequences.)

(SM6)' $Cone_{st}(P^{st}(f))$ and $Cone_{st}(f)$ are naturally weakly-equivalent to $Cone_{can}(f)$.

Remark 4.5. (SM4) along with the axioms above imply that if $f : X \rightarrow Y$ is a monomorphism, $Coker(f)$ is stably weakly equivalent to $Cone_{st}(f)$ and also to $Cone_{st}(P^{st}(f))$ where $P^{st}(f)$ is defined as above. Moreover the above axioms imply that a strict fibration (cofibration) sequence when viewed as a diagram in the stable model category on $\mathcal{C}$ may be identified with a stable fibration (cofibration, respectively) sequence. This is true in the setting of both Γ-spaces as we show in detail in section 5. (To see this simply observe any monomorphism of Γ-spaces induces a monomorphism of the associated spectra. Therefore, it is possible to replace any monomorphism of Γ-spaces by a stable co-fibration up-to natural stable weak-equivalence.) Moreover this facilitates work with stable fibration and cofibration sequences and enables us to obtain the spectral sequences in sections 3 and 4.

Definition 4.6. We define stable cofibration sequences in $\mathcal{C}$ to be diagrams $T' \rightarrow T \rightarrow T'' \rightarrow \Sigma T'$ that are isomorphic in the homotopy category $H\mathcal{C}^{st}$ (see below) to diagrams of the form: $T' \xrightarrow{i} T \rightarrow Cone_{st}(i) \rightarrow \Sigma T'$. One may define stable fibration sequences in $\mathcal{C}$ to be diagrams $\Omega T'' \rightarrow T' \rightarrow T \xrightarrow{f} T''$ that are isomorphic in $H\mathcal{C}$ to diagrams of the form: $\Omega T'' \rightarrow fib_h^{st}(Q(f)) \rightarrow Q(T) \xrightarrow{f} Q(T'')$.

Remark 4.7. In view of the axioms above, one may identify stable cofibration sequences with stable fibration sequences.

Axioms on homotopy limits and colimits. We will conclude this list of hypotheses by axiomatizing the existence of homotopy colimits and homotopy limits of small diagrams in $\mathcal{C}$ with suitable properties. For this we invoke standard material from [B-K]. Let I denote a small category; for each object $i\varepsilon I$, I/i $(I\backslash i)$ will denote the nerve of the comma category denoted I/i ($I\backslash i$, respectively) also. We let $(\mathcal{C})^{I^o}$ $((\mathcal{C})^I)$ denote the category of all contravariant (covariant, respectively) functors from I taking values in $\mathcal{C}$. We define the functor $\underset{I}{\text{hocolim}} : (\mathcal{C})^{I^o} \rightarrow \mathcal{C}$ exactly as a co-end in Chapter I. Now the above functor is left-adjoint to the functor $\mathcal{H}om(I/-\otimes\mathcal{S}, -) : \mathcal{C} \rightarrow (\mathcal{C})^{I^o}$ that sends an object $K\varepsilon\mathcal{C}$ to the simplicial object $\{\mathcal{H}om(I/n \otimes \mathcal{S}, K)|n\}$. ($\mathcal{H}om$ is the internal Hom in the category $\mathcal{C}$.) We require the following hypotheses:

(HCl):there exists a simplicial model structure on $(\mathcal{C})^{I^o}$ with weak-equivalences being stable weak-equivalences in $\mathcal{C}$ in each simplicial degree so that $\underset{I}{\text{hocolim}}$ sends weak-equivalences to stable weak-equivalences. Moreover $\underset{I}{\text{hocolim}}$ sends diagrams $\{A' \rightarrow A \rightarrow A'' \rightarrow \Sigma A'\}$ in $(\mathcal{C})^{I^o}$ that are triangles in $\mathcal{C}$ in each simplicial degree to a triangle in $\mathcal{C}$. In addition, we require, in case $I = \Delta$ (so that $(\mathcal{C})^{I^o}$ = the category of simplicial objects in $\mathcal{C}$) that there exist a spectral sequence:

$$E^2_{s,t} = H_s(\{\pi_t(S_n)|n\}) \Rightarrow \pi_{s+t}(\underset{\Delta}{\text{hocolim}}\{S_n|n\})$$

(The homotopy groups are defined below.) We define the functor $\underset{I}{\text{holim}} : (\mathcal{C})^I \rightarrow \mathcal{C}$ as an end in Chapter I. Now the above functor is right-adjoint to the functor $I/-\otimes : \mathcal{C} \rightarrow (\mathcal{C})^{I^o}$ that sends an object $K\varepsilon\mathcal{C}$ to the diagram $\{I/n \otimes K|n\}$. We require the following hypotheses:

(Hl):there exists a simplicial model structure on $(\mathcal{C})^I$ with weak-equivalences being stable weak-equivalences in $\mathcal{C}$ in each cosimplicial degree so that $\underset{I}{\text{holim}} \circ Q^{st}$ sends weak-equivalences to stable weak-equivalences. Moreover $\underset{I}{\text{holim}} \circ Q^{st}$ sends diagrams $\{\Omega A'' \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow\}$ in $(\mathcal{C})^I$ that are triangles in $\mathcal{C}$ in each degree to a triangle in $\mathcal{C}$. Moreover we require that, in case $I = \Delta$ (so that $(\mathcal{C})^I$ = the category of cosimplicial objects in $\mathcal{C}$) there exist a spectral sequence with $E_2^{s,t} = H^s(\{\pi_t(C^n)|n\}) \Rightarrow \pi_{-s+t}(\underset{I}{\text{holim}}\, C^{\cdot})$. The $E_2^{s,t}$-term is the s-th (co-)homology of the cosimplicial Abelian group $\{\pi_t(C^n)|n\}$.

In addition we will require the following axiom that enables one to compare two homotopy inverse limits or colimits.

Let I denote a small category and let $f : I \rightarrow J$ denote a covariant functor. We say f is *left-cofinal* if for every object $j\varepsilon J$, the nerve of the obvious comma-category f/j is contractible. Now let $F : J \rightarrow \mathcal{C}$ be a functor.

(cofinality). We require that the induced map $\underset{J}{\text{holim}}\, F \rightarrow \underset{I}{\text{holim}}\, F \circ f$ is a stable weak-equivalence if the functor f is left-cofinal.

REMARK 4.8. The hypothesis that $\mathcal{C}^I$ and $\mathcal{C}^{I^{op}}$ are simplicial model categories is satisfied if the category $\mathcal{C}$ is a cofibrantly generated simplicial model category.

The strict homotopy category. Let X, $Y\varepsilon\mathcal{C}$. By (functorially) factoring the map $* \rightarrow X$ we may find a strict weak-equivalence $P(X) \rightarrow X$ with $P(X)$ cofibrant. Let Q denote the functor as in (SM3.1). Now we let $Hom_{\mathcal{C}^{strict}}(X, Y) = \pi_0(Map(P(X), Q(Y))$. It follows readily from the axioms of the (strict) simplicial model category structure that this depends only on X and Y. One defines the strict homotopy category $H\mathcal{C}^{strict}$ to have the same objects as $\mathcal{C}$, but where the morphisms are defined as above.

4.0.11. We define the stable homotopy category $H\mathcal{C}^{st}$ by

$$Hom_{H\mathcal{C}^{st}}(X,Y) = Hom_{H\mathcal{C}^{strict}}(P^{st}(X), Q^{st}Y)$$

and with the same objects as $\mathcal{C}$. (Recall the right hand side $= \pi_0(Map(P^{st}(X), Q^{st}Y)$. Recall also that the functor Map associated to the strict and stable simplicial model structures coincide.)

Now the functor Q^{st} sends stable fibrations to stable fibrations and hence preserves stable fibration sequences. Given a stable cofibration sequence as above, one may also obtain a triangle: $P^{st}(X') \to P^{st}(X) \to P^{st}(X'') \to P^{st}(\Sigma X')$. Therefore if we define $Ext^{-n}(X,Y) = Hom_{D(\mathcal{C})}(\Sigma^n X, Y)$, then $\{Ext^{-n}|n\}$ is a cohomological functor from $D(\mathcal{C})$ to the category of Abelian groups sending distinguished triangles in each argument to long exact sequences of Abelian groups. (The derived category is defined by localizing $H\mathcal{C}^{st}$ with respect to stable weak-equivalences.)

Let $K \varepsilon \mathcal{C}$. We define

$$\pi_n(K) = Hom_{H\mathcal{C}^{strict}}(\Sigma^n \mathcal{S}, Q^{st}(K)) \cong Hom_{D(\mathcal{C})}(\Sigma^n \mathcal{S}, K) \tag{4.0.12}$$

where $\mathcal{S}$ is defined in (M3.1). If $K' \overset{\simeq}{\to} K$ is a stable weak-equivalence, the induced maps $\pi_n(K) \to \pi_n(K')$ are all isomorphisms. Moreover if $K' \to K \to K'' \to \Sigma K'$ is a stable cofibration sequence, one obtains a long-exact sequence:

$$... \to \pi_n(K') \to \pi_n(K) \to \pi_n(K'') \to \pi_{n-1}(K') \to ...$$

These are clear since $\mathcal{S}$ is stably cofibrant and $Q^{st}(L)$ for any L is stably fibrant. We will show below that a map f is a stable weak-equivalence if and only if it induces an isomorphism on π_n for all n.

PROPOSITION 4.9. *(i) Given any object $Z \varepsilon \mathcal{C}$, there exists a collection $\{n_s|s\}$ of integers and a map $\epsilon : \underset{n_s}{\sqcup} \Sigma^{n_s} \mathcal{S} \to Q^{st} Z$ that induces an epimorphism on all π_n. (We use the notation: Σ^{n_s} = the n_s-fold iterate of Σ if $n_s \geq 0$ and = the $-n_s$ fold iterate of Ω if $n_s < 0$.)*

(ii) Given an object $Z \varepsilon \mathcal{C}$, there exists a simplicial object $S(Z)_\bullet$ in $\mathcal{C}$ along with an augmentation $\epsilon : S(Z)_0 \to Q^{st} Z$ so that each term $S(Z)_k$ is of the form in (i) and (ii) $\underset{\Delta}{hocolim}(\epsilon)$ is a stable-weak-equivalence. Moreover $\underset{\Delta}{hocolim} S(Z)_\bullet$ is stably cofibrant.

(iii) $f : X \to Y$ in $\mathcal{C}$ induces an isomorphism on all π_n if and only if f is a stable weak-equivalence.

PROOF. (i) is clear from the definition of π_n. Now we let $S(Z)_0$ to be the term given in (i). (ii) is a special case of Proposition 2.7 where the site $\mathfrak{S}$ is the punctual site and $\mathcal{A} = \mathcal{S}$.

It is clear that if f is a stable weak-equivalence, it induces an isomorphism on all π_n. Therefore, it suffices to prove the converse. Let $S(Z)_\bullet \to Z$ denote a simplicial object chosen as in (ii). Let $P(Z) = \underset{\Delta}{\mathrm{hocolim}} S(Z)_\bullet$. Now consider $Map(P(Z),\ Q^{st}(f)) : Map(P(Z),\ Q^{st}X) \to Map(P(Z),\ Q^{st}Y)$. One may identify this with $\underset{\Delta}{\mathrm{holim}}\, Map(S(Z)_\bullet,\ Q^{st}(f))$: $\underset{\Delta}{\mathrm{holim}}\, Map(S(Z)_\bullet,\ Q^{st}X) \to \underset{\Delta}{\mathrm{holim}}\, Map(S(Z)_\bullet,\ Q^{st}Y)$. Since f induces an isomorphism on all π_n, $Map(S(Z)_n,\ Q^{st}(f))$ is a weak-equivalence for all n; it follows from the hypothesis (Hl) that so is $\underset{\Delta}{\mathrm{holim}}\, Map(S(Z)_\bullet, Q^{st}(f))$. □

THEOREM 4.10. *A stable simplicial model category and the category of presheaves with values in such category define a strongly triangulated category in the sense of Chapter I.*

PROOF. The assertion about the category of presheaves follows exactly as in Theorem (4.1). Therefore we will skip this and prove only the assertion that a stable simplicial model category is a strongly triangulated category. Clearly the axiom (STR0) is implied by the axiom (PM0). The axioms (STR1) through (STR5) are shown to be satisfied by Chapter I, 2.1, 2.2 and Proposition 2.7 with $\mathcal{H}^n = \pi_{-n}$ and with $\mathbf{A}$ = the category of all abelian groups. The admissible mono-morphisms (epi-morphisms) in (STR6) are the stable cofibrations (stable fibrations, respectively). The hypotheses in the axiom (STR6) is implied by the stable simplicial model structure. The cylinder and co-cylinder objects were already defined in (1.1) Chapter I. Now the axioms (STR7.1) through (STR7.3) are clear. We let the functor Q in (STR8.1) be the functor Q^{st} as in (SM4). The cofibrant (fibrant) objects in (STR8.1) and (STR8.2) are the stably cofibrant (stably fibrant ones, respectively) in the sense of the stable model structure. Now the axioms in (STR8.1) through (STR8.4) are implied by the stable model structure and Proposition 2.5 above. □

DEFINITION 4.11. An *enriched stable simplicial model category* $\mathcal{C}$ is a stable simplicial model category provided with a symmetric monoidal bi-functor $\otimes$ satisfying the following:

i) the axioms in Chapter I, (M0) through (M4.6) with the strong triangles defined to be stable cofibration (or equivalently stable fibration) sequences and with the bi-functor $\otimes : (pointed \quad simplicial \quad sets) \times \mathcal{C} \to \mathcal{C}$ in Chapter I, (M4.0) defined by (SM3.4).

ii) the axiom Chapter I, (M5) with $Q = Q^{st}$, $e = Q^{st}$ and $m = P^{st}$.

DEFINITION 4.12. Let $Presh$ denote a category of presheaves on a site so that it falls into any one of the three situations considered in the previous three sections. Let $\mathcal{F}$ denote the free functor defined there. Let $\mathcal{A}$ denote an algebra in $Presh$. Now we define $\mathcal{RH}om_{\mathcal{A}}(M, N) = \underset{\Delta}{\text{holim}}\, \Delta\{\mathcal{H}om_{\mathcal{A}}(P(M)_\bullet, \mathcal{G}Q(N))$ where $P(M)_\bullet \to M$ is a resolution defined as in Proposition 2.4 using the free functor. One defines $RMap_{\mathcal{A}}$ in a similar manner.

CHAPTER III

Homological algebra in enriched monoidal categories

1. Basic Spectral Sequences

In this section we provide several spectral sequences that are crucial for the development of a satisfactory theory of Grothendieck-Verdier duality as in chapter IV.

1.1. Basic Hypotheses. Throughout the remaining chapters, we will assume that the following hypotheses are satisfied:

$\mathfrak{S}$ will denote a site as in Chapter II, section 1 and *either* (i) $Presh$ is an enriched unital monoidal category of presheaves on $\mathfrak{S}$ *or*

(ii) $Presh = C(Mod(\mathfrak{S}, \mathcal{R}))$ for a ringed site $(\mathfrak{S}, \mathcal{R})$ with $\mathcal{R}$ a commutative sheaf of Noetherian rings and that $\mathcal{A}$ is a sheaf of algebras over an operad in $Presh$ in the sense of Chapter II, section 3. (Recall that, in this context, the functor Q as in Chapter I, (STR8.1) is the identity.)

1.2. Terminology. In the situation in (i) we will let $\underline{S}$ denote the unit of the symmetric monoidal structure on $Presh$. $\{\mathcal{H}^n|n\}$ will denote a cohomological functor as in Chapter II, 2.1.1. In this case, if $\mathcal{A}$ is a given algebra in $Presh$, using the observation that the functor Q (in Chapter I, (STR8.1)) is compatible with the monoidal structure, we will replace $\mathcal{A}$ by $Q(\mathcal{A})$ and we will henceforth consider only modules over $Q(\mathcal{A})$ of the form $Q(M)$ for some $M \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ or $M \varepsilon Mod_r(\mathfrak{S}, \mathcal{A})$. *However, we will denote $Q(\mathcal{A})$ by $\mathcal{A}$ and $Q(M)$ by M for simplicity. Moreover, if necessary, by replacing an object $M \varepsilon Mod_l(\mathfrak{S}, Q(\mathcal{A}))$ by $Q(M)$, one may assume that every object in $Mod_l(\mathfrak{S}, \mathcal{A})$ will have a canonical Cartan filtration.* The same applies to $Mod_r(\mathfrak{S}, \mathcal{A})$. $\mathcal{F} : Presh \to Mod_l(\mathfrak{S}, \mathcal{A})$ ($Mod_r(\mathfrak{S}, \mathcal{A})$) will denote the free functor defined by $\mathcal{F}(M) = \mathcal{A} \otimes M$, ($\mathcal{F}(N) = N \otimes \mathcal{A}$, $M, N \varepsilon Presh$. We will let $\otimes$ denote $\underset{\underline{S}}{\otimes}$ and $\mathcal{H}om$ defined as the internal hom in $Presh$. (See Chapter II, (1.2.3).) In the situation in (ii), we let $\mathcal{S} = \mathcal{R}$ and if $\mathcal{A}$ is a sheaf of algebras over an operad $\{\mathcal{O}(k)|k\}$, we let $\mathcal{F}(M) = F_{\mathcal{A},l}(F_{\mathcal{O}(1)}(M))$ ($=F_{\mathcal{A},r}(F_{\mathcal{O}(1)}(N))$) as in Chapter II, section 3. Now $\otimes$ will denote $\underset{\mathcal{R}}{\otimes}$ and $\mathcal{H}om$ will denote $\mathcal{H}om_{\mathcal{R}}$ (which is the internal hom in $C(Mod(\mathfrak{S}, \mathcal{R}))$. Moreover, the functors $\underset{\mathcal{A}}{\otimes} : Mod_l(\mathfrak{S}, \mathcal{A}) \times Mod_r(\mathfrak{S}, \mathcal{A}) \to Presh$, $\mathcal{H}om_{\mathcal{A}} : Mod_l(\mathfrak{S}, \mathcal{A})^{op} \times Mod_l(\mathfrak{S}, \mathcal{A}) \to Presh$ and $\mathcal{H}om_{\mathcal{A}} : Mod_r(\mathfrak{S}, \mathcal{A})^{op} \times Mod_r(\mathfrak{S}, \mathcal{A}) \to Presh$ will denote the ones defined as in Chapter II, section 1. The external hom in the category $Presh$ will be denoted Hom and the functor $T^n : Presh \to Presh$ (where T is as in Chapter I, (STR2)) will be denoted $[n]$.

1.2.1. Throughout, a map $f : P' \to P$ of objects in $Presh$ will be called a quasi-isomorphism, if it induces a quasi-isomorphism of the stalks. This will be denoted $\simeq$.

1.2.2. One may observe readily that, for each object U in the site $\mathfrak{S}$, the object $j^{\#}_{U!} j^*_U(\mathcal{S})$ is a *compact object* in $Presh$ in the sense that giving any map from it to a filtered colimit of objects in $Presh$ is equivalent to giving a map to one of the objects forming the filtered colimit.

REMARK 1.1. Observe that the hypotheses in (i) are in fact satisfied in both the situations considered in Chapter II, sections 2 and 4. We begin with the following spectral sequences and then proceed to obtain a vast generalization of them.

PROPOSITION 1.2. *Assume the above situation. Let $M \varepsilon Mod_r(\mathfrak{S}, \mathcal{A})$ and $N, N' \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$. Then there exist spectral sequences*

$$E^2_{s,t} = \mathcal{T}or^{\mathcal{H}^*(\mathcal{A})}_{s,t}(\mathcal{H}^*(M),\ \mathcal{H}^*(N)) \Rightarrow \mathcal{H}^{-s+t}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N)$$

$$E_2^{s,t} = Ext^{s,t}_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(N), \mathcal{H}^*(N')) \Rightarrow \mathcal{H}^{s+t}(RHom_{\mathcal{A}}(N, N')) \text{ and}$$

$$E_2^{s,t} = \mathcal{E}xt^{s,t}_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(N), \mathcal{H}^*(N')) \Rightarrow \mathcal{H}^{s+t}(\mathcal{RH}om_{\mathcal{A}}(N, N')).$$

The first always converges strongly, while the last two converge conditionally, in general, in the sense of [Board]. The identification of the $E^2_{s,t}$-term ($E_2^{s,t}$-term) is as the t-th graded piece of the s-th Tor (the t-th graded piece of the s-th Ext or $\mathcal{E}xt$, respectively).

PROOF. (See [Qu] chapter II, section 6.8, Theorem 6 and also [K-M] Chapter V for a similar result for simplicial rings.) Let $Q \to N$ denote a quasi-isomorphism with Q locally projective and flat; let $P_\bullet = P(M)_\bullet \to M$ denote a simplicial resolution as in Chapter II, Proposition 2.4. Recall each term $P(M)_n$ is a sum of terms of the form $\mathcal{F}(j^{\#}_{U_s!} j^*_{U_s}(\mathcal{S}))$. Now we consider the first spectral sequence. We consider the simplicial object $P_\bullet \underset{\mathcal{A}}{\otimes} Q$ in *Presh*.

As n varies $\{\mathcal{H}^t(P_n \underset{\mathcal{A}}{\otimes} Q)|n\}$ forms a simplicial Abelian sheaf. We take the homology of this simplicial Abelian sheaf. The required spectral sequence is given by the spectral sequence for the homotopy colimit as in Chapter I:

$$E_2^{s,t} = H_s(\mathcal{H}^t(P_\bullet \underset{\mathcal{A}}{\otimes} Q)) \Rightarrow \mathcal{H}^{-s+t}(\underset{\Delta}{\text{hocolim}}(P_\bullet \underset{\mathcal{A}}{\otimes} Q)).$$

It suffices to identify the abutment with $\mathcal{H}^{-s+t}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N)$ and the E^2-term with $\mathcal{T}or^{\mathcal{H}^*(\mathcal{A})}_{s,t}(\mathcal{H}^*(M)$, $\mathcal{H}^*(N))$. Observe that each term P_n is of the form $\mathcal{F}(P'_n)$ for some P'_n an object in *Presh* which is a sum of terms $j_{U!} j^*_U(\mathcal{S})$. Therefore, $P_n \underset{\mathcal{A}}{\otimes} Q \simeq P'_n \otimes Q \cong \underset{s \varepsilon S}{\sqcup} j_{U_s!} j^*_{U_s}(Q)$ for each n. Now $\mathcal{H}^*(P_n) = \mathcal{H}^*(P'_n \otimes \mathcal{A}) \cong \underset{s \varepsilon S}{\oplus} j_{U_s!} j^*_{U_s} \mathcal{H}^*(\mathcal{A}) \cong \mathcal{H}^*(P'_n) \otimes \mathcal{H}^*(\mathcal{A})$. Therefore, $\mathcal{H}^*(P_n \underset{\mathcal{A}}{\otimes} Q)$ is isomorphic to:

$$\mathcal{H}^*(P_n) \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \mathcal{H}^*(Q).$$

Recall $\mathcal{H}^*(Q) \cong \mathcal{H}^*(N)$ and that $\{\mathcal{H}^*(P_n)|n\}$ is a flat resolution of $\mathcal{H}^*(M)$. Therefore, one obtains the required identification of the E^2-terms. To identify the abutment it suffices to show that $\underset{\Delta}{\text{hocolim}}\{P_\bullet \underset{\mathcal{A}}{\otimes} Q\} \simeq \underset{\Delta}{\text{hocolim}} P_\bullet \underset{\mathcal{A}}{\otimes} Q \simeq M \underset{\mathcal{A}}{\otimes} Q$. The last quasi-isomorphism follows since Q is flat and the first follows from the fact that the homotopy colimit commutes with co-equalizers. This establishes the first spectral sequence.

Next we consider the last two spectral sequences. We begin with the identification:

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F}(j^{\#}_{U_s!} j^*_{U_s}(\mathcal{S})),\ \mathcal{G}^n N) \simeq \mathcal{H}om_{\mathcal{S}}(j^{\#}_{U_s!} j^*_{U_s}(\mathcal{S}),\ \mathcal{G}^n N)$$

$$\simeq \mathcal{H}om_{\mathcal{S}}(j^*_{U_s}(\mathcal{S}),\ j^*_{U_s} \mathcal{G}^n N) \simeq j^*_{U_s} \mathcal{G}^n N)$$

The first identification follows in the situations of Chapter II, sections 2 or 4 by Chapter II, Proposition 2.1 (i) while it follows in the situation of Chapter II, section 3 by Chapter II, Proposition 3.8 (vi). Therefore

$$\mathcal{H}^*(\mathcal{H}om_{\mathcal{A}}(\mathcal{F}(j^{\#}_{U_s!}j^*_{U_s}(\mathcal{S})),\, \mathcal{G}^n N)) \simeq \mathcal{H}^*(j^*_{U_s}\mathcal{G}^n N)$$

$$\simeq \mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(j^{\#}_{U_s!}j^*_{U_s}\mathcal{H}^*(\mathcal{A}),\, \mathcal{H}^*(\mathcal{G}^n N)) \simeq \mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(j^{\#}_{U_s!}j^*_{U_s}\mathcal{H}^*(\mathcal{A})\ ,\, \mathcal{G}^n\mathcal{H}^*(N))$$

Next observe that $\{\mathcal{H}om_{\mathcal{A}}(P(M)_k, \mathcal{G}^n N)|k,n\}$ is a double cosimplicial object; we take its diagonal. The required spectral sequence is simply the spectral sequence for the homotopy limit of the corresponding cosimplicial object. Since

$$\underset{\Delta}{\text{holim}}\, \Delta\{\mathcal{H}om_{\mathcal{A}}(P(M)_k, \mathcal{G}^n N)|k,n\} \simeq \underset{\Delta}{\text{holim}}\, \underset{\Delta}{\text{holim}}\{\mathcal{H}om_{\mathcal{A}}(P(M)_k, \mathcal{G}^n N)|k,n\}$$

where the outer (inner) $\underset{\Delta}{\text{holim}}$ is in the direction of n (k, respectively), the latter identifies with $\mathcal{RH}om_{\mathcal{A}}(M,N)$ and thereby provides the identification of the abutment.

Taking the diagonal of the double cosimplicial Abelian sheaf $\{\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(P(M))_k, \mathcal{G}^n(\mathcal{H}^*(N)))|k,n\}$ provides the term $\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M), \mathcal{H}^*(N))$. Taking the cohomology of this cosimplicial Abelian sheaf one obtains the identification of the E_2-terms. □

2. Stronger spectral sequences

DEFINITION 2.1. Let $L\varepsilon Presh$. A *non-decreasing* filtration on L is given by a collection $\{F_kL|k\}$ of objects in $Presh$ provided with the following structure:

(i) for each k and each U in the site $\mathfrak{S}$, there exist admissible monomorphisms $i_{k,k+1} : j^{\#}_{U!}j^*_U(F_kL) \to j^{\#}_{U!}j^*_U(F_{k+1}L)$ and $i_k : j^{\#}_{U!}j^*_U(F_kL) \to j^{\#}_{U!}j^*_U(L)$ so that $i_k = i_{k+1} \circ i_{k,k+1}$. (Here $j_U : U \to \mathfrak{S}$ is the obvious map associated to an object in the site $\mathfrak{S}$. j^*_U is the restriction to $\mathfrak{S}/U$ and $j^{\#}_{U!}$ is its left adjoint.)

(ii) on taking the direct limits over all neighborhoods of any point p in the site $\mathfrak{S}$, the admissible monomorphisms in (i) induce admissible monomorphisms $i_{k,k+1,p} : i_{p*}i^*_p(F_kL) \to i_{p*}i^*_p(F_{k+1}L)$ and $i_{k,p} : i_{p*}i^*_p(F_kL) \to i_{p*}i^*_p(L)$. (Here i^*_p is the restriction functor from presheaves on the site $\mathfrak{S}$ to presheaves on the point p and i_{p*} is its right adjoint.)

(iii) A non-decreasing filtration $\{F_kL|k\}$ on L as above is *exhaustive* (*complete*) if the natural map
$\underset{k\to\infty}{\text{colim}}\mathcal{H}^n(F_kL) \to \mathcal{H}^n(L)$ (the natural map $\mathcal{H}^n(L) \to \mathcal{H}^n(\underset{-\infty\leftarrow k}{\text{holim}}\, L/F_kL)$, respectively) is an isomorphism of sheaves for all n. Such a filtration is *strongly separated* if for each integer q, there exists an integer N_q so that $\mathcal{H}^q(F_kL) = 0$ for all $k < N_q$. It is *separated* if $L = \underset{\alpha}{\sqcup} L_\alpha$ with each $L_\alpha\varepsilon Presh$ and strongly separated. (If $L\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$, we will in fact require that each summand $L_\alpha\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$.)

(iv) Let $L\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$. A non-decreasing filtration $\{F_kL|k\}$ on L as above is *compatible with the Cartan filtration on* $\mathcal{A}$ if:

$Gr(L) = \underset{k}{\sqcup} F_kL/F_{k-1}L$ belongs to $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$. (Equivalently, the pairing $\mathcal{A}{\otimes}L \to L$ sends $F_i\mathcal{A}{\otimes}F_kL \to F_{i+k}L$ where $\{F_i(A)|i\}$ denotes the Cartan filtration on $\mathcal{A}$ and $\{F_kL|k\}$ denotes the given filtration on L.)

PROPOSITION 2.2. *Let M, $N\varepsilon Presh$ be provided with exhaustive filtrations. Then the induced product filtration on $M \otimes N$ (defined by $F_k(M\otimes N) = Image(\underset{i+j=k}{\sqcup} F_iM \otimes F_jN \to M \otimes N)$) is also exhaustive. It will be separated if either of the two holds:*

- *the given filtration on M is separated and $N = \underset{\alpha}{\sqcup}\Sigma^{n_\alpha}\underline{\mathcal{S}}$ or*
- *the given filtration on N is separated and $M = \underset{\alpha}{\sqcup}\Sigma^{n_\alpha}\mathcal{S}$*

Here $\mathcal{S}\varepsilon Presh$ is the unit as in 1.0.2.

PROOF. Since colimits commute with sums and $\otimes$, it is clear that the induced filtration on $M \otimes N$ is exhaustive. To see it is also separated, one may proceed as follows. Recall $M \otimes \Sigma^n \mathcal{S} = \Sigma^n M$ provided with the induced filtration which is clearly separated if the given filtration on M is separated. One considers the other situation similarly. $\square$

2.0.3. *Convention.* Throughout the remaining sections, the full sub-category of $Mod_l(\mathfrak{S}, \mathcal{A})$ consisting of objects provided with a an exhaustive and separated non-decreasing filtration will be denoted $Mod_l^{filt}(\mathfrak{S}, \mathcal{A})$.

THEOREM 2.3. *Assume as in the above situation that $\mathcal{A}$ is an algebra in $Presh$. Then the following hold:*

Let $M\varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ be provided with a non-decreasing exhaustive filtration.Then there exists a locally projective and flat object P_0'' provided with a non-decreasing exhaustive and filtration compatible with the Cartan filtration on $\mathcal{A}$ and a filtration preserving map $P_0'' \rightarrow M$ which induces a stalk-wise surjection $\mathcal{H}^n(P_0'')_k) \rightarrow \mathcal{H}^n(F_k M)$ for all n and all k. Moreover, P_0'' is locally projective and flat in $Mod_l(\mathfrak{S}, \mathcal{A})$ and the filtration on P_0'' is separated if the given filtration on M is separated.

PROOF. We will first consider the case when $Presh$ is a unital symmetric monoidal category. Let $j_U : U \rightarrow X$ denote an object in the site $\mathfrak{S}$, let n denote an integer and let $M\varepsilon Presh$. Now we will let

$$S(n, U)(M) = Hom_{Presh}(\Sigma^n j_{U!}^{\#} j_U^*(\mathcal{S}), M) \tag{2.0.4}$$

and

$$P_0' = \underset{n\varepsilon\mathbb{Z}}{\sqcup} (\underset{U in \mathfrak{S}}{\sqcup} \underset{S(n,U)(M)}{\sqcup} \Sigma^n j_{U!}^{\#} j_U^*(\mathcal{S})) \tag{2.0.5}$$

with the filtration on it defined by $F_k P_0' = \underset{n\varepsilon\mathbb{Z}}{\sqcup} (\underset{U in \mathfrak{S}}{\sqcup} \underset{S(n,U)(F_k M)}{\sqcup} \Sigma^n j_{U!}^{\#} j_U^*(\mathcal{S}))$. One defines a map $u'_{-1} : P_0' \rightarrow M$ by mapping the summand indexed by $\alpha : \Sigma^n j_{U!}^{\#} j_U^*(\mathcal{S}) \rightarrow M$ to M by the map α. Now u'_{-1} is a map of filtered objects. The definition of the filtration $\{F_k P_0' | k\}$ shows that each

$$\mathcal{H}^n(F_k(u'_{-1})) : \mathcal{H}^n(F_k(P_0')) \rightarrow \mathcal{H}^n(F_k M) \tag{2.0.6}$$

is a surjection for each k and n. Moreover, the filtration on P_0' is exhaustive since each object $j_{U!}^{\#} j_U^*(\mathcal{S})$ was observed to be compact.

Next we let

$$P_0'' = \mathcal{F}(P_0')(= \mathcal{A} \otimes P_0'); \tag{2.0.7}$$

u'_{-1} induces a map $u''_{-1} : P_0'' \rightarrow M$ obtained as the composition $\mathcal{A} \otimes P_0' \overset{id_{\mathcal{A}} \otimes u'_{-1}}{\rightarrow} \mathcal{A} \otimes M \rightarrow M$. We filter P_0'' using the product filtration with the Cartan filtration on $\mathcal{A}$ and the above filtration on P_0'. This is clearly exhaustive; in view of Proposition 2.2 and the observation that the Cartan filtration (on $\mathcal{A}$) is clearly strongly separated, it is also separated. Since the map $\mathcal{S} \otimes P_0' \rightarrow \mathcal{A} \otimes P_0' \rightarrow \mathcal{A} \otimes P_0' \rightarrow P_0'$ is the identity and is filtration preserving, (2.0.6) shows that the induced map $\mathcal{H}^n(F_k(P_0'')) \rightarrow \mathcal{H}^n(F_k M)$ is also surjective for each k and n.

Next we consider the operadic case. In this case we will let

$$S(n, U)(M) = Hom_{Mod(\mathfrak{S};\mathcal{R})}(j_{U!} j_U^*(\mathcal{R})[n], M) \tag{2.0.8}$$

and

$$P_0' = \underset{n\varepsilon\mathbb{Z}}{\oplus} (\underset{U}{\oplus} \underset{S(n,U)(M)}{\oplus} j_{U!} j_U^*(\mathcal{R}))[n] \tag{2.0.9}$$

with the filtration on it defined by $F_kP_0' = \underset{n\varepsilon\mathbb{Z}}{\oplus}(\underset{U}{\oplus} \underset{S(n,U)(F_kM)}{\oplus} \Sigma^n j_{U!}j_U^*(\mathcal{R})[n])$. (Here U varies over a cofinal set of open neighborhoods of every point of X.) One defines a map $u'_{-1}: P_0' \to M$ by mapping the summand indexed by $\alpha: j_{U!}j_U^*(\mathcal{R})[n] \to M$ to M by the map α. Now u'_{-1} is a map of filtered objects. The definition of the filtration $\{F_kP_0'|k\}$ shows that each

$$\mathcal{H}^n(F_k(u'_{-1})) : \mathcal{H}^n(F_k(P_0')) \to \mathcal{H}^n(F_kM) \tag{2.0.10}$$

is a surjection. Next we let

$$P_0'' = F_{\mathcal{A},l}(F_{A(1)}(P_0'))(= \mathcal{A} \triangleleft (\mathcal{O}(1) \otimes P_0')); \tag{2.0.11}$$

u'_{-1} induces a map $u''_{-1}: P_0'' \to M$ obtained as the composition $\mathcal{A}\triangleleft(\mathcal{O}(1)\otimes P_0') \overset{id_{\mathcal{A}}\triangleleft(id_{\mathcal{O}(1)}\otimes id)}{\to} \mathcal{A}\triangleleft(\mathcal{O}(1) \otimes M) \to \mathcal{A} \triangleleft M \to M$. (The last map is the $\mathcal{A}$-module structure on M, while the one before that is the $\mathcal{O}(1)$-module structure on $For(M)\varepsilon Mod(\mathfrak{S}; \mathcal{O}(1))$.) We filter P_0'' using the product filtration with the Cartan filtration on $\mathcal{A}$ and $A(1)$ and the above filtration on P_0'. Since the given filtration on M is compatible with the given filtration on $\mathcal{A}$ and $\mathcal{O}(1)$, it follows that the map $u''_{-1}: P_0'' \to M$ is a map of objects in $Mod_l(\mathfrak{S}; \mathcal{A})$. This filtration will be exhaustive and separated by similar reasons. Moreover, since the composition $P_0' \cong \mathcal{R} \triangleleft P_0' \to \mathcal{A} \triangleleft (\mathcal{O}(1) \otimes P_0') \to P_0'$ is the identity and is also filtration preserving, (2.0.10) shows that the induced map $\mathcal{H}^n(F_k(P_0'')) \to \mathcal{H}^n(F_kM)$ is also surjective for each k and n. □

PROPOSITION 2.4. *Assume the above situation. Let $M\varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ be provided with an exhaustive filtration. Then there exist locally projective and flat objects $P_i\varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$, $i \geq 0$ provided with non-decreasing filtrations $\{F_k(P_i)|i\}$ and maps $d_i: P_i \to P_{i-1}$ in $Mod_l(\mathfrak{S}, \mathcal{A})$, $i \geq 1$, and a map $d_{-1}: P_0 \to M$ so that the following conditions hold:*

(i) for each i, $Gr(P_i)\varepsilon Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$ is locally projective and flat

(ii) the maps d_i preserve the filtrations

(iii) $d_i \circ d_{i+1} = *$

(iv) for each fixed n and k,

$$\dots \overset{\mathcal{H}^n(F_k(d_{i+1}))}{\to} \mathcal{H}^n(F_k(P_i)) \overset{\mathcal{H}^n(F_k(d_i))}{\to} \mathcal{H}^n(F_k(P_{i-1})) \to \dots \overset{\mathcal{H}^n(F_k(d_{-1}))}{\to} \mathcal{H}^n(F_kM)$$

is exact stalkwise.

(v) Moreover, the filtration $\{F_k(P_n)|n\}$ on each P_n is exhaustive and separated.

PROOF. We define P_i' and P_i using ascending induction on i. We will let P_0' be as defined in (2.0.5) or (2.0.9). We let P_0'' as in (2.0.7) or (2.0.11). Next we let

$$P_0 = Cocyl(u''_{-1}) \tag{2.0.12}$$

with the induced map $u_{-1}: P_0 \to M$. We provide P_0 with the induced filtration. i.e. $F_kP_0 = Cocyl(F_k(u''_{-1})) = (F_kM)^I \underset{F_kM}{\times} F_k(P_0'')$. Recall $fib_h(u_{-1}) = fib(u_{-1}) = u_{-1}^{-1}(*)$; this is filtered by $F_kfib(u_{-1}) = fib(F_k(u_{-1}))$. It follows that the induced filtration on P_0 is exhaustive. It is also separated by Proposition 2.2. Let Gr_k denote the associated graded term with respect to the above filtration. Observe that $F_{k-1}P_0 \to F_kP_0 \to Gr_kP_0$ is a triangle in $Presh$ and that there exists a natural quasi-isomorphism $fib_h(Gr_k(u_{-1})) \simeq Gr_k(fib(u_{-1}))$. Furthermore, P_0 is a flat object in $Mod_l(\mathfrak{S}, \mathcal{A})$ which is locally projective; this follows from the fact that P_0 is naturally homotopy equivalent to $P_0'' = \mathcal{F}(P_0')$. Similarly $Gr(P_0)$ is a flat object in $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$ which is locally projective - this follows from the fact that $Gr(P_0)$ is naturally homotopy equivalent to $Gr(P_0'') = \mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P_0'))$. Finally, it follows from the construction that, for each k and n, the induced map of Abelian sheaves $\mathcal{H}^n(F_k(u_{-1})) : \mathcal{H}^n(F_kP_0) \rightarrow \mathcal{H}^n(F_kM)$ is an epi-morphism.

Now repeat the same construction with M replaced by $fib(u_{-1})$ provided with the above filtration. (Strictly speaking one needs to first apply the functor e as in Chapter I, (STR6) to P_0 before proceeding with the construction; however, for the sake of simplicity, we will not mention the functor e explicitly.) This provides an object $P_1 \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ provided with a non-decreasing filtration $\{F_kP_1|k\}$ and a filtered map $u_0 : P_1 \rightarrow fib(u_{-1})$ so that the following hold:

(i) $F_{k-1}P_1 \rightarrow F_kP_1 \rightarrow Gr_kP_1$ is a triangle in $Presh$ for all $k\varepsilon\mathbb{Z}$

(ii) P_1 is a flat and locally projective object in $Mod_l(\mathfrak{S}, \mathcal{A})$; similarly $Gr_{\mathcal{C}}(P_1)$ is a flat and locally projective object in $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$.

(iii) there exists a natural quasi-isomorphism $fib_h(Gr_k(u_0)) \simeq Gr_k(fib_h(u_0))$

(iv) the diagrams $F_k(fib_h(u_0)) = fib_h(F_k(u_0)) \rightarrow F_k(P_1) \overset{F_k(u_0)}{\rightarrow} F_kfib_h(u_{-1})$ are triangles for all k. The same conclusion holds for the diagram: $fib_h(u_0) \rightarrow P_1 \overset{u_0}{\rightarrow} fib_h(u_{-1})$ as well as for $Gr_k(fib_h(u_0)) \simeq fib_h(Gr_k(u_0)) \rightarrow Gr_k(P_1) \overset{Gr_k(u_0)}{\rightarrow} Gr_k(fib_h(u_{-1}))$

(v) for each k and n, the induced map of Abelian sheaves $\mathcal{H}^n(F_k(u_{-1})) : \mathcal{H}^n(F_kP_1) \rightarrow \mathcal{H}^n(F_k(fib_h(u_{-1})))$ is an epi-morphism.

Continuing this way we obtain a collection of flat objects $\{P_n|n \geq 0\}$ in $Mod_l(\mathfrak{S}, \mathcal{A})$ that are flat and locally projective. Moreover, there exists a non-decreasing exhaustive and separated filtration $\{F_kP_n|k\}$ on each P_n so that the above conditions hold with P_n (P_{n-1}) replacing P_1 (P_0, respectively). In this situation, one may now observe the following:

The map $u_i : P_i \rightarrow fib(u_{i-1})$ is the one corresponding to u_{-1} when $fib(u_{i-1})$ (P_i) replaces N (P_0, respectively).

For $i \geq 0$, $d_{i+1} : P_{i+1} \rightarrow P_i$ be the composition $P_{i+1} \overset{u_i}{\rightarrow} fib_h(u_{i-1}) \rightarrow P_i$ and $d_{-1} = u_{-1} : P_0 \rightarrow M$. Now one may readily verify the conditions of the proposition. $\square$

2.1. Let M and $\{P_i|i\}$ be as in the proposition above. First one observes that $\{P_i \overset{d_i}{\rightarrow} P_{i-1}|i \geq 0\}$, $\{F_k(P_i) \overset{F_k(d_i)}{\rightarrow} F_k(P_{i-1}|i \geq 0\}$ and $\{Gr_k(P_i) \overset{Gr_k(d_i)}{\rightarrow} Gr_k(P_{i-1})|i \geq 0\}$ are *complexes* i.e. the composition of the successive differentials is $*$. (This follows from the construction where the map d_i factors through the fiber of u_{i-1} and d_{i-1} is the composition of u_{i-1} and another map. Observe that this is true, though we have omitted the functors e throught the discussion.) Therefore, one may apply the denormalization functor DN to it to obtain a simplicial object $DN(P_\bullet)$ provided with a non-decreasing filtration $\{F_k(DN(P_\bullet))|k\}$ by sub-simplicial objects. Now one may take the homotopy colimits to obtain:

$$\text{(2.1.1)} \qquad \begin{gathered} \underset{\Delta}{\text{hocolim}} DN(P_\bullet) \simeq M, \underset{\Delta}{\text{hocolim}} F_k(DN(P_\bullet)) \simeq F_k(M) \quad \textit{and} \\ \underset{\Delta}{\text{hocolim}} Gr_k(DN(P_\bullet)) \simeq F_kM/F_{k-1}M \quad \textit{for} \quad \textit{all} \quad k \end{gathered}$$

The first two follow readily from the observation that the spectral sequence for the homotopy colimit of the above simplicial objects degenerates in view of the conclusions (iv) and (v) in the Proposition. We proceed to establish the third quasi-isomorphism. Since the maps $F_{k-1}DN(P_\bullet)_n \overset{i_k}{\rightarrow} F_kDN(P_\bullet)_n$ are admissible monomorphisms in $Presh$ for all k and all n, we see that $Cone(i_k)_n \simeq Gr_k(DN(P_\bullet))_n$ for all k and all n. (This follows from the hypothesis (STR6) of Chapter I. Strictly speaking one needs to consider $Cone(m(i_k))_n$.) Here $Cone(i_k)_\bullet$ is the simplicial object defined by $\{Cone(i_k)_n|n\}$. Since hocolim preserves quasi-isomorphisms, it follows that $\underset{\Delta}{\text{hocolim}} Gr_k DN(P_\bullet) \simeq \underset{\Delta}{\text{hocolim}} Cone(i_k)_\bullet \overset{\Delta}{\cong} Cone(\underset{\Delta}{\text{hocolim}}(i_k)) \simeq Cone(F_{k-1}M \rightarrow F_k M) \simeq F_k(M)/F_{k-1}(M) = Gr_k(M)$.

DEFINITION 2.5. The simplicial object $DN(P_\bullet)$ defined in the last proposition will be denoted $\mathcal{P}(M)_\bullet$ henceforth. This will be referred to as *a filtered simplicial resolution* of the filtered object M.

2.1.2. Let M, $N \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ be provided with non-decreasing filtrations compatible with the Cartan filtration on $\mathcal{A}$. Now we define an *induced filtration* on $\mathcal{H}om_{\mathcal{A}}(M, N)$ and on $Hom_{\mathcal{A}}(M, N)$ as follows. Let $K \varepsilon Presh$ and let k denote a fixed integer. We let $K \otimes M$ be filtered by $F_i(K \otimes M) = Image(K \otimes F_i(M) \rightarrow K \otimes M)$. We let

$F_k Hom_{\mathcal{A}}(K \otimes M, N) = \{f : K \otimes M \rightarrow N \varepsilon Hom_{\mathcal{A}}(K \otimes M, N) | f_{K \otimes F_i M}$ factors through the obvious map $F_{i+k}N \rightarrow N\}$.

Now fix k, M and N. Consider the functor $K \rightarrow F_k Hom_{\mathcal{A}}(K \otimes M, N)$, $Presh \rightarrow (sets)$. Since the functor $\otimes$ preserves colimits in either argument, it is clear that the above functor sends colimits in K to limits. Now, we let $F_k \mathcal{H}om_{\mathcal{A}}(M, N)$ be defined by:

$$Hom_{\mathcal{A},k}(K \otimes M, N) \cong Hom_{Presh}(K, F_k \mathcal{H}om_{\mathcal{A}}(M, N)).$$

(It should be clear that $\{F_k \mathcal{H}om_{\mathcal{A}}(M, N)|k\}$ defines a filtration of $\mathcal{H}om_{\mathcal{A}}(M, N)$.) Let M and N be provided with non-decreasing filtrations compatible with the Cartan filtration on $\mathcal{A}$. Let $\mathcal{P}(M)_\bullet \rightarrow M$ denote a simplicial resolution chosen as above applied to M instead of L. Each $\mathcal{P}(M)_k \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$; it is provided with a non-decreasing filtration compatible with the structure maps of the augmented simplicial object $\mathcal{P}(M)_\bullet \rightarrow M$ and compatible with the Cartan filtration on $\mathcal{A}$. The above filtration, along with the one on N, defines an induced filtration on each $\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_k, \mathcal{G}^n N) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_k, \mathcal{G}^n N)$ and hence on

$$R\mathcal{H}om_{\mathcal{A}}(M, N) = \underset{\Delta}{\text{holim}}\, \Delta\{\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_k, \mathcal{G}^n N)|n, k\}$$

We will denote this by $\{F_k R\mathcal{H}om_{\mathcal{A}}(M, N)|k\}$. One defines a similar filtration on on $RMap_{\mathcal{A}}(M, N)$.

LEMMA 2.6. *Let $M \varepsilon Presh$ be provided with a non-decreasing filtration $\{F_k M|k\}$. Assume the filtration is separated. Let $j_U : U \rightarrow X$ be in the site $\mathfrak{S}$ and let n denote an integer. A map $f \varepsilon S(n, U)(F_k M)$ will be called a trivial map if $\mathcal{H}^*(f)$ is the trivial map. Then, after identifying the trivial map with the base point, one obtains the isomorphisms*

$$\underset{k \rightarrow \infty}{colim} S(n, U)(F_k M) \cong \underset{k}{\sqcup} S(n, U)(F_k M)/S(n, U)(F_{k-1}M)$$

(Observe that $S(n, U)(F_{k-1}M)$ is a subset of $S(n, U)(F_k M)$ for each k. Each $S(n, U)(F_k M)$ is pointed with the trivial map being the base point. The quotient on the right hand side is the set theoretic quotient where all maps in $S(n, U)(F_{k-1}M)$ are identified with the base point.)

Let $f : M' \rightarrow M$ denote a filtration preserving map between objects in $Presh$ provided with filtrations as above. If the filtrations on M' and M are exhaustive (separated), so is the induced filtrations on $Cocyl(f)$ and $fib_h(f)$.

PROOF. Fix an integer k. Suppose $f \varepsilon S(n, U)(F_k M)$ be a non-trivial map, i.e. $\mathcal{H}^*(f) \neq *$. The hypothesis that the filtration is separated shows that, there exists a smallest integer $m \geq k$ so that $f \varepsilon S(n, U)(F_m M)$. Now f does not belong to $S(n, U)(F_{m-1}M)$. Therefore, f represents a non-trivial class in $S(n, U)(F_m M)/S(n, U)(F_{m-1}M)$. The map

$S(n,U)(F_mM) \rightarrow S(n,U)(F_mM)/S(n,U)(F_{m-1}M)$ is *bijective* on all maps $f \varepsilon S(n,U)(F_mM) - S(n,U)(F_{m-1}M)$. This provides the required isomorphisms.

The last assertion follows readily by considering the long-exact sequence on applying the cohomology functor $\{\mathcal{H}^q|q\}$ to the triangle $fib_h(f) \rightarrow M' \simeq Cocyl(f) \rightarrow M \rightarrow Tfib_h(f)$ and since $\mathcal{H}^q$ is assumed to commute with sums. □

PROPOSITION 2.7. *Assume in addition to the hypotheses of Proposition 2.4 that the filtration on M is separated.*

(i) If $F_k\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_i, \mathcal{G}^nN)$ denotes the k-th term of the filtration, then

$$F_{t-1}\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_i, \mathcal{G}^nN) \rightarrow F_t\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_i, \mathcal{G}^nN) \rightarrow Gr_t\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_i, \mathcal{G}^nN) \tag{2.1.3}$$

is a triangle in Presh.

(ii) Moreover, there exists a quasi-isomorphism

$$Gr_t(\mathcal{RH}om_{\mathcal{A}}(M,N)) \simeq \mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(M), Gr(N))_t \quad \textit{and} \tag{2.1.4}$$

a triangle:

$$F_{t-1}\mathcal{RH}om_{\mathcal{A}}(M,N) \rightarrow F_t\mathcal{RH}om_{\mathcal{A}}(M,N) \rightarrow Gr_t\mathcal{RH}om_{\mathcal{A}}(M,N) \tag{2.1.5}$$

PROOF. Throughout the proof we will let $\mathcal{S}$ denote the unit of $Presh$ (i.e. for the symmetric monoidal structure) in the situations of Chapter II, sections 2 and 4; in the situation of Chapter II, section 3, it will denote $\mathcal{R}$. Observe that the triangle in (2.1.5) is obtained from the triangle in (2.1.3) by taking the diagonal followed by homotopy limits. Moreover, by (2.1.1), $\{Gr(\mathcal{P}(M)_i)|i\}$ is a resolution of $Gr(M)$. Therefore, it suffices to prove (i) and show the existence of a natural quasi-isomorphism for all t, i and n:

$$Gr_t(\mathcal{H}om_{\mathcal{A}}(\mathcal{P}(M)_i, \mathcal{G}^nN) \simeq \mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(\mathcal{P}(M))_i, \mathcal{G}^nGrN)_t \tag{2.1.6}$$

Next recall that $\{\mathcal{P}(M)_i|i \geq 0\}$ is defined using ascending induction on i as in Proposition 2.4. We let P_i in Proposition 2.4 be given by $P(M)_i$. Now $\mathcal{P}(M)_\bullet = DN(P(M)_\bullet)$. Observe that $\mathcal{H}om(DN(P(M)_\bullet), L) = DN(\{\mathcal{H}om(P(M)_\bullet, L)\})$ for any $L \varepsilon Presh$ where the DN on the right is the denormalization functor sending co-chain complexes to cosimplicial objects. Therefore, to prove (i), it suffices to prove the corresponding statement when $\mathcal{P}(M)_\bullet$ has been replaced by $P(M)_\bullet$. Recall

$$P(M)_i = \sqcup_{m\varepsilon\mathbb{Z}} \sqcup_{U\varepsilon\mathfrak{S}} \sqcup_{S(n,U)(fib_h(u_{i-1}))} \mathcal{F}j^{\#}_{U!}j^*_U(\Sigma^m\mathcal{S}) \tag{2.1.7}$$

where the free functor $\mathcal{F}$ is defined as in (1.0.2) . Let

$$P(M)'_i = \sqcup_{m\varepsilon\mathbb{Z}} \sqcup_{U\varepsilon\mathfrak{S}} \sqcup_{S(n,U)(fib_h(u_{i-1}))} j^{\#}_{U!}j^*_U(\Sigma^m\mathcal{S}).$$

Now $P(M)'_i$ is filtered by the filtration:

$$F_kP(M)'_i = \sqcup_{m\varepsilon\mathbb{Z}} \sqcup_{U\varepsilon\mathfrak{S}} \sqcup_{S(n,U)(F_k(fib_h(u_{i-1})))} j^{\#}_{U!}j^*_U(\Sigma^m\mathcal{S})$$

and $P(M)_i = \mathcal{F}(P(M)'_i)$ is given the filtration induced from the Cartan filtration of $\mathcal{A}$ and the above filtration on $\mathcal{P}(M)'_i$.

Now we will fix an i and $j_U : U \rightarrow X$ in the site $\mathfrak{S}$. Let $S_k(U) = \sqcup_n S(n,U)(F_kfib_h(u_{i-1}))$ and $S(U) = \operatorname{colim}_{k\rightarrow-\infty} S_k(U)$. Since the induced filtration on $fib_h(u_{i-1})$ is also separated, Lemma 2.6 shows $S(U) = \underset{k}{\sqcup} S_k(U)/S_{k-1}(U)$ modulo the identification of the trivial map with the base point. Next consider the situations in Chapter II, sections 2 or 4 where *Presh* is provided with a unital symmetric monoidal structure. In this case, one obtains the quasi-isomorphism (making use of the filtration preserving quasi-isomorphism: $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}(M), M') \cong \mathcal{H}om_{\mathcal{S}}(M, M')$, $M \varepsilon Presh$, $M' \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$):

$$F_t(\mathcal{H}om_{\mathcal{A}}(P(M)_i, \mathcal{G}^n N) \simeq \prod_k \prod_{j_U} \prod_{S_k(U)/S_{k-1}(U)} j_{U*}\mathcal{G}^n F_{t+k}N, \text{ for each } t.$$

Therefore, in this case, one obtains a quasi-isomorphism:

$$Gr_t(\mathcal{H}om_{\mathcal{A}}(P(M)_i, \mathcal{G}^n N) \cong \prod_k \prod_{j_U} \prod_{S_k(U)/S_{k-1}(U)} j_{U*}\mathcal{G}^n Gr_{t+k}N$$

Since the map $F_{t+k-1}N \to F_{t+k}N$ is an admissible monomorphism, the diagram in (i) is indeed a triangle. On the other hand, in the same situation,

$$\mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A}),t}(Gr(P(M))_i, \mathcal{G}^n GrN) \cong \prod_k \prod_{j_U} \prod_{S_k(U)/S_{k-1}(U)} j_{U*}\mathcal{G}^n Gr_{t+k}N$$

as well. This proves the proposition in the situations of Chapter II, sections 2 or 4. In the situation of Chapter II, section 3, where $\mathcal{A}$ is assumed to be a sheaf of differential graded algebras over an operad, Chapter II, Proposition 3.7 shows that one instead obtains a filtration preserving chain homotopy equivalence between the corresponding terms, that is natural in the arguments M and N. Therefore one obtains the required quasi-isomorphism in this case as well. This proves the isomorphism in (2.1.6). □

REMARK 2.8. Now fix an integer t_0. The given filtrations on M and N induce a non-decreasing filtration F_t on $F_{t_0}(\mathcal{RH}om_{\mathcal{A}}(M, N))$. The same proof as above now shows one obtains

$$Gr_t(F_{t_0}\mathcal{RH}om_{\mathcal{A}}(M, N)) \simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M), Gr(N))_t, \quad t \le t_0 \quad \textit{and} \tag{2.1.8}$$
$$\simeq * \quad t > t_0 \tag{2.1.9}$$

and therefore a triangle:

$$F_{t-1}\mathcal{RH}om_{\mathcal{A}}(M, N) \to F_t\mathcal{RH}om_{\mathcal{A}}(M, N) \to Gr_t\mathcal{RH}om_{\mathcal{A}}(M, N), \quad t \le t_0 \tag{2.1.10}$$

LEMMA 2.9. *Let $\phi^* : Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A})) \to Mod_l(\mathfrak{S}, GEM(\mathcal{H}^*(\mathcal{A})))$ denote the functor sending an object $\tilde{M}$ to $GEM(\mathcal{H}^*(\mathcal{A})) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} \tilde{M}$. If $\alpha : \tilde{M} \to \tilde{M}'$ denotes a quasi-isomorphism of objects in $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$, the induced map $\phi^*(\alpha)$ is also a quasi-isomorphism. Similar conclusions hold for the category of right-modules.*

PROOF. Consider a commutative square:

$$\begin{array}{ccc} P & \xrightarrow{\alpha'} & P' \\ {\scriptstyle \epsilon_{\tilde{M}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\tilde{M}'}} \\ \tilde{M} & \xrightarrow{\alpha} & \tilde{M}' \end{array}$$

with P, P' flat objects in $Mod_l(\mathfrak{S}, \mathcal{A})$ and where the vertical maps are quasi-isomorphisms. Now $L\phi^*(\tilde{M}) = GEM(\mathcal{H}^*(\mathcal{A})) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} P$ and $L\phi^*(\tilde{M}') = GEM(\mathcal{H}^*(\mathcal{A})) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} P'$; the first spectral sequence in Proposition 1.2 computes both terms. Consider the spectral sequence for the first term:

$$E_2^{s,t} = Tor_{s,t}^{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(GEM(\mathcal{H}^*(\mathcal{A})), \mathcal{H}^*(P)) \Rightarrow \mathcal{H}^{-s+t}(GEM(\mathcal{H}^*(\mathcal{A})) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} P).$$

This spectral sequence degenerates since $\mathcal{H}^*(GEM(\mathcal{H}^*(\mathcal{A}))) \cong \mathcal{H}^*(\mathcal{A})$. The same conclusions hold for the corresponding spectral sequence for the second term. It follows that the required conclusions hold with $L\phi^*$ in the place of ϕ^*.

Next recall that $P = \underset{\Delta}{\text{hocolim}} P_\bullet$, with $P_\bullet$ a simplicial object in $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$ with each P_n being flat. The augmentation $\epsilon_{\tilde{M}}$ is induced by a map of simplicial objects $\epsilon_{\tilde{M}} : P_\bullet \to K(\tilde{M}, 0)$, the right-hand-side being the obvious constant simplicial object. Recall that spectral sequence above is the spectral sequence for the homotopy colimit as in section 1. Therefore the above simplicial map induces a map of the above spectral sequence to the corresponding spectral sequence for the homotopy colimit of the constant simplicial object $GEM(\mathcal{H}^*(\mathcal{A})) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} K(\tilde{M}, 0)$. Clearly the spectral sequence for the above constant simplicial object also degenerates thereby showing the augmentation $L\phi^*(\tilde{M}) \to \phi^*(\tilde{M})$ is a quasi-isomorphism. $\square$

2.1.11. We let $\phi_* : Mod_l(\mathfrak{S}, GEM(\mathcal{H}^*(\mathcal{A}))) \to Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$ denote the obvious functor sending an object K in the first category to an object in the second category using the map ϕ.

PROPOSITION 2.10. *(i) Let* $M \varepsilon Mod_r^{fil}(\mathfrak{S}, \mathcal{A})$, $N \varepsilon Mod_l^{fil}(\mathfrak{S}, \mathcal{A})$. *Let* $\bar{M} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *and*
$\bar{N} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *so that* $Gr_F(M) \simeq \phi_*(Sp(\bar{M}))$ *and* $Gr_F(N) \simeq \phi_*(Sp(\bar{N}))$. *Then there exist quasi-isomorphisms:*

$$Gr_F(M) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Gr_F(N) \simeq Sp(\bar{M}) \underset{Sp(\mathcal{H}^*(\mathcal{A}))}{\overset{L}{\otimes}} Sp(\bar{N})$$

(ii) Let M, $N \varepsilon Mod_l^{filt}(\mathfrak{S}, \mathcal{A})$. *Let* $\bar{M} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *and* $\bar{N} \varepsilon D(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *so that* $Gr_F(M) \simeq \phi_*(Sp(\bar{M}))$ *and* $Gr_F(N) \simeq \phi_*(Sp(\bar{N}))$. *Then there exist quasi-isomorphisms:*

$$\mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M), Gr(N)) \simeq \mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\bar{M}), Sp(\bar{N}))$$

PROOF. Observe that the maps

$$Gr_F(M) \simeq Gr_F(M) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Gr_{\mathcal{C}}(\mathcal{A}) \to Gr_F(M) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} GEM(\mathcal{H}^*(\mathcal{A})) = L\phi^*(Gr_F(M))$$

and

$$Gr_F(N) \simeq Gr_F(N) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Gr_{\mathcal{C}}(\mathcal{A}) \to Gr_F(N) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} GEM(\mathcal{H}^*(\mathcal{A})) = L\phi^*(Gr_F(N))$$

are quasi-isomorphisms. (This follows readily from the degeneration of the first spectral sequence in Proposition 1.2 .) The given quasi-isomorphisms $Gr_F(M) \simeq \phi_* Sp(\bar{M})$ and $Gr_F(N) \simeq \phi_* Sp(\bar{N})$ show that

$$L\phi^*(Gr_F(M)) \simeq L\phi^*(\phi_*(Sp(\bar{M}))) \text{ and } L\phi^*(Gr_F(N)) \simeq L\phi^*(\phi_*(Sp(\bar{N}))).$$

Finally observe that there exist natural maps $L\phi^*(\phi_*(Sp(\bar{M}))) \to Sp(\bar{M})$ and $L\phi^*(\phi_*(Sp(\bar{N}))) \to Sp(\bar{N})$. These maps are quasi-isomorphisms, once again by the degeneration of the spectral sequences in Proposition 1.2. It follows that

$$Gr_F(M) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Gr_F(N) \simeq Gr_F(M) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Sp(\mathcal{H}^*(\mathcal{A})) \underset{Sp(\mathcal{H}^*(\mathcal{A}))}{\overset{L}{\otimes}} Gr_F(N) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\overset{L}{\otimes}} Sp(\mathcal{H}^*(\mathcal{A}))$$

$$\simeq Sp(\bar{M}) \underset{Sp(\mathcal{H}^*(\mathcal{A}))}{\overset{L}{\otimes}} Sp(\bar{N}).$$

The first assertion follows.

Next we consider the second assertion. Let $\mathcal{P}(M)_\bullet \to M$ denote the complex constructed as in Definition (2.5). Now

$$\mathcal{H}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)_i,\, \mathcal{G}^n Gr(N))) \simeq \mathcal{H}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)_i,\, \mathcal{G}^n \phi_*(Sp(\bar{N}))))$$

since $Gr(P(M))_i$ is locally projective in $Mod_l(Gr_{\mathcal{C}}(\mathcal{A}))$. The latter term is quasi-isomorphic to

$$\mathcal{H}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)_i,\, \phi_*\mathcal{G}^n(Sp(\bar{N}))))$$

$$\cong \mathcal{H}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)_i,\, \phi_*(\mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\mathcal{H}^*(\mathcal{A})), \mathcal{G}^n(Sp(\bar{N}))))))$$

$$\cong \mathcal{H}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)_i,\, \mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\mathcal{H}^*(\mathcal{A})), \mathcal{G}^n(Sp(\bar{N})))))$$

where $\mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\mathcal{H}^*(\mathcal{A})), \mathcal{G}^n(Sp(\bar{N})))$ has the structure of a sheaf of left-modules over $Gr_{\mathcal{C}}(\mathcal{A})$ induced from the structure of a sheaf of right-modules over $Gr_{\mathcal{C}}(\mathcal{A})$ on $Sp(\mathcal{H}^*(\mathcal{A}))$. By Chapter II, (2.0.9) with $\mathcal{B}$ replaced by $Sp(\mathcal{H}^*(\mathcal{A}))$ and $\mathcal{A}$ replaced by $Gr_{\mathcal{C}}(\mathcal{A})$, the last term above is quasi-isomorphic to

$$\mathcal{H}om_{Sp(\mathcal{H}*(\mathcal{A}))}(\mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)'_i)) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} (Sp(\mathcal{H}^*(\mathcal{A}))), \mathcal{G}^n Sp(\bar{N}))$$

Here $\mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})}$ is the free functor associated to $Gr_{\mathcal{C}}(\mathcal{A})$. Recall

$$P(M)'_k = \underset{m\varepsilon\mathbb{Z}}{\sqcup} \quad \underset{U\varepsilon\mathfrak{S}}{\sqcup} \underset{S(m,U)(fib_h(u_{i-1}))}{\sqcup} j^{\#}_{U!} j^*_U(\Sigma^m \mathcal{S})$$

which is filtered as in Theorem (2.3). Therefore

$$\mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(\mathcal{P}(M)')) \cong \underset{m\varepsilon\mathbb{Z}}{\sqcup} \underset{U\varepsilon\mathfrak{S}}{\sqcup} \underset{S(m,U)(Gr(fib_h(u_{i-1})))}{\sqcup} j^{\#}_{U!} j^*_U(\Sigma^m Gr_{\mathcal{C}}(\mathcal{A})).$$

We define a complex $\overset{\bar{d}_{i+1}}{\to} \bar{P}_i \overset{\bar{d}_i}{\to} \bar{P}_{i-1} \overset{\bar{d}_{i-1}}{\to} \dots \overset{\bar{d}_{-1}}{\to} \bar{M}$ of Abelian sheaves as follows. We let $\bar{P}_0 = \mathcal{H}^*(P_0)$, $\bar{P}_i = \mathcal{H}^*(P_i)$, $\bar{u}_i = \mathcal{H}^*(u_i) : \bar{P}_{i+1} = \mathcal{H}^*(P_{i+1}) \to \mathcal{H}^*(fib_h(u_{i-1})) \cong ker(\mathcal{H}^*(u_{i-1}))$ and $\bar{d}_i = \mathcal{H}^*(d_i) : \mathcal{H}^*(P_i) \to \mathcal{H}^*(P_{i-1})$. Now one may observe that

$$\dots \bar{P}_i \overset{\bar{d}_i}{\to} \bar{P}_{i-1} \overset{\bar{d}_{i-1}}{\to} \dots \overset{\bar{d}_0}{\to} \bar{P}_0 \overset{\bar{d}_{-1}}{\to} \bar{M}$$

is a resolution of $\bar{M}$ by a complex of sheaves of $\mathcal{H}^*(\mathcal{A})$-modules. Moreover, there exists a natural map (see Chapter I, (ST8)) $\mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})} Gr(\mathcal{P}(M)'_i = Gr(\mathcal{P}(M))_i \to \phi_* Sp(\bar{P}_i)$ of objects in $Mod_l(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$; this map is a quasi-isomorphism. Therefore, there exists a quasi-isomorphism:

$$\mathcal{F}_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(P(M)')_i) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} (Sp(\mathcal{H}^*(\mathcal{A})))$$

$$\overset{\simeq}{\to} \phi_*(Sp(\bar{P}))_i \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} (Sp(\mathcal{H}^*(\mathcal{A}))) \simeq \phi^*\phi_*(Sp(\bar{P})_i) \simeq Sp(\bar{P})_i$$

where one obtains the last quasi-isomorphism as in (i). By first applying the denormalization functor and then taking the homotopy limit over Δ, one completes the proof of (ii). □

Definition 2.11. Let $M\varepsilon Mod^{fil}_r(\mathfrak{S}, \mathcal{A})$. We will consider the following two conditions on the given filtration F:

(i) $Gr(M) = \{F_nM/F_{n-1}M|n\} \varepsilon Mod_r(\mathfrak{S}, Gr_{\mathcal{C}}(\mathcal{A}))$ and

(ii) $Gr(M) \simeq Sp(\bar{M})$, $\bar{M} \varepsilon D(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$.

We say M is of *finite tor dimension or f.t.d* if $\bar{M}$ is. We say M is *globally of f.t.d* if in addition $\bar{M}^{\cdot}$ is globally of $f.t.d.$ Similar definitions apply to $N \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$. We say M is *pseudo-coherent* (*perfect*) if the hypotheses (i) is satisfied and $Gr(M) \simeq Sp(\bar{M})$, $\bar{M} \varepsilon D((Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ with $\bar{M}$ pseudo-coherent (perfect, respectively).

PROPOSITION 2.12. *Let $\bar{M} \varepsilon D_b(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ and $\bar{N} \varepsilon D_b(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. Assume that $\bar{M}$ is globally of finite tor dimension. Then there exists a quasi-isomorphism:*

$$Sp(\bar{M}) \overset{L}{\underset{Sp(\mathcal{H}^*(\mathcal{A}))}{\otimes}} Sp(\bar{N}) \simeq Sp(\bar{M} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N})$$

PROOF. First assume that both $\bar{M}$ and $\bar{N}$ are complexes concentrated in degree 0. Now we show that there exists a natural map

$$GEM(\bar{M}) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} GEM(\bar{N}) \rightarrow GEM(\bar{M} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}) \tag{2.1.12}$$

The hypothesis in Chapter I, (ST6) shows there exists a commutative diagram:

$$\begin{array}{ccc}
GEM(\bar{M}) \otimes GEM(\mathcal{H}^*(\mathcal{A})) \otimes GEM(\bar{N}) & \rightarrow & GEM(\bar{M}) \otimes GEM(\bar{N}) \\
\downarrow & & \downarrow \\
GEM(\bar{M} \otimes \mathcal{H}^*(\mathcal{A}) \otimes \bar{N}) & \rightarrow & GEM(\bar{M} \otimes \bar{N})
\end{array}$$

The horizontal map in the first row is given by $\lambda_{GEM(\bar{M})} \otimes id_{GEM(\bar{N})}$, with $\lambda_{GEM(\bar{M})}$: $GEM(\bar{M}) \otimes GEM(\mathcal{H}^*(\mathcal{A})) \rightarrow GEM(\bar{M})$ the induced module structure on $GEM(\bar{M})$ and the horizontal map in the second row is given by $GEM(\lambda_{\bar{M}} \otimes id_{\bar{N}}$, with $\lambda_{\bar{M}} : \bar{M} \otimes \mathcal{H}^*(\mathcal{A}) \rightarrow \bar{M}$ being the module structure on $\bar{M}$. A similar commutative square also exists where the top horizontal map is given by $id_{GEM(\bar{M})} \otimes \lambda_{GEM(\bar{N})}$, with $\lambda_{GEM(\bar{N})}$: $GEM(\mathcal{H}^*(\mathcal{A})) \otimes GEM(\bar{N}) \rightarrow GEM(\bar{N})$ the induced module structure on $GEM(\bar{N})$ and where the bottom row is given by $GEM(id_{\bar{M}} \otimes \lambda_{\bar{N}})$, with $\lambda_{\bar{N}}$ being the module structure on $\bar{N}$. The definition of $GEM(\bar{M}) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} GEM(\bar{N})$ as in Chapter II, (1.2.2) and (1.2.7), shows that the map in (2.1.12) exists.

Next consider the case when $\bar{M}$ is a presheaf of graded *flat* modules over $\mathcal{H}^*(\mathcal{A})$. Now the first spectral sequence in Proposition 1.2 computes

$$\mathcal{H}^*(GEM(\bar{M}) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} GEM(\bar{N})^i) \cong \mathcal{H}^*(GEM(\bar{M})) \underset{\mathcal{H}^*(GEM(\mathcal{H}^*(\mathcal{A})))}{\otimes} \mathcal{H}^*(GEM(\bar{N})^i) \cong \bar{M} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}^i$$

for each i. One may directly compute $\mathcal{H}^*(GEM(\bar{M} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}^i)) \cong \bar{M} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}^i$ for each fixed i. (See for example the proof of 2.17 below.) It follows that in this case the map in (2.1.12) is a quasi-isomorphism.

Now one may observe from (2.1.12) and Appendix B that there exists a map:

$$DN(GEM(\bar{M})) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} DN(GEM(\bar{N})) \rightarrow DN(GEM(\bar{M})) \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} GEM(\bar{N}))$$

(Here DN is the functor considered in Appendix B, 0.1.) Next we will assume that $\bar{M}$ and $\bar{N}$ are bounded above by an integer m. In this case we may find a resolution of the chain complex $\bar{M}[m_h]$ by a chain-complex $\bar{F}_\bullet$ all whose terms as in Theorem 2.3. By Chapter I, (ST9)

$$Sp(\bar{M}) \simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(GEM(\bar{M}[m_h])) \simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(\bar{F}.) \text{ and}$$

$$Sp(\bar{N}) \simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(GEM(\bar{N}[m_h])).$$

Therefore

$$Sp(\bar{M}) \overset{L}{\underset{Sp(\mathcal{H}^*(\mathcal{A}))}{\otimes}} Sp(\bar{N})$$

$$\simeq \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(GEM(\bar{F}.)) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} \Omega^m \underset{\Delta}{\mathrm{hocolim}} DN(GEM(\bar{N}[m_h]))$$

$$\simeq \Omega^{2m} \underset{\Delta}{\mathrm{hocolim}} \Delta[DN(\bar{F}) \underset{GEM(\mathcal{H}^*(\mathcal{A}))}{\otimes} DN(GEM(\bar{N}[m_h]))]$$

$$\simeq \Omega^{2m} \underset{\Delta}{\mathrm{hocolim}} DN(GEM(TOT(\bar{F}[m_h] \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}[m_h])))$$

$$= \Omega^{2m} \underset{\Delta}{\mathrm{hocolim}} DN(GEM(TOT(\bar{F} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N})[m_h])) \simeq Sp(TOT(\bar{F} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}) = Sp(\bar{M} \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \bar{N}).$$

(Here TOT denotes the total complex.) □

Proposition 2.13. *Let $\bar{M} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ and $\bar{N} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. Assume that $\bar{M}$ is globally of finite tor dimension. Then there exists a quasi-isomorphism:*

$$\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\bar{M}),\ Sp(\bar{N})) \simeq Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\bar{M},\ \bar{N}))$$

Proof. We will first consider the case when the site $\mathfrak{S}$ is punctual, $\bar{M} = \bar{P}$ is a projective module over $\mathcal{H}^*(\mathcal{A})$ and $\bar{N}$ is a single module over $\mathcal{H}^*(\mathcal{A})$. Now the right-hand-side identifies with $Sp(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, \bar{N}))$ and the left-hand-side identifies with $\mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(P, Sp(\bar{N}))$ where $P \to Sp(M)$ is a quasi-isomorphism with P a projective object in $D(Mod_l(\mathfrak{S}, Sp(\mathcal{H}^*(\mathcal{A}))))$. Using the observation that $\bar{P}$ is a split summand of a free $\mathcal{H}^*(\mathcal{A})$-module, one may now obtain a quasi-isomorphism: $\mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(P, Sp(\bar{N})) \simeq \mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\bar{P}), Sp(\bar{N})) = \mathcal{H}om_{GEM(\mathcal{H}^*(\mathcal{A}))}(GEM(\bar{P}), GEM(\bar{N}))$. Using the definition of the latter as an equalizer (see Chapter II, (1.2.2) and (1.2.8)), one may now obtain a natural map

$$Sp(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, \bar{N})) = GEM(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, \bar{N})) \to \mathcal{H}om_{GEM(\mathcal{H}^*(\mathcal{A}))}(GEM(\bar{P}), GEM(\bar{N})).$$

One may compute the cohomology sheaves of the left-hand-side as in Proposition 2.17 below and one may compute the cohomology sheaves of the right-hand-side by the third spectral sequence in Proposition 1.2. It follows the above map is a quasi-isomorphism.

Next we consider the case when $\bar{M}$ is a sheaf of graded modules over $\mathcal{H}^*(\mathcal{A})$ that is stalk-wise projective (as a module over the corresponding stalks of $\mathcal{H}^*(\mathcal{A})$) and $\bar{N}$ is a single sheaf. Now the right-hand-side identifies with $Sp(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, \mathcal{G}\bar{N}))$. Using the first case, one may identify the left-hand-side now with $\mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\bar{P}), \mathcal{G}Sp(\bar{N}))$. Since the homotopy inverse limits commute with themselves and with products, one may identify the former (the latter) with

$$\underset{\Delta}{\mathrm{holim}}\, GEM(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, G^n\bar{N}))$$

$(\underset{\Delta}{\mathrm{holim}}\, \mathcal{H}om_{GEM(\mathcal{H}^*(\mathcal{A}))}(GEM(\bar{P}), G^nGEM(\bar{N}))$, respectively)

Next we proceed to show these are quasi-isomorphic. Clearly it suffices to show

$$GEM(\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, G\bar{N})) \tag{2.1.13}$$

and

$$\mathcal{H}om_{GEM(\mathcal{H}^*(\mathcal{A}))}(GEM(\bar{P}), GGEM(\bar{N})) \tag{2.1.14}$$

are quasi-isomorphic. Recall that $G = \bar{p}_* \circ U \circ a \circ \bar{p}^*$ as in Chapter II, (1.0.1). Now (2.1.13) identifies with $GEM(\bar{p}_* U(\mathcal{H}om_{a\circ\bar{p}^*\mathcal{H}^*(\mathcal{A})}(a \circ \bar{p}^*(\bar{P}), a \circ \bar{p}^*(\bar{N}))))$ which, by the first case considered above identifies with

$\bar{p}_* U \mathcal{H}om_{GEM(a\circ\bar{p}^*\mathcal{H}^*(\mathcal{A}))}(GEM(a \circ \bar{p}^*(\bar{P}), a \circ \bar{p}^*(\bar{N})))$.
(2.1.14) identifies with $\bar{p}_* U(\mathcal{H}om_{a\circ\bar{p}^*(GEM(\mathcal{H}^*(\mathcal{A})))}(a \circ \bar{p}^* GEM(\bar{P}), a \circ \bar{p}^* GEM(\bar{N})))$.

Moreover, as in the first case above, one may show there exists a natural map from the former to the latter. Next observe that $\bar{p}^* GEM(\bar{P})$ is a projective module over $\bar{p}^* GEM(\mathcal{H}^*(\mathcal{A}))$ while $GEM(\bar{p}^*(\bar{P}))$ is a projective module over $GEM(\bar{p}^*(\bar{P}))$. Now consider the third spectral sequence in Proposition 1.2 applied to these. It follows readily that they degenerate at the E_2-terms and the above map induces an isomorphism there. It follows that the terms in (2.1.13) and (2.1.14) are quasi-isomorphic, thereby proving the proposition in this case.

Next consider the case where everything remains as above, except that $\bar{N}$ is a bounded complex that is trivial in negative degrees. In this case $Sp(\bar{N}) = \prod_i \underset{\Delta}{\text{holim}}\, DN(GEM(\bar{N}(i)))$, if $\bar{N} = \prod_i \bar{N}(i)$. The above $\underset{\Delta}{\text{holim}}$ comes out of the $\mathcal{H}om$ and commutes with the $\underset{\Delta}{\text{holim}}$ associated to the Godement resolution. Therefore, this case follows readily from the previous one.

Next we assume $\bar{M}$ is a bounded complex that is globally of finite tor-dimension. We may now replace $\bar{M}$ by a bounded complex $\bar{P}$ each term of which is stalk-wise projective over the corresponding stalk of $\mathcal{H}^*(\mathcal{A})$. By applying appropriate shifts (see the proof of the previous proposition), one may now write $Sp(\bar{P}) = \Omega^m \underset{\Delta}{\text{hocolim}} DN(GEM(P[p_h]))$. Then $\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(Sp(\bar{M}), Sp(\bar{N}))$ identifies with $\underset{\Delta}{\text{holim}}\{\Sigma^m \mathcal{H}om_{Sp(\mathcal{H}^*(\mathcal{A}))}(DN(GEM(P[p_h])), G^n DNGEM(\bar{N}))|n\}$. Since each term of the simplicial object $DN(GEM(P[p_h]))$ is stalkwise projective over the stalks of $\mathcal{H}^*(\mathcal{A})$, one may apply the previous case along with the results on shifts and suspension in Appendix B to identify it with $\underset{\Delta}{\text{holim}}\{Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, G^n(\bar{N})))|n\}$. The case when $\bar{N}$ is not necessarily trivial in negative degrees is also handled by applying certain shifts. (See Appendix B.) □

LEMMA 2.14. *Let $\bar{M} \varepsilon Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$. Then the following are true:*

(i) $\bar{M}$ has a resolution by sheaves of the form $\tilde{F}_n = \underset{\alpha}{\oplus} j^{\#}_{U_\alpha !}((\underline{\mathcal{H}^*(\mathcal{A})})_{|U_\alpha})$,

where each $U_\alpha \varepsilon \mathfrak{S}$.

(ii) If $\bar{M}$ is locally of finite type, for each point $\bar{x}$ of $\mathcal{S}$, there exists a neighborhood U_x of x in $\mathcal{S}$ so that each $\tilde{F}_n$ has only finitely many summands

(iii) If $\tilde{M}$ is of f.t.d, we may find a resolution $\bar{F}_\bullet \to \bar{M}$, so that the following conditions are also satisfied:

for each point $\bar{x}$ of X there is a neighborhood U_x and an integer $m_{\bar{x}} >> 0$ so that

(a) $(\bar{F}_i)_{\bar{x}} = 0$ if $i > N_{\bar{x}}$, $\bar{F}_i$ for $i < m_{\bar{x}}$ are as in (i) and

(b) $(\bar{F}_{m_{\bar{x}}}) = ker(\underset{\alpha}{\oplus} j_{U_\alpha}!((\underline{\bar{F}}_{m_{\bar{x}}-1})_{|U_\alpha}) \rightarrow \underset{\beta}{\oplus} j_{U_\beta}!((\underline{\bar{F}}_{m_{\bar{x}}-2})_{|U_\beta}))$

which is a sheaf of flat graded $\mathcal{H}^*(\mathcal{A})$-modules. (If the site $\mathfrak{S}$ is quasi-compact, one may find a common m that works for all points $\bar{x}$.)

PROOF. Since the site $\mathfrak{S}$ has enough points, the above three statements are clear. □

LEMMA 2.15. *Let $\{K^{i,j}|i,j\}$ denote a double complex in $Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$ so that the differentials in the indices i and j are of degree $+1$. Assume that $K^{i,j} = 0$ if $i < 0$ or $j < 0$. Now $DN \circ DN(\widehat{G}EM(K))$ is a double cosimplicial object in $Mod_l(GEM(\mathcal{H}^*(\mathcal{A})))$. If Tot_1 and Tot_2 denote the functor Tot (which is the Tot functor as in [B-K]), applied in the first and second degrees respectively, one obtains a natural quasi-isomorphism:*

$$Tot_1 \circ Tot_2 DN \circ DN(GEM(K)) \simeq Tot\Delta(DN \circ DN(GEM(K)))$$

$$\simeq Tot(DN(TOT(\{GEM(K^{i,j})|i,j\})))$$

where $TOT(\{GEM(K^{i,j})|i,j\})$ is the total co-chain complex defined by

$$(TOT(\{GEM(K^{i,j})|i,j\}))^k = \underset{u+v=k}{\Pi} GEM(K^{u,v})$$

REMARK 2.16. Observe that $TOT(\{GEM(K^{i,j})|i,j\}) = GEM(TOT(\{K^{i,j}|i,j\}))$.

PROOF. This is clear since we are working in an Abelian category. □

PROPOSITION 2.17. *(i) Let $\bar{M} = \underset{i}{\Pi}\bar{M}(i)\varepsilon D(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ be globally of finite tor dimension,*
$\bar{N} = \underset{i}{\Pi}\bar{N}(i)\varepsilon Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$. Then

$$\mathcal{H}^{s+t}(Gr_t[Sp(\bar{M} \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \bar{N})]) \cong \mathcal{H}^s([\bar{M} \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \bar{N}](t)) \cong \mathcal{H}^s([\bar{F} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}](t))$$

$$\cong Tor_{-s,t}^{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N})$$

(ii) Let $\bar{M} = \underset{i}{\Pi}\bar{M}(i)\varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ be globally of finite tor dimension,

$\bar{N} = \underset{i}{\Pi}\bar{N}(i)\varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. Then

$$\mathcal{H}^{s+t}(Gr_t[Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N}))]) \cong \mathcal{H}^s([\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N})](t)) \cong \mathcal{H}^s([\mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{F}, \mathcal{G}^\bullet\bar{N})](t))$$

$$\cong \mathcal{E}xt^{s,t}_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N})$$

PROOF. This follows from the following computation. Let $\bar{K} = \underset{i}{\Pi}\bar{K}(i)\varepsilon D(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. By applying some shifts as in Appendix B, one may assume without loss of generality that this co-chain complex is trivial in negative degrees. Now recall that $Sp(\bar{K}) = \underset{i}{\Pi}\underset{\Delta}{\text{holim}}(EM_i(\bar{K}(i))$. Moreover, $\mathcal{H}^k(EM_i(\bar{K}(i)) \cong \bar{K}(k)$ if $i = k$ and $\cong 0$ otherwise. Moreover, observe that the filtration on $Sp(\bar{K})$ is given by $Sp(\bar{K})_t = \underset{i \leq t}{\Pi} EM_i(\bar{K}(i))$. Therefore the spectral sequence for the homotopy inverse limit in Chapter I, (Hl) shows that

$$\mathcal{H}^{s+t}(Gr_t[Sp(\bar{K})]) \cong \mathcal{H}^{s+t}(\underset{\Delta}{\text{holim}}(EM_t(\bar{K}(t)))) \cong \mathcal{H}^s(\mathcal{H}^t(EM_t(\bar{K}(t)))) \cong \mathcal{H}^s(\bar{K}(t))$$

Let $\bar{P} \to \bar{M}$ denote a quasi-isomorphism from a bounded complex of sheaves of $\mathcal{H}^*(\mathcal{A})$-modules that is stalkwise projective. Take $\bar{K} = \bar{P} \underset{\mathcal{H}^*(\mathcal{A})}{\otimes} \bar{N}$ ($\bar{K} = \mathcal{H}om_{\mathcal{H}^*(\mathcal{A})}(\bar{P}, \mathcal{G}(\bar{N}))$) to obtain the first (second, respectively) result. □

THEOREM 2.18. *(i) Let* $M \varepsilon Mod_r^{fil}(\mathfrak{S}, \mathcal{A})$, $N \varepsilon Mod_l^{fil}(\mathfrak{S}, \mathcal{A})$ *and let* $\bar{M} \varepsilon D(Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$, $\bar{N} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *so that* $Gr(M) \simeq Sp(\bar{M})$ *and* $Gr(N) \simeq Sp(\bar{N})$.

In this situation, there exists a spectral sequence:

$$E^2_{s,t} = \mathcal{T}or^{\mathcal{H}^*(\mathcal{A})}_{s,t}(\bar{M}, \bar{N}) \Rightarrow \mathcal{H}^{-s+t}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N)$$

Moreover, this spectral sequence converges strongly if at least one of M *or* N *is of finite tor dimension.*

(ii) Let $M \varepsilon Mod_l^{fil}(\mathfrak{S}, \mathcal{A})$, $N \varepsilon Mod_l^{fil}(\mathfrak{S}, \mathcal{A})$ *and let* $\bar{M} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$, $\bar{N} \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ *so that* $Gr(M) \simeq Sp(\bar{M})$ *and* $Gr(N) \simeq Sp(\bar{N})$. *Assume further that both* $\bar{M}$ *and* $\bar{N}$ *are globally of finite tor dimension (and in particular, bounded).*

In this situation, there exists a spectral sequence:

$$E_2^{s,t} = \mathcal{E}xt^{s,t}_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N}) \Rightarrow \mathcal{H}^{s+t}(\mathcal{RH}om_{\mathcal{A}}(M,\ N))$$

In general, this spectral sequence converges only conditionally in the sense of [Board]. However, this spectral sequence converges strongly in the following cases:

(a) if M *is perfect (with no further hypotheses) or*

(b) if $\mathcal{H}^*(\mathcal{A})$ *is locally constant on the site* $\mathfrak{S}$ *and* $\bar{M}$ *is constructible.*

PROOF. Let $\mathcal{P}(M)_\bullet \to M$ denote a *resolution* as in Proposition 2.4. Consider (i). Now we filter $M \overset{L}{\underset{\mathcal{A}}{\otimes}} N = \underset{\Delta}{\text{hocolim}} \mathcal{P}(M)_\bullet \underset{\mathcal{A}}{\otimes} N$ by the filtration induced from the given filtrations on M, N and the Cartan filtration on $\mathcal{A}$. Now we obtain the identification:

$$Gr(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N) = Gr(\underset{\Delta}{\text{hocolim}} \mathcal{P}(M)_\bullet \underset{\mathcal{A}}{\otimes} N)$$

$$\simeq \underset{\Delta}{\text{hocolim}} Gr(\mathcal{P}(M)_\bullet \underset{\mathcal{A}}{\otimes} N) \simeq \underset{\Delta}{\text{hocolim}} [Gr\mathcal{P}(M)_\bullet \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} Gr(N)]$$

$$\simeq (\underset{\Delta}{\text{hocolim}} Gr\mathcal{P}(M)_\bullet) \underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes} Gr(N) \simeq Gr(M) \overset{L}{\underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes}} Gr(N)$$

The first $\simeq$ is clear since $\underset{\Delta}{\text{hocolim}}$ commutes with taking the associated graded terms, while the second $\simeq$ follows from the observation that taking the associated graded terms commutes with co-equalizers, the third follows from the commutativity of $\underset{\Delta}{\text{hocolim}}$ with $\underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes}$ and the fourth follows from (2.1.1).

i.e. $F_{t-1}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N) \to F_t(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N) \to Gr_t(M \overset{L}{\underset{\mathcal{A}}{\otimes}} N) = [Gr(M) \overset{L}{\underset{Gr_{\mathcal{C}}(\mathcal{A})}{\otimes}} Gr(N)]_t$

is a triangle. We take $\mathcal{H}^*$ of the above triangle to obtain a long exact sequence and the associated exact-couple. This provides the required spectral sequence. Now the identification of the E^2-terms follows from Proposition 2.10 (i), Proposition 2.12 and Proposition 2.17.

The strong convergence of the spectral sequence is clear from the hypotheses that either M or N is of finite tor dimension.

Now we consider (ii). $Gr\mathcal{RH}om_{\mathcal{A}}(M,N) \simeq \mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(M), Gr(N))$ by (2.1.4). i.e. we obtain the triangle

$$F_{t-1}\mathcal{RH}om_{\mathcal{A}}(M,N) \to F_t\mathcal{RH}om_{\mathcal{A}}(M,N) \to \mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A}),t}(Gr(M), Gr(N)).$$

On taking the cohomology sheaves, we get a long exact sequence which provides the exact couple for the spectral sequence in (ii). It suffices to identify the E_2-terms of this spectral sequence. Now Proposition 2.10 (ii) and Proposition 2.13 show that

$$F_t\mathcal{RH}om_{Gr_{\mathcal{C}}(\mathcal{A})}(Gr(M), Gr(N)) \simeq Sp(F_t\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N}))$$

Proposition 2.17(ii) computes the cohomology sheaves of the last term to obtain the identification of the E_2-terms. Under either of the assumptions one may show that there exists an integer $N >> 0$ so that $E_2^{s,t} = 0$ if $s > N$. Observe that the system of neighborhoods of any point have uniform finite cohomological dimension. Therefore the spectral sequence converges strongly under the given hypotheses. □

REMARK 2.19. Recall the results in Remark 2.8. These show that, under the same hypotheses as in the theorem, for any fixed t_0, one obtains a spectral sequence:

$$\begin{aligned} E_2^{s,t} &\cong \mathcal{E}xt^{s,t}_{\mathcal{H}^*(\mathcal{A})}(\bar{M}, \bar{N}), \quad t \le t_0 \\ &\cong 0, \quad t > t_0 \\ &\Rightarrow \mathcal{H}^{s+t}(F_{t_0}\mathcal{RH}om_{\mathcal{A}}(M,N)) \end{aligned}$$

In particular taking $t_0 = 0$, one obtains a spectral sequence whose $E_2^{s,t}$ terms are trivial if $s < 0$ or $t > 0$ i.e. the spectral sequence is a fourth quadrant spectral sequence. The convergence of this spectral sequence is *conditional*, in general, under the same hypotheses as in (ii) of the above theorem. However, [Board] Theorem (7.2) shows that if M', N' are two objects in $Mod_l(\mathfrak{S}, \mathcal{A})$ satisfying the hypotheses of (ii) in the above theorem provided with maps $M' \to M$, $N' \to N$ inducing an isomorphism of the corresponding E_2-terms of the above spectral sequence, then one obtains an isomorphism of the abutments. In a similar manner, one obtains a spectral sequence

$$\begin{aligned} E^2_{s,t} &\cong \mathcal{T}or^{\mathcal{H}^*(\mathcal{A})}_{s,t}(\bar{M}, \bar{N}), \quad t \le t_0 \\ &\cong 0, \quad t > t_0 \\ &\Rightarrow \mathcal{H}^{-s+t}(F_{t_0}\mathcal{T}or^{\mathcal{A}}(M,N)) \end{aligned}$$

3. Triangulated category structure on the derived category of objects with finite tor dimension or objects that are perfect

We end this chapter by defining a derived category associated to the category of objects that are globally of f.t.d or perfect in the sense of Definition 2.11.

DEFINITION 3.1. Assume the situation in section 1. (i) If $\mathcal{A} \varepsilon Presh$ is an algebra, we will let $Mod_l^{f.t.d}(\mathfrak{S}, \mathcal{A}))$ $(Mod_l^{perf}(\mathfrak{S}, \mathcal{A}))$ denote the following category. An *object* of $Mod_l^{f.t.d}(\mathfrak{S}, \mathcal{A})$ $(Mod_l^{perf}(\mathfrak{S}, \mathcal{A}))$ is an object $M \varepsilon Mod_l(\mathfrak{S}, \mathcal{A})$ which is globally of f.t.d (perfect, respectively) together with the a non-decreasing exhaustive and separated filtration F compatible with the Cartan filtration on $\mathcal{A}$ along-with the choice of an $\bar{M} \varepsilon D_b(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$ globally of f.t.d (perfect) so that $Sp(\bar{M}) \simeq Gr_F(M)$. In case $\mathcal{H}^*(\mathcal{A})$ is locally constant on the site $\mathfrak{S}$, we will define an $M\varepsilon Mod_l(\mathfrak{S},\mathcal{A})$ to be constructible, if $\mathcal{H}^*(M)$ is constructible as a sheaf of modules over $\mathcal{H}^*(\mathcal{A})$. The full sub-category of $D(Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A}))$ consisting of objects that are also constructible will be denoted $D(Mod_l^{c,f.t.d}(\mathfrak{S},\mathcal{A}))$.

(ii) Given two such objects M, N, we let $Hom(M,N)$ denote the subset of all maps $f\varepsilon Hom_{\mathcal{A}}(M,N)$ so that f preserves the given filtrations on M and N i.e. $Hom(M,N) = F_0Hom_{\mathcal{A}}(M,N)$.

(iii) Given a map $f : M \to N$ as in (ii), f is a *filtered quasi-isomorphism* if it induces a quasi-isomorphism $F_iM \to F_iN$ for all i. (Observe that this implies $Gr_F(M) \to Gr_F(N)$ is also a quasi-isomorphism; conversely, if the filtrations are bounded below in the sense $F_iM = F_{i-1}M$ for all $i << 0$ and similarly for N, the last condition is equivalent to f being a filtered quasi-isomorphism.)

(iv) In the situation of Chapter II, sections 2 or 4, we observe that $F_0Hom_{\mathcal{A}}(M,N) = Map(\mathcal{S}, F_0\mathcal{H}om_{\mathcal{A}}(M,N))$. In this case we define the homotopy category associated to $Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A})$ ($Mod_l^{perf}(\mathfrak{S},\mathcal{A})$) to be given by the same objects as $Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A})$ ($Mod_l^{perf}(\mathfrak{S},\mathcal{A})$, respectively) and with morphisms $\mathcal{H}^0(Map(\mathcal{S}, F_0\mathcal{H}om_{\mathcal{A}}(M,N)))$. In the situation of Chapter II, section 3, we define the homotopy category associated to $Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A})$ ($Mod_l^{perf}(\mathfrak{S},\mathcal{A})$) by the same objects as $Mod_l(\mathfrak{S},\mathcal{A})$ ($Mod_l^{perf}(\mathfrak{S},\mathcal{A})$, respectively) and where the morphisms are given by $\mathcal{H}^0(F_0Map_{\mathcal{A}}(M,N))$.

PROPOSITION 3.2. *Assume the above situation.*

(i) The $\bar{M}$ in (i) in the above definition is uniquely determined by the given filtration

(ii) The homotopy categories defined above are additive

(iii) The class of filtered quasi-isomorphisms admits a calculus of left and right fractions.

PROOF. Observe that $\mathcal{H}^*(Gr_F(M)) \cong \mathcal{H}^*(Sp(\bar{M})) \cong \bar{M}$. Therefore $\bar{M}$ is uniquely determined by the given filtration. This proves (i). In the situation of Chapter II, sections 2 or 4, observe that $F_0\mathcal{H}om_{\mathcal{A}}(M,N)\varepsilon Presh$ and therefore $\mathcal{H}^0(Map(\mathcal{S}, F_0\mathcal{H}om_{\mathcal{A}}(M,N)))$ is an abelian group. In the case of Chapter II, section 3, it is clear that $\mathcal{H}^0(F_0(Map_{\mathcal{A}}(M,N)))$ may be identified with certain chain homotopy classes of filtration preserving maps M to N in $Mod_l(\mathfrak{S},\mathcal{A})$. Therefore this group is also abelian. Moreover, the category $Mod_l(\mathfrak{S},\mathcal{A})$ is clearly closed under sums and one may readily verify now that the homotopy category is additive. In order to prove (iii), we simply remark that the proof in Chapter II, lemma (4.3) carries over to the filtered setting, since all the constructions there preserve filtrations. □

DEFINITION 3.3. $D(Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A}))$ ($D(Mod_l^{perf}(\mathfrak{S},\mathcal{A}))$) will denote the localization of the homotopy category associated to $Mod_l^{f.t.d}(\mathfrak{S},\mathcal{A})$ ($Mod_l^{perf}(\mathfrak{S},\mathcal{A})$) by inverting filtered quasi-isomorphisms.

PROPOSITION 3.4. *Let D denote one of the above derived categories. Now*

(i) $Hom_D(M,N) \cong \mathcal{H}^0(Map(\mathcal{S}, F_0R\mathcal{H}om_{\mathcal{A}}(M,N)))$ in the situation of Chapter II, sections 2 or 4 and $\cong \mathcal{H}^0(F_0\mathcal{R}\mathcal{M}ap_{\mathcal{A}}(M,N))$ in the situation of Chapter II, section 3. (ii) The above derived category has the structure of a triangulated category.

PROOF. We will only consider the first situation, since the proof of the second situation is similar. If $M' \to M$ and $N \to N''$ are filtered quasi-isomorphisms, the spectral sequence in Remark 2.19 shows that the induced maps $\mathcal{H}^0(Map(\mathcal{S}, F_0R\mathcal{H}om_{\mathcal{A}}(M',N))) \to \mathcal{H}^0(Map(\mathcal{S}, F_0R\mathcal{H}om_{\mathcal{A}}(M,N))) \to \mathcal{H}^0(Map(\mathcal{S}, F_0R\mathcal{H}om_{\mathcal{A}}(M,N'')))$ are isomorphisms. It follows that the natural map from $\mathcal{H}^0(F_0Map_{\mathcal{A}}(M,N))$ to $\mathcal{H}^0(Map(\mathfrak{S}, F_0\mathcal{RH}om_{\mathcal{A}}(M,N)))$ factors through the derived category defined above. Now an argument as in the proof of Chapter II, Proposition 2.7 completes the proof of (i).

Since the homotopy category is additive, so is the derived category. Now it suffices to define the triangles. We define these to be diagrams of the form: $X \xrightarrow{u'} Y \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X$ where the maps all preserve the given filtrations and which are isomorphic in the filtered derived category above to diagrams of the form: $X \xrightarrow{u} Y \xrightarrow{v} Cone(u) \xrightarrow{w} \Sigma X$ (Here all the maps are again supposed to preserve the obvious filtrations.) We skip the verification that these satisfy the usual axioms on distinguished triangles. $\square$

3.1. Next assume one of the following: under the hypotheses that $Presh$ is a unital symmetric monoidal category, $\mathcal{A}$ is a commutative algebra in $Presh$ or under the hypotheses that $Presh = C(Mod(\mathfrak{S}, \mathcal{R}))$ for a commutative ringed site $(\mathfrak{S}, \mathcal{R})$, $\mathcal{A}$ is an E^∞-sheaf of algebras over an E^∞-operad. We may identify $Mod_l(\mathfrak{S}, \mathcal{A})$ and $Mod_r(\mathfrak{S}, \mathcal{A})$ and denote them by $Mod(\mathfrak{S}, \mathcal{A})$. The results of the last section show the following:

There exists bi-functors

$$\underset{\mathcal{A}}{\overset{L}{\otimes}} : D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A})) \times D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A})) \rightarrow D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A})) \tag{3.1.1}$$

$$\mathcal{RH}om_{\mathcal{A}} : D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A}))^{op} \times D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A})) \rightarrow D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A})) \tag{3.1.2}$$

so that $\mathcal{RH}om_{\mathcal{A}}(M, \mathcal{RH}om_{\mathcal{A}}(U,V)) \simeq \mathcal{RH}om_{\mathcal{A}}(M \underset{\mathcal{A}}{\overset{L}{\otimes}} U, V)$. Similar conclusions hold for the category
$D(Mod^{perf}(\mathfrak{S}, \mathcal{A}))$. i.e. The categories $D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{A}))$ and $D(Mod^{perf}(\mathfrak{S}, \mathcal{A}))$ are tensor categories with an internal hom defined by $\mathcal{RH}om_{\mathcal{A}}$. In the next chapter, we will establish the formalism of Grothendieck-Verdier duality in the setting of the above derived categories.

3.1.3. We end this chapter with a *summary of the basic results.*

- There are essentially two distinct frameworks for the rest of the paper: a category of presheaves on a site that is an enriched unital symmetric monoidal t-category and presheaves (and sheaves) of modules over an E^∞-operad on a site. Though the latter is not an enriched unital symmetric monoidal category, it is a sub-category of the category of complexes of sheaves of modules over a ringed site in the usual sense. The latter is an enriched unital symmetric monoidal t-category: this observation enables one to apply the techniques for enriched unital symmetric monoidal t-categories to presheaves and sheaves of modules over an E^∞-operad. For example, one may obtain a sheaf of E^∞-DGAs associated to the motivic complex on the étale, Nisnevich or Zariski site of a scheme and one may consider the category of sheaves of E^∞-modules over it. (See [J-6].)
- In the first case one can consider either a category of presheaves on a site which is an enriched (unital symmetric) monoidal category *or* one can consider a category of presheaves on a site taking values in an enriched unital symmetric monoidal category. Presheaves taking values in a stable simplicial model category (for example the stable simplicial model category of Γ-spaces, symmetric spectra) form an example of the latter. The A^1-local presheaves of spectra in the sense of [M-V] form an example of the former. (Once the axioms on the strong t-structure are verified in this case, the entire theory of Grothendieck-Verdier duality developed here, will apply to this case.)

- In all the above situations, one has an associated homotopy category (which is additive) and a derived category which is obtained by localizing the homotopy category by inverting a class of morphisms that are quasi-isomorphisms. The additivity of the homotopy category follows as in the discussion following the proof of Lemma (4.3) in Chapter II for the case of enriched unital symmetric monoidal categories: this is clear in the case of presheaves of algebras and modules over an E^{∞}-operad. One may also verify readily that finite sums are canonically quasi-isomorphic to finite products in all of the above cases.
- Assume any one of the above situations and that $\mathcal{A}$ is either an algebra with respect to the unital symmetric monoidal structure in that case or that $\mathcal{A}$ is an algebra over the given E^{∞}-operad in the operadic case. Let $\mathcal{S}$ denote the unit for the symmetric monoidal structure and let it denote the sheaf of rings $\mathcal{R}$ as in chapter II, section 3 (i.e. in the operadic case.) Let $Mod_l(\mathfrak{S}, \mathcal{A})$ denote the category of all left-modules over $\mathcal{A}$ and let $D(Mod_l(\mathfrak{S}, \mathcal{A}))$ denote the associated derived category. Let $D(Mod(\mathfrak{S}, \mathcal{S}))$ denote the derived category of modules over $\mathcal{S}$. In this case there exists a free-functor $\mathcal{F} : D(Mod(\mathfrak{S}, \mathcal{S})) \rightarrow D(Mod_l(\mathfrak{S}, \mathcal{A}))$ adjoint to the forgetful functor $U : D(Mod_l(\mathfrak{S}, \mathcal{A})) \rightarrow D(Mod(\mathfrak{S}, \mathcal{S}))$.

CHAPTER IV

Grothendieck-Verdier duality

1. Introduction

In this chapter we complete the theory of Grothendieck-Verdier duality in the setting of enriched symmetric monoidal t-categories. We show that the familiar six derived functors of Grothendieck may be defined in this setting with reasonable properties. The key to much of these is the frame-work developed in the first three chapters; in particular the spectral sequences in chapter 3 play a key role.

Throughout this section we will closely follow the framework and terminology adopted in Chapter II, section 1. In addition to the hypotheses and conventions there, we will adopt the following as well.

1.0.1. We will often impose various other hypotheses on the sites. Some of our results are often easier to establish if all the objects in a given site have finite L-cohomological dimension for some (possibly empty) set of primes L in the following sense: an object U in the site $\mathfrak{S}$ has finite L-cohomological dimension , if there exists an integer $N >> 0$ so that for every abelian l-torsion sheaf F, $l \varepsilon L$, $H^i_{\mathfrak{S}}(U, F) = 0$ for all $i > N$. (Here $H^i_{\mathfrak{S}}$ denotes the cohomology computed on the site $\mathfrak{S}$.) (If L is empty, the above hypothesis will mean that for every abelian sheaf F on the site $\mathfrak{S}$, $H^i_{\mathfrak{S}}(U, F) = 0$ for sufficiently large i.) Nevertheless, since we will need to consider schemes defined over arbitrary base schemes (for example, fields that have in general infinite cohomological dimension), we will never make this hypothesis a requirement. (On the other hand, when considering the right derived functor of the direct image functor and the direct image functor with proper supports, there is no loss of generality in making a similar assumption: see (2.2) below.)

1.0.2. There are often properties that we can require of morphisms between sites. Some of these are left as primitive, as the meaning may change from one situation to another. For example, the notion of a morphism being *proper*, *of finite type*, an *open immersion or imbedding* are left as primitive. If the sites are associated to schemes or algebraic spaces, these will have the familiar meaning.

Examples of sites. Clearly most of the sites that one encounters often satisfy these hypotheses: these include the big and small étale, Nisnevich and Zariski sites as well as the *h-topology or site* in [MV] associated to algebraic spaces of finite type over a Noetherian base scheme. In addition, one can also consider the familiar sites associated to locally compact Hausdorff topological spaces as shown in 2.13.

1.0.3. We will assume that if $\mathfrak{S}$ is a site (as above), $Presh(\mathfrak{S})$ denotes a category of presheaves on the site $\mathfrak{S}$ satisfying either one of the two hypotheses as in Chapter III, and $\mathcal{B}$ is an algebra in $Presh(\mathfrak{S})$. Recall this means it is either an enriched unital symmetric monoidal t-category and $\mathcal{A}$ is an algebra in the underlying symmetric monoidal category or that $Presh(\mathfrak{S}) = C(Mod(\mathfrak{S}, \mathcal{R}))$ for a sheaf of commutative Noetherian rings $\mathcal{R}$ and that $\mathcal{A}$ is a sheaf of algebras over an E^∞-operad. In the either case, we will let $\mathcal{S}$ denote the unit of the category $Presh(\mathfrak{S})$, i.e. in the first case $\mathcal{S}$ will denote the unit of the given unital symmetric monoidal structure and in the second case it will denote $\mathcal{R}$. (The existence of such a unit will simplify the proofs often.) $\mathcal{H}^*$ will denote the corresponding cohomology functor taking values in an Abelian category $\mathbf{A}$: we will require that this satisfy the hypothesis as in Chapter II, (2.1.1.*). The homotopy category and the derived category associated to $Presh(\mathfrak{S})$ will be again as in Chapter II. If $\mathcal{B}$ is an algebra, the derived category associated to $Mod_l(\mathfrak{S}, \mathcal{B})$ will often be denoted by $D(Mod_l(\mathfrak{S}, \mathcal{B}))$. In case X is the terminal object of the site $\mathfrak{S}$, we will often denote the ringed site $(\mathfrak{S}, \mathcal{B})$ by $(X, \mathcal{B})$ and the above derived category by $D(Mod_l(X, \mathcal{B}))$. We will also consider the derived categories $D(Mod_l^{c,f.t.d}(X, \mathcal{B}))$ (in case $\mathcal{H}^*(\mathcal{B})$ is locally constant on the site) and also $D(Mod_l^{perf}(X, \mathcal{B}))$ in the sense of last chapter.

Let $\mathfrak{S}$ and $\mathfrak{S}'$ denote two sites as above provided with presheaves of algebras $\mathcal{B}$ (on $\mathfrak{S}$) and $\mathcal{B}'$ on $\mathfrak{S}'$. Let X and X' denote the corresponding terminal objects. If $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is a map of such ringed sites, we define several derived functors associated to f in this section. The main result we obtain shows that these derived functors satisfy the usual formalism of Grothendieck-Verdier duality. These may be stated as follows. (Throughout, we will require the hypotheses as in 2.2 hold in the following statements.)

THEOREM 1.1. *(See 2.9.) Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map as in 2.4. (i) Under the hypotheses of 2.5 through 2.7, there exists a functor*

$$Rf_!^{\#} : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$$

which satisfies a projection formula as in 2.17. (ii) In case f is proper, $Rf_!^{\#}$ may be identified with Rf_* = the derived functor of the direct image functor. (iii) Moreover, the functor $Rf_!^{\#}$ has a right-adjoint $Rf_{\#}^! : D(Mod_l(X', \mathcal{B}')) \to D(Mod_l(X, \mathcal{B}))$.

(iv) Let Sp denote the functor in Chapter I, Definition 4.6. Then there exist natural isomorphisms of functors $Rf_!^{\#} \circ Sp \simeq Sp \circ Rf_!^{\#} : D(Mod_l(X, \mathcal{H}^*(\mathcal{B}))) \to D(Mod_l(X', Sp(\mathcal{H}^*(\mathcal{B}'))))$ and $Rf_{\#}^! \circ Sp \simeq Sp \circ Rf_{\#}^! : D(Mod_l(X', \mathcal{H}^*(\mathcal{B}'))) \to D(Mod_l(X, Sp(\mathcal{H}^*(\mathcal{B}))))$.

1.0.4. Next we consider dualizing presheaves both in the relative and absolute situation. We will assume throughout that all maps are compactifiable in the sense of 2.4. Furthermore we will assume that $(\mathbf{S}, \mathcal{A})$ is a commutative base-ringed site and that all ringed sites we consider are commutative and defined over it. We let $D(Mod_l^?(\mathfrak{S}, \mathcal{B}))$ denote either $D(Mod_l^{perf}(\mathfrak{S}, \mathcal{B}))$ in general or $D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{B}))$ when $\mathcal{H}^*(\mathcal{B})$ is locally constant on the site $\mathfrak{S}$. (We will similarly let $D(Mod_l^?(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$ denote either one of the derived categories $D(Mod_l^{perf}(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$ and $D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$.) In this case we let $D_{\mathcal{B}} = Rp_{\#}^!(\mathcal{A})$ and call it the *dualizing presheaf.* We let $\mathbb{D}_{\mathcal{B}} : D(Mod_l(X, \mathcal{B})) \to D(Mod_r(X, \mathcal{B}))$ denote the functor $F \to \mathcal{R}\mathcal{H}om_{\mathcal{B}}(F, D_{\mathcal{B}})$. Now we obtain the bi-duality theorem.

THEOREM 1.2. **Bi-duality Theorem** *(See Theorem 4.7.) Assume in addition to the above situation that the following hypothesis holds.*

Let $D_{\mathcal{H}^*(\mathcal{B})}$ denote the dualizing complex (defined in the usual sense) for the category $D(Mod_l^?(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$ of complexes of sheaves of $\mathcal{H}^*(\mathcal{B})$-modules. Let $\bar{F} \varepsilon D(Mod_l^?(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$ be so that the natural map $\bar{F} \to D_{\mathcal{H}^*(\mathcal{B})}(D_{\mathcal{H}^*(\mathcal{B})}(\bar{F})$ is a quasi-isomorphism.

Let $F \varepsilon D(Mod_l^?(\mathfrak{S}, \mathcal{B}))$ so that $Gr(F) \simeq Sp(\bar{F})$. Then the natural map $F \to \mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(F))$ is a quasi-isomorphism.

The above theorem applies in (at least) the following three situations:

(i) Consider schemes or algebraic spaces of finite type over a base scheme $\mathbf{S}$. Assume all the schemes and algebraic spaces are provided with the étale topology and L is a non-empty set of primes different from the residue characteristics. Let $\mathcal{A}$ denote a presheaf of commutative algebras on $\mathbf{S}$ so that for each n, $\mathcal{H}^n(\mathcal{A})$ is locally constant on the étale topology of $\mathbf{S}$ and has L-primary torsion. Now the hypotheses in the Bi-duality theorem are satisfied by any $\bar{F}\varepsilon D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. (See [SGA]$4_{1/2}$ p. 250.) Therefore, the bi-duality theorem holds for any $F\varepsilon D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{A}))$. The bi-duality theorem also holds for suitable L-completions of a presheaf of algebras $\mathcal{A}$. See 6.1 for a detailed discussion of this application.

(ii) Next assume $\bar{F}\varepsilon D(Mod_l^{perf}(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})))$ and that $D_{\mathcal{H}^*(\mathcal{B})}$ is locally quasi-isomorphic to $\mathcal{H}^*(\mathcal{B})$ modulo certain shift. In this case, the conclusion of the theorem holds for any $F\varepsilon D(Mod_l^{perf}(\mathfrak{S}, \mathcal{B}))$ so that $Gr(F) \simeq Sp(\bar{F})$.

(iii) Consider locally compact Hausdorff topological spaces over a base space $\mathbf{S}$ of the same type. Assume that L is a (possibly empty) set of primes for which all the spaces are of finite L-cohomological dimension. (Recall that if L is empty, this means all the spaces are of finite cohomological dimension.) Let $\mathcal{A}$ denote a presheaf of commutative algebras on $\mathbf{S}$ so that each $\mathcal{H}^n(\mathcal{A})$ is locally constant and of L-primary torsion. Then the hypotheses in the bi-duality theorem are satisfied by any $\bar{F}\varepsilon D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. (See [K-S-2] chapter III.) Therefore, the bi-duality theorem applies to the case when $\mathcal{A}$ is the constant presheaf of spectra representing a generalized cohomology theory, for example topological (complex) K-theory. The details are worked out at the end of this chapter. (See 6.4.)

In the following theorem, if $(\mathfrak{S}, \mathcal{A})$ is a ringed site, $D(Mod_l^?(\mathfrak{S}, \mathcal{A}))$ will denote either $D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{A}))$ or $D(Mod_l^{perf}(\mathfrak{S}, \mathcal{A}))$.

THEOREM 1.3. *(Grothendieck-Verdier duality)*

Assume in addition to the situation of the above theorem that $f : (X, \mathcal{H}^*(\mathcal{B})) \to (X', \mathcal{H}^*(\mathcal{B}'))$ *is either of finite tor dimension or perfect.*

(i) If $Rf_! : D(Mod_l^?(X, \mathcal{B})) \to D(Mod_l^?(X', \mathcal{B}'))$ *is defined by* $\mathbb{D}_{\mathcal{B}'} \circ Rf_* \circ \mathbb{D}_{\mathcal{B}}$*, there exists a natural isomorphism* $Rf_! \simeq Rf_!^{\#}$. *If* $Rf^! = \mathbb{D}_{\mathcal{B}} \circ Lf^* \circ \mathbb{D}_{\mathcal{B}'}$*, there exists a natural isomorphism* $Rf^! \simeq Rf^!_{\#}$ *of functors* $D(Mod_l^?(X', \mathcal{B}')) \to D(Mod_l^?(X, \mathcal{B}))$.

(ii) There exist the following natural isomorphisms of functors:

$Rf_* \circ \mathbb{D}_{\mathcal{B}} \simeq \mathbb{D}_{\mathcal{B}} \circ Rf_!^{\#} : D(Mod_l(X, \mathcal{B})) \to D_r(X', \mathcal{B}')$, $Rf^!_{\#} \circ \mathbb{D}_{\mathcal{B}'} \simeq \mathbb{D}_{\mathcal{B}} \circ Lf^* : D(Mod_l(X', \mathcal{B}')) \to D(Mod_r(X, \mathcal{B}))$,

$\mathbb{D}_{\mathcal{B}'} \circ Rf_* \simeq Rf_! \circ \mathbb{D}_{\mathcal{B}} : D(Mod_l^?(X, \mathcal{B})) \to D(Mod_r^?(X', \mathcal{B}'))$ *and* $Lf^* \circ \mathbb{D}_{\mathcal{B}'} \simeq \mathbb{D}_{\mathcal{B}} \circ Rf^! : D(Mod_l^?(X', \mathcal{B}')) \to D(Mod_r^?(X, \mathcal{B}))$

If X belongs to the site $\mathfrak{S}$, we define the generalized homology of X with respect to $\mathcal{A}$ to be the hypercohomology of X with respect to $Rp^!_{\#}(\mathcal{A})$. (Here X denotes the terminal object of the site $\mathfrak{S}$.) We say that X has Poincaré-Verdier duality if there exists a class $[X]\varepsilon\mathcal{H}^{-n}(\mathbb{H}(X; Rp^!_{\#}(\mathcal{A})))$ so that cap-product with this class induces an isomorphism $\mathcal{H}^k(\mathbb{H}(X; p^*(\mathcal{A}))) \to \mathcal{H}^{-n+k}(\mathbb{H}(X; Rp^!_{\#}(\mathcal{A})))$. We conclude by showing that if we are considering schemes over a base-scheme provided with the étale site, Poincaré-Verdier duality in the above sense implies an isomorphism between the functors $Rf^!$ and $f^* \circ \Sigma^n$. We also derive various other formal consequences of Grothendieck-Verdier duality.

Next we will provide a quick summary of the various sections. In the second section we consider the derived functors of the direct image and inverse image functors associated to a map of ringed sites as in 2.1 or 2.4. We also define the hypercohomology spectrum functor, the derived functor of the direct image functor with proper supports and obtain a projection formula. In the third section we show, under the hypothesis that the sites are locally coherent and coherent, that there exists a right adjoint to the derived functor of the direct image with proper supports.

In the fourth section, we define various *dualizing presheaves* and end with the bi-duality theorem. In the fifth section we derive the Grothendieck-Verdier formalism of duality between the various derived functors: we show that all the familiar results on Grothendieck-Verdier duality carry over to our general setting. (In turn, these are applied in the next chapter to provide micro-local character-cycles for constructible sheaves with values in complex K-theory.) We end by considering some concrete examples in section six.

2. The derived functors of the direct and inverse image functors

2.1. Maps of ringed sites. Let $Presh$ ($Presh'$) denote the category of presheaves on a site $\mathfrak{S}'$ ($\mathfrak{S}$, respectively) as in Chapter III, 1.2. We will further assume *one* of the following:

- Both are unital symmetric monoidal t-categories. X' (X) is the terminal object of the site $\mathfrak{S}'$ ($\mathfrak{S}$) and $\mathcal{B}'$ ($\mathcal{B}$) is a presheaf of algebras in $Presh(\mathfrak{S}')$ ($Presh(\mathfrak{S})$, respectively) *or*
- $\mathcal{O}'$ ($\mathcal{O}$) is an E^∞-operad on the ringed site $(\mathfrak{S}', \mathcal{R}')$ ($(\mathfrak{S}, \mathcal{R})$, respectively). $\mathcal{B}'$ ($\mathcal{B}$) is a presheaf of algebras over the operad $\mathcal{O}'$ ($\mathcal{O}$, respectively) and X' (X) is the terminal object of the site $\mathfrak{S}'$ ($\mathfrak{S}$, respectively)

DEFINITION 2.1. In the first case, a map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ of ringed sites is a map of sites $f : \mathfrak{S} \to \mathfrak{S}'$ so that the induced functors: $f_* : Presh(\mathfrak{S}) \to Presh(\mathfrak{S}')$, $f^* : Presh(\mathfrak{S}') \to Presh(\mathfrak{S})$ satisfy the following conditions. f_* preserves admissible monomorphisms and commutes with the functors EM_n, $n \varepsilon \mathbb{Z}$, while f^* preserves the monoidal structure. We let $\mathcal{S}$ ($\mathcal{S}'$) denote the unit of $Presh(\mathfrak{S})$ ($Presh(\mathfrak{S}')$, respectively). The inverse-image functor $Mod(\mathfrak{S}', \mathcal{S}') \to Mod(\mathfrak{S}, \mathcal{S})$ induced by f will be denoted f^{-1} and we require that $f^{-1}(\mathcal{S}') = \mathcal{S}$. Moreover, in case $\mathcal{B} = f^{-1}(\mathcal{B}')$, we require that f^* also preserves the strongly triangulated structure and strong t-structure.

In the second case, a map $f : (X, B) \to (X', \mathcal{B}')$ is given by a map of sites $f : (\mathfrak{S}, \mathcal{R}) \to (\mathfrak{S}', \mathcal{R}')$ so that $f^{-1}(\mathcal{R}') = \mathcal{R}$, $f^{-1}(\mathcal{O}'(k)) = \mathcal{O}(k)$ for all $k \geq 0$. In addition one is given a map $\mathcal{B}' \to f_*(\mathcal{B})$ of algebras over the operad $\mathcal{O}'$.

In this context $Mod_l(X, \mathcal{B})$ and $Mod_l(X', \mathcal{B}')$ will denote the category of sheaves of modules over $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ respectively. f_* (f^*) now induces a functor $f_* : Mod_l(X, \mathcal{B}) \to Mod_l(X', \mathcal{B}')$ ($f^* : Mod_l(X', \mathcal{B}') \to Mod_l(X, \mathcal{B})$, respectively).

(Following 1.0.3, we let $\mathcal{S}$ ($\mathcal{S}'$) denote the unit of $Presh(\mathfrak{S})$ ($Presh(\mathfrak{S}')$, respectively). Recall that in the second case this is $\mathcal{R}$ ($\mathcal{R}'$, respectively).)

REMARK 2.2. Often we may assume that there is a base-ringed site $(\mathbf{S}, \mathcal{A})$ and that the given map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is a map of ringed sites over $(\mathbf{S}, \mathcal{A})$.

EXAMPLE 2.3. As an example of the first case of maps of ringed sites, we may consider the following. Let $\mathcal{C}$ denote a fixed enriched unital symmetric monoidal t-category. Let $\mathcal{B}$ ($\mathcal{B}'$) denote a presheaf of algebras with values in $\mathcal{C}$ on the site $\mathfrak{S}$ ($\mathfrak{S}'$). Then any map $f : \mathfrak{S} \to \mathfrak{S}'$ for which there exists an induced map $\mathcal{B}' \to f_*(\mathcal{B})$ of presheaves of algebras defines a map of ringed sites $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ if X' (X) denotes the terminal object of the site $\mathfrak{S}'$ ($\mathfrak{S}$, respectively).

2.2. Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites in the above sense. If $F \varepsilon D(Mod_l(X, \mathcal{B}))$, we define $Rf_*(F) \varepsilon D(Mod_l(X', \mathcal{B}'))$ as in Chapter II, (1.1). The definition of $\underset{\Delta}{\text{holim}}$ as an end shows one may identify $Rf_*(F)$, upto a natural quasi-isomorphism, with $f_* \underset{\Delta}{\text{holim}}\, G^\bullet F$. In this context we will always make the following assumption:

there exists an integer $N >> 0$ so that $R^s f_*(\bar{M}) = 0$ for all $s > N$ and all $\bar{M} \varepsilon Mod_l(X, \mathcal{H}^*(\mathcal{B}))$.

One may define the hypercohomology of an object $U \varepsilon \mathfrak{S}/X$ with respect to an $F \varepsilon D(Mod_l(X, \mathcal{B}))$ by $\mathbb{H}(U, F) = \underset{\Delta}{\text{holim}}\{\Gamma(U, \mathcal{G}^n F)|n\}$. Observe that there exist spectral sequences:

$$E_2^{s,t} = R^s f_* \mathcal{H}^t(F) \Rightarrow \mathcal{H}^{s+t}(Rf_*(F)) \quad \textit{and} \tag{2.2.1}$$

$$E_2^{s,t} = H^s(U, \mathcal{H}^t(F)) \to H^{s+t}(\mathbb{H}(U, F)) \tag{2.2.2}$$

In view of the hypothesis 2.2, the first spectral sequence converges strongly. Since we do not assume a similar condition of finite cohomological dimension on the objects of the site, the second spectral sequence does *not* converge strongly in general.

2.3. One may define $Lf^* : D(Mod_l(X', \mathcal{B}')) \to D(Mod_l(X, \mathcal{B}))$ by

$$Lf^*(K) = \mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\overset{L}{\otimes}} f^{-1}(K),\ K \varepsilon D(Mod_l(X', \mathcal{B}'))$$

where the left derived functor $\underset{f^{-1}(\mathcal{B}')}{\overset{L}{\otimes}}$ is defined as $\underset{\Delta}{\text{hocolim}}(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(P(K)_\bullet))$ where $P(K)_\bullet \to K$ is a flat resolution as in Chapter II, Proposition 2.4. Now there exist spectral sequences:

$$E^2_{s,t} = L^s f^*(\mathcal{H}^t(N)) \Rightarrow \mathcal{H}^{-s+t}(Lf^*(N)), \quad N \varepsilon D(Mod_l(X'; \mathcal{B}')) \tag{2.3.1}$$

PROPOSITION 2.4. *(i) If $\Omega F'' \to F' \to F \to F''$ is a triangle in $D(Mod_l(X, \mathcal{B}))$, $Rf_*(\Omega F'') \to Rf_*(F') \to Rf_*(F) \to Rf_*(F'')$ is a triangle in $D(Mod_l(X', \mathcal{B}'))$. Moreover, if $F' \to F$ is a quasi-isomorphism in $D(Mod_l(X, \mathcal{B}))$, the induced map $Rf_*(F') \to Rf_* F$ is a quasi-isomorphism in $D(Mod_l(X', \mathcal{B}'))$.*

(ii) If $F' \to F \to F'' \to \Sigma F'$ is a triangle in $D(Mod_l(X', \mathcal{B}'))$, the induced diagram $Lf^*(F') \to Lf^*(F) \to Lf^*(F'') \to Lf^*(\Sigma F')$ is a triangle in $D(Mod_l(X, \mathcal{B}))$. Moreover, if $F' \to F$ is a quasi-isomorphism in $D(Mod_l(X', \mathcal{B}'))$, the induced map $Lf^* F' \to Lf^* F$ is a quasi-isomorphism in $D(Mod_l(X, \mathcal{B}))$.

PROOF. These are immediate from our definitions, and the hypotheses on homotopy limits and homotopy colimits. □

Next we recall the functors $R\mathcal{H}om_{\mathcal{B}}$ for an algebra $\mathcal{B} \varepsilon Presh(\mathfrak{S})$. We define $R\mathcal{H}om_{\mathcal{B},l}$ ($R\mathcal{H}om_{\mathcal{B},r}$) to be the functor $R\mathcal{H}om_{\mathcal{B}}$ applied to the category $Mod_l(X, \mathcal{B})$ ($Mod_r(X, \mathcal{B})$, respectively). We proceed to consider variants of these presently.

LEMMA 2.5. *Let $\mathcal{B}'$, $\mathcal{B}$ denote two algebras in $Presh(\mathfrak{S})$. We will let $Mod_{\mathcal{B},l;\mathcal{B}',r}(\mathfrak{S})$ denote the category of objects in $Presh(\mathfrak{S})$ that have the structure of a presheaf of left-$\mathcal{B}$ and right $\mathcal{B}'$-bi-modules. Let $N \varepsilon Mod_{\mathcal{B},l;\mathcal{B}',r}(\mathfrak{S})$ and $P \varepsilon Mod_{\mathcal{B},r}(\mathfrak{S})$. Assume there exists a map $\mathcal{B}' \to \mathcal{B}$ of algebras. Then*

(i) $\mathcal{RH}om_{\mathcal{B}',r}(M \overset{L}{\underset{\mathcal{B}}{\otimes}} N, P) \simeq \mathcal{RH}om_{\mathcal{B},r}(M, \mathcal{RH}om_{\mathcal{B}',r}(N,P))$ *and*

(ii) $RMap_{\mathcal{B}',r}(M \overset{L}{\underset{\mathcal{B}}{\otimes}} N, P) \simeq RMap_{\mathcal{B},r}(M, \mathcal{RH}om_{\mathcal{B}',r}(N,P))$

PROOF. Let $P(M)_\bullet \to M$ denote a simplicial resolution as in Chapter II, Proposition 2.4 by objects in $Mod_r(\mathfrak{S}, \mathcal{B}')$ and let $P(N)_\bullet \to N$ denote a corresponding simplicial resolution in $Mod_r(\mathfrak{S}, \mathcal{B})$. Let $\{\mathcal{G}^n P|n\}$ denote the Godement resolution. Now

$$\mathcal{RH}om_{\mathcal{B}',r}(M \overset{L}{\underset{\mathcal{B}}{\otimes}} N, P) = \underset{\Delta}{\text{holim}}\, \Delta\{\mathcal{H}om_{\mathcal{B}',r}(P(M)_\bullet \underset{\mathcal{B}}{\otimes} P(N)_\bullet, P)\}$$

$$\simeq \underset{\Delta}{\text{holim}}\, \Delta\{\mathcal{H}om_{\mathcal{B},r}(P(M)_\bullet., \Delta\mathcal{H}om_{\mathcal{B}',r}(P(N)_\bullet., \mathcal{G}^n QP)\}$$

$$\simeq \underset{\Delta}{\text{holim}}\{\mathcal{H}om_{\mathcal{B},r}(P(M)_\bullet, \underset{\Delta}{\text{holim}}\{\mathcal{H}om_{\mathcal{B}',r}(P(N)_\bullet, \mathcal{G}^n QP)\})\}$$

$$\simeq \mathcal{RH}om_{\mathcal{B},r}(M, \mathcal{RH}om_{\mathcal{B}',r}(N,P))$$

The first $\simeq$ follows from Chapter II, (2.0.15) while the second $\simeq$ follows from chapter I, cofinality of the homotopy limits. The last $\simeq$ is clear from the definition of the above derived functors. This proves (i). The proof of (ii) is similar. □

PROPOSITION 2.6. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites as before. Then one obtains the quasi-isomorphism:*

(i) $Rf_*\mathcal{RH}om_{\mathcal{B},r}(Lf^*M, N) \simeq \mathcal{RH}om_{\mathcal{B}',r}(M, Rf_*N)$, $M \varepsilon D(Mod_r(\mathfrak{S}', \mathcal{B}'))$ *and* $N \varepsilon D(Mod_r(\mathfrak{S}, \mathcal{B}))$.

Under the same hypotheses, one also obtains:

(ii) $RMap_{\mathcal{B},r}(Lf^*M, N) \simeq RMap_{\mathcal{B}',r}(M, Rf_*N)$. *(i.e. The functor Rf_* is right adjoint to Lf^*.)*

PROOF. We will let $\mathcal{S}$ ($\mathcal{S}'$) denote a unit for the category $Presh(\mathfrak{S})$ ($Presh(\mathfrak{S}')$, respectively) as in 1.0.3. Clearly it suffices to show that one obtains a quasi-isomorphism after applying the functor $RMap_{\mathcal{S}}(K, -)$ to both sides, where $K \varepsilon Presh_{\mathcal{C}}(\mathfrak{S}')$. On applying this functor to the left-hand-side, one obtains:

$$RMap_{\mathcal{S}}(K, Rf_*\mathcal{RH}om_{\mathcal{B},r}(Lf^*(M), N)) \simeq RMap_{\mathcal{S}'}(f^{-1}(K), \mathcal{RH}om_{\mathcal{B},r}(Lf^*(M), N))$$

$$\simeq RMap_{\mathcal{B},r}(f^{-1}(K) \overset{L}{\underset{\mathcal{S}'}{\otimes}} Lf^*(M), N) = RMap_{\mathcal{B},r}(f^{-1}(K) \overset{L}{\underset{\mathcal{S}'}{\otimes}} f^{-1}(M) \overset{L}{\underset{f^{-1}(\mathcal{B}')}{\otimes}} \mathcal{B}, N)$$

$$\simeq RMap_{f^{-1}(\mathcal{B}'),r}(f^{-1}(K) \overset{L}{\underset{\mathcal{S}'}{\otimes}} f^{-1}(M), \mathcal{RH}om_{\mathcal{B},r}(\mathcal{B}, N))$$

$$\simeq RMap_{f^{-1}(\mathcal{B}'),r}(f^{-1}(K) \overset{L}{\underset{\mathcal{S}'}{\otimes}} f^{-1}(M),\ N) \simeq RMap_{\mathcal{B}',r}(K \overset{L}{\underset{\mathcal{S}}{\otimes}} M,\ Rf_*N)$$

$$\simeq RMap_{\mathcal{S}}(K, \mathcal{RH}om_{\mathcal{B}',r}(M, Rf_*N))$$

The first $\simeq$ follows from the adjunction between f^{-1} and f_* while the second and third follow from Lemma (2.5). The next $\simeq$ follows from Chapter II, Proposition 2.1 (i), while the one following it results from the adjunction between f^{-1} and f_*. Finally the last $\simeq$ follows by another application of Lemma 2.5. Chapter II, 2.4.2, shows how to obtain the second assertion from the first. $\square$

PROPOSITION 2.7. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$, $g : (X', \mathcal{B}') \to (X'', \mathcal{C})$ denote maps of ringed sites. Then the natural map $R(g \circ f)_* F \to Rg_* \circ Rf_* F$, $F \varepsilon D(Mod_l(X, \mathcal{B}))$ is a quasi-isomorphism.*

PROOF. This results readily from the following observations:

(i) $R(g \circ f)_* F = \underset{\Delta}{\mathrm{holim}}\{(g \circ f)_* \mathcal{G}^n F|n\} \simeq g_* \circ f_* \underset{\Delta}{\mathrm{holim}}\{\mathcal{G}^n F|n\}$ and

(ii) the natural map $g_* \circ f_* \underset{\Delta}{\mathrm{holim}}\{\mathcal{G}^n F|n\} \to g_* \underset{\Delta}{\mathrm{holim}}\{\mathcal{G}^m f_*(\underset{\Delta}{\mathrm{holim}}\{\mathcal{G}^n F|n\})|m\}$ is a quasi-isomorphism.

The first is clear. Now observe that both sides of (ii) are functorial in F and send triangles in F to triangles. This shows that there exist spectral sequences:

$E_2^{s,t} = R^s(g \circ f)_* \mathcal{H}^t(F)\tilde{} \Rightarrow \mathcal{H}^{s+t}(R(g \circ f)_*(F))\tilde{}$ and

$E_2^{s,t} = \mathcal{H}^s(Rg_* \circ Rf_* \mathcal{H}^t(F))\tilde{} \Rightarrow \mathcal{H}^{s+t}(Rg_* \circ Rf_*(F))\tilde{}$

These spectral sequences converge strongly in general in view of the hypothesis 2.2. Therefore the above spectral sequences reduce the proof to that of abelian sheaves. In this case the quasi-isomorphism $R(g_* \circ f_*)(F) = R(g \circ f)_*(F) \overset{\simeq}{\to} Rg_* \circ Rf_*(F)$ is clear, since $Rf_*(F) = f_*\{\mathcal{G}^n F|n\}$ is a complex of sheaves, each term of which is *flabby* on the site $\mathfrak{S}'$. $\square$

THEOREM 2.8. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites.*

(i) Suppose $\mathfrak{S}$ is algebraic and $\mathfrak{S}'$ is locally coherent. Let $\{F_\alpha|\alpha\}$ denote a filtered direct system of objects in $D(Mod_l(X, \mathcal{B}))$. Now the natural map $\underset{\alpha}{colim} Rf_*(F_\alpha) \overset{\simeq}{\to} Rf_*(\underset{\alpha}{colim}\, F_\alpha)$ is a quasi-isomorphism in general (under the hypothesis 2.2). The functor Rf_* commutes upto quasi-isomorphism with finite sums and hence with all small sums.

(ii) Suppose $\mathfrak{S}$ is coherent. Then, for each n, the functor $F \to \mathcal{H}^*(\mathbf{H}(X, F))$, F, $D(Mod_l(X, \mathcal{B})) \to$ (abelian groups) commutes with filtered direct limits under the hypothesis of finite L-cohomological dimension as in 1.1 on the site $\mathfrak{S}$ and the presheaves all have l-torsion cohomology sheaves. For each n, the functor $F \to \mathcal{H}^n(\mathbf{H}(X, F))$ commutes with finite sums.

PROOF. Under the hypothesis of 2.2, the spectral sequence in (0.5.2) converges strongly for all F. Therefore the above spectral sequence reduces the first assertions to abelian sheaves which are clear. The assertion about Rf_* and $F \mapsto \mathbf{H}^(X, F)$ commuting with finite products is clear. Finite sums were observed to be finite products in our basic framework - see chapter I, Proposition 2.4. Under the hypothesis on finite L-cohomological dimension, the spectral sequence in (0.5) converges strongly reducing the statements in (ii) to the corresponding ones for abelian sheaves which were observed to be true by the discussion in chapter II, (1.0.6). $\square$

We end this section by considering *the derived functor of the direct image functor with proper supports.* For this we will consider collections of ringed sites $\{(X, \mathcal{B})\}$, with $\mathcal{B}$ a sheaf of algebras in $Presh(X)$ so that the following hypotheses are satisfied:

2.4. Every map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is *compactifiable*, i.e. fits in a commutative triangle

$$\begin{tikzcd}
(X, \mathcal{B}) \arrow[rr, "j"'] \arrow[dr, "f"'] & & (\bar{X}, \bar{\mathcal{B}}) \arrow[dl, "\bar{f}"] \\
& (X', \mathcal{B}') &
\end{tikzcd}$$

with $\bar{f}$ proper and j an open imbedding and $\mathcal{B} \cong j^*(\bar{\mathcal{B}})$.

In this situation, we define $Rf_!^{\#}F = R\bar{f}_*(j_{U!}^{\#}F)$, $F\varepsilon Presh(\mathfrak{S}')$. This will be called *the derived functor of the direct image functor with proper supports associated to f.*

We will further assume that, so defined $Rf_!^{\#}$ has the following properties:

2.5. if $j_U : U \to X$ is an object in the site $\mathfrak{S}$ and $f_U = f \circ j_U$, $Rf_{U!}^{\#}F \simeq Rf_!^{\#} j_{U!}^{\#}F$, $F\varepsilon D(Mod_l(U, \mathcal{B}))$

2.6. if f is proper, $Rf_!^{\#}(F) \simeq Rf_*(F)$, $F\varepsilon D(Mod_l(X, \mathcal{B}))$

2.7. Moreover, $Rf_!^{\#}$ is independent (upto natural quasi-isomorphism) of the factorization of f into $\bar{f}$ and j.

We will next consider, under what hypotheses, the properties 2.4 through 2.7 hold.

For this we begin by considering *proper* base-change. Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites as before. Let $F\varepsilon D(Mod_l(X, \mathcal{B}))$. We say that the pair (F, f) is *cohomologically proper* if for every map $g : (Y', \mathcal{C}') \to (X', \mathcal{B}')$ of ringed spaces, the induced map

(2.7.1) $$Lg^*(Rf_*F) \to Rf'_*Lg'^*F$$

is a quasi-isomorphism, where the maps g', f' are defined by the cartesian square:

$$\begin{tikzcd}
(Y, \mathcal{C}) \arrow[r, "g'"] \arrow[d, "f'"'] & (X, \mathcal{B}) \arrow[d, "f"] \\
(Y', \mathcal{C}') \arrow[r, "g"] & (X', \mathcal{B}')
\end{tikzcd}$$

where $\mathcal{C} = f'^{-1}(\mathcal{C}') \underset{f'^{-1}(g^{-1}(\mathcal{B}'))}{\otimes} g'^{-1}(\mathcal{B})$.

2.8. We say *that proper-base-change* holds for all presheaves F we consider, if the conclusions above are valid for every proper map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ and for any map $g : (Y', \mathcal{C}') \to (X', \mathcal{B}')$ of sites.

2.9. An alternate, somewhat weaker, hypothesis is the following: let

$$\begin{tikzcd} & (\bar{X}, \bar{\mathcal{B}}) \arrow[d, "\bar{f}"] \\ (U, \mathcal{B}_{|U}) \arrow[ur, "\bar{j}"] \arrow[r, "j"'] & (X, \mathcal{B}) \end{tikzcd}$$

denote a commutative triangle with $\bar{j}$ and j open imbeddings and $\bar{f}$ proper. Then the natural map $j_{U!}^{\#} \to R\bar{f}_* \circ \bar{j}_{U!}^{\#}$ is an isomorphism of functors: $D(Mod_l(U, \mathcal{B}_U) \to D(Mod_l(X, \mathcal{B}))$.

PROPOSITION 2.9. *Assume that the hypothesis (2.4) holds and that either (2.8) or (2.9) holds for every sheaf F we consider. Then $Rf_!\#$ is independent of the factorization of f as above. This also implies the hypotheses in 2.5 and 2.6.*

PROOF. This is a standard proof. Consider two compactifications $(X, \mathcal{B}) \xrightarrow{j_1} (\bar{X}_1, \bar{\mathcal{B}}_1)$ and $(X, \mathcal{B}) \xrightarrow{j_2} (\bar{X}_2, \bar{\mathcal{B}}_2)$. Take the product $j_1 \times j_2$ composed with the diagonal $(X, \mathcal{B}) \to (X \times X, p_1^{-1}(\mathcal{B}) \otimes p_2^{-1}(\mathcal{B}))$. Let the closure of X in $\bar{X}_1 \times \bar{X}_2$ by the above map be denoted $\bar{X}$; we provide $\bar{X}$ with the presheaf of algebras which is the restriction of $p_1^{-1}(\mathcal{B}) \otimes p_2^{-1}(\mathcal{B})$. As a result one may assume without loss of generality that there exists a proper map $p : (\bar{X}_1, \bar{\mathcal{B}}_1) \to (\bar{X}_2, \mathcal{B}_2)$ so that $p \circ j_1 = j_2$. Now proper base-change shows that there is a natural quasi-isomorphism: $j_{2!}^{\#}(F) \simeq Rp_*(j_{1!}^{\#}(F))$. Alternatively, the weaker hypothesis 2.9 also shows the same. Now compose with $R\bar{f}_*$ to complete the proof of the independence on the factorization of f. Clearly this implies that if f is proper, $Rf_!^{\#}(F) \simeq Rf_*(F)$ and thereby proves 2.6. To prove the hypothesis 2.5, observe that one may factor the map $U \xrightarrow{j_0} X \xrightarrow{j_1} \bar{X}$ as the composition of an open imbedding $U \xrightarrow{j} \bar{U}$ followed by a proper map $p : \bar{U} \to \bar{X}$. Now both 2.8 and 2.9 once again provide a quasi-isomorphism: $Rp_* \circ j_!^{\#}(F) \simeq j_{1!}^{\#} \circ j_{0!}^{\#}$. Finally compose with $R\bar{f}_*$ to obtain 2.5. □

2.10. Moreover, 2.4 through 2.7 imply that if $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ and $g : (X', \mathcal{B}') \to (X'', \mathcal{B}'')$ are maps of ringed sites as above, then $Rg_!^{\#} \circ Rf_!^{\#}(F) \simeq R(g \circ f)_!^{\#}(F)$, $F \varepsilon D(Mod_l(X, \mathcal{B}))$. This may be established exactly as Proposition 2.7. (See, for example, [SGA]4 Exposé XVII.) We skip the details.

Next we will specialize to various special sites to apply the above results. We will mention at least three distinct situations where the hypotheses in 2.4 through 2.7 are satisfied.

2.11. The simplest situation is where, for every ringed site $(\mathfrak{S}, \mathcal{B})$ we consider, $\mathfrak{S}$ is *proper* over a base $\mathbf{S}$ and where for every morphism between ringed sites $(\mathfrak{S}, \mathcal{B}) \xrightarrow{f} (\mathfrak{S}', \mathcal{B}')$, the underlying map $f : \mathfrak{S} \to \mathfrak{S}'$ of sites is *proper.* For example: we restrict to proper schemes or proper algebraic spaces over a Noetherian separated base scheme. In this case $Rf_!^{\#}$ identifies with Rf_*.

2.12. Assume next that the sites we consider are all sites associated to schemes or algebraic spaces of finite type over a Noetherian base scheme $\mathbf{S}$ provided with a presheaf of algebras $\mathcal{A}$. For example, the sites could be the small or big Zariski, étale or the Nisnevich sites, the flat site, or the recent *h-site* as in [Voe-1] of schemes of finite type over a Noetherian base scheme S. We will further assume that the morphism $f : \mathfrak{S} \to \mathfrak{S}'$ of sites is induced by a map of schemes $f : X \to Y$ over $\mathbf{S}$. (Here X (Y) is the terminal object of $\mathfrak{S}$ $(\mathfrak{S}'$, respectively).) Furthermore, if $p_X : X \to \mathbf{S}$ is the structure map of X, we let the site $\mathfrak{S}$ be provided with the pre-sheaf of algebras $\mathcal{B} = p_X^{-1}(\mathcal{A})$. We say a map $f : X \to Y$ of schemes (or algebraic spaces) is *compactifiable*, if it can be factored as the composition of an open immersion $j : X \to \bar{X}$ and a proper map $\bar{f} : \bar{X} \to Y$. (If we restrict to schemes that are quasi-projective over the base scheme $\mathbf{S}$, this is always possible.) Now every map

$f : (\mathfrak{S}, \mathcal{B}) \to (\mathfrak{S}', \mathcal{B}')$ of ringed sites is compactifiable in the sense of 2.4 if the map of schemes (or algebraic spaces) $f : X \to Y$ is compactifiable.

PROPOSITION 2.10. *Assume we are in the situation of 2.12 and that proper-base-change holds for all sheaves of $\mathcal{H}^*(\mathcal{A})$-modules. Then proper base-change in the sense of 2.7.1 holds.*

PROOF. Observe that, under the hypotheses of 2.12, the functor Lg^* (Lg'^*) may be identified with g^{-1} $({g'}^{-1}$, respectively). There exist spectral sequences:

$$E_2^{s,t} = g^* R^s f_*(\mathcal{H}^t(F))\tilde{\ } \Rightarrow \mathcal{H}^{s+t}(g^* Rf_* F) \text{ and } E_2^{s,t'} = R^s f'_* g'^* \mathcal{H}^t(F)\tilde{\ } \to \mathcal{H}^{s+t}(Rf'_* g'^* F)\tilde{\ }$$

for any $F \varepsilon D(Mod_l(X, \mathcal{B}))$. These spectral sequences converge strongly in view of the hypothesis 2.2. In this case, it suffices to show that one obtains an isomorphism at the E_2-terms. This is clear from the proper base-change for all sheaves of $\mathcal{H}^*(\mathcal{A})$-modules. $\square$

COROLLARY 2.11. *(i) Assume we are in the situation of 2.12 and that proper base change holds for all sheaves of $\mathcal{H}^*(\mathcal{A})$-modules. Assume all maps of schemes or algebraic spaces we consider are compactifiable and the sites we consider are all coherent in the sense of Chapter II. Then the functor $Rf_!^\# : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', , \mathcal{B}')$ is well defined and has the properties in 2.4 through 2.7 for all maps $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ of sites. Moreover, it commutes upto quasi-isomorphism with filtered colimits and sums upto quasi-isomorphism in general.*

(ii) In particular, the conclusions above hold if the sites considered are the small étale sites associated to schemes or algebraic spaces, all maps are compactifiable and if $\mathcal{H}^n(\Gamma(U, \mathcal{B}))$ is torsion for all n, all U in the corresponding site of the base scheme $\mathbf{S}$.

PROOF. (i) The first statement is clear from Proposition 2.9. The last statement in (i) is clear from 2.8. (ii) follows from (i) since, proper base-change holds for all torsion abelian sheaves on the étale site. $\square$

2.13. We will next consider sites which are the usual sites associated to locally compact Hausdorff topological spaces over a base topological space $\mathbf{S}$ which is also assumed to be locally compact and Hausdorff. We will assume that all spaces are of finite cohomological dimension. If X is a topological space, the associated site will be denoted simply by X. Let $\mathcal{A}$ denote a presheaf of algebras on $\mathbf{S}$. If $p : X \to \mathbf{S}$ is the structure map of X, we will let $\mathcal{B} = p^{-1}(\mathcal{A})$. The morphism $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ of sites will be the one associated in the obvious manner to a continuous map $f : X \to Y$ of topological spaces. In this case we may define a functor $f_!^\# : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$ (intrinsically) by
(2.13.1)

$$\Gamma(V, f_!\#(M)) = \{s \varepsilon \Gamma(V, f_* M) | f : support(s) \to Y \quad is \quad proper\}, \quad M \varepsilon D(Mod_l(X, \mathcal{B}))$$

(Recall that a continuous map $f : X \to Y$ of topological spaces is proper if and only if the image of closed sets is closed.) So defined, one may readily verify that if $f = j : X \to Y$ is an open imbedding, then $f_!^\#$ is merely extension by zero. Moreover, if f is *proper*, $f_!^\# = f_*$. Therefore, it follows that if the map f admits a factorization $f = \bar{f} \circ j$ with $\bar{f}$ proper and j an open imbedding, the functor $f_!^\# = \bar{f}_* \circ j_!^\#$ and therefore the right-hand-side is independent of the factorization of $f = \bar{f} \circ j$.

Assume as above that $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is a map of ringed spaces with X, X' locally compact Hausdorff topological spaces. Assume that f factors as the composition $(X, \mathcal{B}) \xrightarrow{j} (\bar{X}, \bar{\mathcal{B}}) \xrightarrow{\bar{f}} (X', \mathcal{B}')$ with j an open imbedding and $\bar{f}$ proper. Now we define

$$Rf_!^\# : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}')) \text{ by } Rf_!^\#(M) = R\bar{f}_*(\circ j_!^\#)(M)), \, M \varepsilon D(Mod_l(X, \mathcal{B})).$$

One may readily see, in view of the hypothesis on finite cohomological dimension, that the last functor is independent of the factorization.

PROPOSITION 2.12. *Recall the functors*

$Sp : D_+(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{B}))) \to D_+(Mod_l(\mathfrak{S}, Sp(\mathcal{H}^*(\mathcal{B}))))$

(and $Sp : D_+(Mod_l(\mathfrak{S}', \mathcal{H}^*(\mathcal{B}'))) \to D_+(Mod_l(\mathfrak{S}', Sp(\mathcal{H}^*(\mathcal{B}'))))$

from Chapter I, Definition 4.6. Let $f : (\mathfrak{S}, \mathcal{B}) \to (\mathfrak{S}', \mathcal{B}')$ *denote a map of ringed sites. Then one obtains natural quasi-isomorphisms:*

(i) $Sp(Rf_*\bar{M}) \simeq Rf_*(Sp(\bar{M}))$, $\bar{M}\varepsilon D_+(Mod_l(\mathfrak{S}, \mathcal{H}^*(\mathcal{B})\check{\ }))$ *and*

$Sp(Lf^*(\bar{N})) \simeq Lf^*(Sp(\bar{N}))$, $\bar{N}\varepsilon D_+(Mod_l(\mathfrak{S}', \mathcal{H}^*(\mathcal{B}')\check{\ }))$

(ii) The same conclusions hold with the functor Rf_* *replaced by* $Rf_!^{\#}$ *if* f *is a compactifiable map of schemes or algebraic spaces in the situation of 2.12, or in the situation as in 2.13*

PROOF. Recall that if $K = \underset{i}{\Pi} K(i)\varepsilon D_+(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$, $Sp(K) = \underset{i}{\Pi} \underset{\Delta}{\operatorname{holim}} DN(EM_i(K(i)))$. Now the hypotheses in 2.1 , shows that the functor f_* commutes with EM_i. f_* also clearly commutes with products and homotopy inverse limits. This proves the first assertion for the derived functor of the direct image functor.

Recall that the functor EM_n is exact and therefore commutes with filtered colimits. Now the hypotheses in 2.1 shows the functor f^{-1} commutes with EM_n. This readily shows that $Sp(f^{-1}(\bar{N})) \simeq f^{-1}(Sp(\bar{N}))$. Now chapter III, Proposition (2.12) with $\bar{M} = \mathcal{H}^*(\mathcal{B})$ and $\mathcal{B}$ replaced by $f^{-1}(\mathcal{B}')$ shows that $Lf^*(Sp(\bar{N})) = Sp(\mathcal{H}^*(\mathcal{B}))\underset{f^{-1}Sp(\mathcal{H}^*(\mathcal{B}'))}{\overset{L}{\otimes}} f^{-1}Sp(\bar{N}) \simeq$ $Sp(\mathcal{H}^*(\mathcal{B})\underset{f^{-1}\mathcal{H}^*(\mathcal{B}')}{\overset{L}{\otimes}} f^{-1}\bar{N}) = Sp(Lf^*(\bar{N}))$. This proves the second quasi-isomorphism in (i).

Now we consider the assertion in (ii). Let $j : X \to \bar{X}$ denote the given open imbedding and let $i : Z = \bar{X} - X \to \bar{X}$ denote the closed imbedding of its complement. Let $\bar{N}\varepsilon D(Mod_l(\bar{X}, \bar{\mathcal{B}}))$. Now one obtains the triangles:

$j_!^{\#} j^*\bar{N} \to \bar{N} \to i_*i^{-1}\bar{N}$ and $j_!^{\#} j^* Sp(\bar{N}) \to Sp(\bar{N}) \to i_*i^{-1}Sp(\bar{N})$.

Since the functor Sp sends triangles to triangles (see Chapter I, Proposition 4.4), one also obtains the triangle: $Sp(j_!^{\#} j^*\bar{N}) \to Sp(\bar{N}) \to Sp(i_*i^{-1}\bar{N})$. The natural quasi-isomorphism of the last and middle terms with the corresponding terms of the previous triangle show that there exists a natural quasi-isomorphism $j_!^{\#} j^* Sp(\bar{N}) \xrightarrow{\simeq} Sp(j_!^{\#} j^*\bar{N})$. This proves the assertion in (ii) in view of (i). □

DEFINITION 2.13. Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites. We say Rf_* is *perfect* (*of finite tor dimension*, respectively) if it sends $D(Mod^{perf}(X, \mathcal{H}^*(\mathcal{B})))$ to $D(Mod^{perf}(X', \mathcal{H}^*(\mathcal{B}')))$ ($D(Mod^{f.t.d}(X, \mathcal{H}^*(\mathcal{B})))$ to $D(Mod^{f.t.d}(X', \mathcal{H}^*(\mathcal{B})))$, respectively). We will similarly define $Rf_!^{\#}$ to be perfect (of finite tor dimension, respectively). We will say f *is perfect (of finite tor dimension)* if both $Rf_!^{\#}$ and Rf_* are perfect (of finite tor dimension, respectively).

PROPOSITION 2.14. *Let* $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ *denote a map of ringed sites.*

(i) If Rf_* is perfect (of finite tor dimension) the functor Rf_* sends $D(Mod_l^{perf}(X, \mathcal{B}))$ to $D(Mod_l^{perf}(X', \mathcal{B}'))$ ($D(Mod_l^{f.t.d}(X, \mathcal{B}))$ to $D(Mod_l^{f.t.d}(X', \mathcal{B}'))$, respectively). The corresponding assertion also holds with $Rf_!^{\#}$ in the place of Rf_*.

*(ii) Lf^* is always perfect (of finite tor dimension). i.e. Lf^* sends $D(Mod_l^{perf}(X', \mathcal{B}'))$ to $D(Mod_l^{perf}(X, \mathcal{B}))$
($D(Mod_l^{f.t.d}(X', \mathcal{B}'))$ to $D(Mod_l^{f.t.d}(X, \mathcal{B}))$, respectively).*

Proof. Let $\{F_kN|k\}$ denote the exhaustive and separated filtration of N with respect to which N is globally of finite tor dimension. Recall this means there exists an object $\bar{N}\varepsilon D(\mathfrak{S}, \mathcal{H}^*(\mathcal{B}))$ globally of finite tor dimension so that $Sp(\bar{N}) \simeq Gr_F(N)$. Now $Rf_!^{\#}(F_{k-1}N) \rightarrow Rf_!^{\#}(F_kN) \rightarrow Rf_!^{\#}(Gr_{F,k}N) \rightarrow Rf_!^{\#}(\Sigma F_{k-1}N)$ is a triangle and the first map is an admissible mono-morphism (since products and homotopy limits preserve admissible mono-morphisms). Therefore $\{Rf_!^{\#}F_kN|k\}$ is a filtration of $Rf_!^{\#}N$ and $GrRf_!^{\#}N \simeq Rf_!^{\#}(Gr_FN) \simeq Rf_!^{\#}(Sp(\bar{N})) \simeq Sp(Rf_!^{\#}(\bar{N}))$. Clearly, the same arguments show that $\{Rf_*F_kN|k\}$ is a filtration of Rf_*N and that $GrRf_*N \simeq Rf_*(Gr_FN) \simeq Rf_*(Sp(\bar{N})) \simeq Sp(Rf_*(\bar{N}))$. To complete the proof of the first assertion, now it suffices show that $Rf_!^{\#}$ sends an exhaustive (separated) filtration to an exhaustive (a separated) filtration. Since $Rf_!^{\#}$ commutes with filtered colimits (as shown in Theorem 2.8), it follows immediately that $Rf_!^{\#}$ sends an exhaustive filtration to an exhaustive filtration. In view of the hypotheses in 2.2 and Theorem 2.8, one may show readily that the separatedness of the filtration on N implies the filtration $\{Rf_!^{\#}F_kN|k\}$ is also separated.

Now we consider the second assertion. We may first replace $M\varepsilon D(Mod_l^{perf}(X', \mathcal{B}'))$ by an object that is also flat over $\mathcal{B}'$ by Chapter II, 2.1.1. Therefore we may assume M itself is flat; now we will show $f^*(M) = \mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)$ belongs to $D(Mod_l^{perf}(X, \mathcal{B}))$. For this observe that the filtration on M is compatible with the Cartan filtrations on $\mathcal{B}$ and $f^{-1}(\mathcal{B}')$. Therefore, $Gr_F(f^*(M)) = Gr_F(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)) = Gr_{\mathcal{C}}(\mathcal{B}) \underset{f^{-1}(Gr_{\mathcal{C}}(\mathcal{B}'))}{\otimes} f^{-1}(Gr_F(M)) = f^*(Gr_F(M))$. Now the morphism $F_{k-1}(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)) \rightarrow F_k(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M))$ is an admissible mono-morphism, since it is the kernel of the admissible epimorphism

$$F_k(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)) \rightarrow Gr_k(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)).$$

(To see the last map is in fact an admissible epimorphism, recall

$$F_k(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)) = Coequalizer(\ F_k(\mathcal{B} \otimes f^{-1}(\mathcal{B}') \otimes f^{-1}(M)) \underset{g}{\overset{f}{\rightrightarrows}} F_k(\mathcal{B} \otimes f^{-1}(M)\ ))$$

while

$$Gr_k(\mathcal{B} \underset{f^{-1}(\mathcal{B}')}{\otimes} f^{-1}(M)) = Coequalizer(\ Gr_k(\mathcal{B} \otimes f^{-1}(\mathcal{B}') \otimes f^{-1}(M)) \underset{g}{\overset{f}{\rightrightarrows}} Gr_k(\mathcal{B} \otimes f^{-1}(M)\ ))$$

and co-equalizers preserve admissible epimorphisms. See axiom (STR6) in Chapter I. (Alternatively, one replaces the co-equalizers above with homotopy co-equalizers as in Chapter II, 1.2.1.) One may see readily that Lf^* sends an exhaustive filtration to an exhaustive filtration; one may use the second strongly convergent spectral sequence of Chapter III, Remark 2.19 to conclude that the induced filtration on $f^*(M)$ is also separated. Therefore, it follows readily that Lf^* sends $D(Mod_l^{perf}(X', \mathcal{B}')) \rightarrow D(Mod_l^{perf}(X, \mathcal{B}))$. The proof in the case of finite tor dimension is similar. $\square$

COROLLARY 2.15. *Suppose $(X', \mathcal{B}')$ is the ringed site associated to the étale site of a scheme or algebraic space X' so that in addition $\mathcal{H}^*(\mathcal{B}) = \underset{n}{\oplus}\mathcal{H}^n(\mathcal{B})$ is locally constant. Let $f : X \to X'$ be a map of algebraic spaces and $\mathcal{B} = f^{-1}(\mathcal{B}')$. Then $Rf_!^{\#}$ induces a functor $D(Mod_l^{f.t.d}(X, \mathcal{B})) \to D(Mod_l^{f.t.d}(X', \mathcal{B}'))$.*

PROOF. In view of the hypothesis in 2.1.1, it is a standard result (which one may readily prove using the projection formula) that $Rf_!^{\#}(\bar{N})$ is of finite tor dimension as an object in $D(Mod_l(X', \mathcal{H}^*(\mathcal{B}')))$ if $\bar{N} \varepsilon D(Mod_l(X, \mathcal{H}^*(\mathcal{B})))$ is of finite tor dimension. □

PROPOSITION 2.16. *(Base-change) Assume the situation of 2.8. Then the map $Lg^* Rf_!^{\#} F \to Rf'^{\#}_! Lg'^* F$ is a quasi-isomorphism provided proper base-change as in 2.8 holds.*

PROOF. This is clear since every morphism is assumed to be compactifiable. □

PROPOSITION 2.17. *(Projection formula). Assume in addition to the above situation that $Rf_!^{\#}$ has finite tor dimension. Then*

$$Rf_!^{\#}(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M) \simeq Rf_!^{\#}(N) \overset{L}{\underset{\mathcal{B}'}{\otimes}} M$$

for $N \varepsilon D(Mod_r(X, \mathcal{B}))$, $M \varepsilon D(Mod_l(X', \mathcal{B}')$ and either N or M is of finite tor dimension.

PROOF. One first observes that there exists a map $Rf_!^{\#}(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M) \to Rf_!^{\#}(N) \overset{L}{\underset{\mathcal{B}'}{\otimes}} M$ that preserves the filtration on either side. (Recall these filtrations are induced in the obvious manner from the canonical Cartan filtrations on N, $\mathcal{B}$, M and $\mathcal{B}'$.) Now consider the spectral sequences obtained from these filtrations:

$$E_2^{s,t} = \mathcal{H}^{s+t}(Gr_t[Rf_!^{\#}(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M)])\,\tilde{}\Rightarrow \mathcal{H}^{s+t}(Rf_!^{\#}(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M))\,\tilde{} \quad \text{and}$$

$$E_2^{s,t} = \mathcal{H}^{s+t}(Gr_t[Rf_!^{\#}(N) \overset{L}{\underset{\mathcal{B}'}{\otimes}} M])\,\tilde{}\Rightarrow \mathcal{H}^{s+t}(Rf_!^{\#}(N) \overset{L}{\underset{\mathcal{B}'}{\otimes}} M)\,\tilde{}$$

The natural map above induces a map of these spectral sequences; the two spectral sequences converge strongly by the hypotheses 2.2 and on finite tor dimension. (See also 2.15 as well as the identification of the E_2-terms below.) Therefore it suffices to show one obtains an isomorphism at the E_2-terms. Now Chapter III, Proposition (2.10)(i), Chapter III, Proposition (2.12) and the proof of Proposition (2.12) above show that

$$Gr[Rf_!^{\#}(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M)] \simeq Rf_!^{\#}[Gr(N \overset{L}{\underset{\mathcal{B}}{\otimes}} f^* M)]$$

$$\simeq Rf_!^{\#}(Gr(N) \overset{L}{\underset{Gr(\mathcal{B})}{\otimes}} Grf^*(M)) \simeq Rf_!^{\#}(Sp(\bar{N}) \overset{L}{\underset{Sp(\mathcal{H}^*(\mathcal{B}))}{\otimes}} Sp(f^*(\bar{M})))$$

$$\simeq Rf_!^{\#}(Sp(\bar{N} \overset{L}{\underset{\mathcal{H}^*(\mathcal{B})}{\otimes}} f^*(\bar{M}))) \simeq Rf_!^{\#}(Sp(\bar{N} \overset{L}{\underset{f^{-1}(\mathcal{H}^*(\mathcal{B}'))}{\otimes}} f^{-1}(\bar{M})))$$

$$\simeq Sp(Rf_!^{\#}(\bar{N} \overset{L}{\underset{f^{-1}(\mathcal{H}^*(\mathcal{B}'))}{\otimes}} f^{-1}(\bar{M}))).$$

By the usual projection formula, one may now identify the last term with $Sp(Rf_!^{\#}(\bar{N}) \overset{L}{\underset{\mathcal{H}^*(\mathcal{B}')}{\otimes}} \bar{M})$. An argument (just as above) using Chapter III, Propositions (2.10)(i) and (2.12) as well as the proof of Proposition (2.12) above identifies this with $Gr[Rf_!^{\#}(N) \overset{L}{\underset{\mathcal{B}'}{\otimes}} M]$. □

3. The right adjoint to the derived direct image functor with proper supports

We will assume throughout this section that $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ is a map of ringed sites and that one may define a functor $Rf_!^{\#} : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$ satisfying the hypotheses of 2.4 through 2.7. The goal of this section is to define a right adjoint to this functor. We will first define a functor $Rf_{\#}^!$ explicitly and show this is in fact a right adjoint for objects of finite tor dimension or objects that are perfect. We also provide a second construction of this functor using a recent theorem of Neeman that applies since the functor $Rf_!^{\#}$ is shown to commute with all (small) sums upto quasi-isomorphism. We will then consider various properties of this functor.

We begin by defining the functor $Rf_{\#}^! : D(Mod_l(X', \mathcal{B}')) \to D(Mod_l(X, \mathcal{B}))$. Let $K \varepsilon D(Mod_l(X', \mathcal{B}'))$. Let $j_U : U \to X$ be in the site $\mathfrak{S}'$ and let $\mathcal{B}_U = j_{U!}^{\#} j_U^*(\mathcal{B})$. We let

$$\Gamma(U, Rf_{\#}^! K) = R\Gamma(X', \mathcal{RH}om_{\mathcal{B}'}(Rf_!^{\#}(\mathcal{B}_U), K)) \tag{3.0.2}$$

Alternatively we may define a sequence of functors $Rf_{\#,n}^! : D(Mod_l(X', \mathcal{B}')) \to D(Mod_l(X, \mathcal{B}))$, $n \geq 1$, by $\Gamma(U, Rf_{\#,n}^! K) = R\Gamma(X', \mathcal{H}om_{\mathcal{B}}(Rf_!^{\#}(\mathcal{B}_U), G^n QK))$. Now $\{\Gamma(U, Rf_{\#,n}^!(K)) | n\}$ forms a cosimplicial object and we let

$$\Gamma(U, Rf^! K) = \underset{\Delta}{\operatorname{holim}} \{\Gamma(U, Rf_{\#,n}^!(K)) | n\} \tag{3.0.3}$$

Let $j : X \to \bar{X}$ denote an open imbedding and $\bar{f} : \bar{X} \to X'$ a proper map so that $f = \bar{f} \circ j$. Let $\bar{\mathcal{B}}$ denote a presheaf of algebras on $\bar{X}$ so that $j^*(\bar{\mathcal{B}}) = \mathcal{B}$ and $\bar{f} : (\bar{X}, \bar{\mathcal{B}}) \to (X', \mathcal{B}')$ is a map of ringed sites. Observe the pairing

$$j_* \circ j_{U*} j_U^* j^*(\bar{\mathcal{B}}) \otimes j_!^{\#} \circ j_{U!}^{\#} j_U^* j^*(\bar{\mathcal{B}}) \otimes j_* \circ j_{U*} j_U^* j^*(\bar{\mathcal{B}}) \to j_!^{\#} \circ j_{U!}^{\#} j_U^* j^*(\bar{\mathcal{B}})$$

that factors in the obvious two ways showing that $j_!^{\#} \circ j_{U!}^{\#} j_U^* j^*(\bar{\mathcal{B}}) \varepsilon Mod_{bi}(X, j_* \circ j_{U*} j_U^* j^*(\bar{\mathcal{B}}))$, i.e. $j_!^{\#} \circ j_{U!}^{\#} j_U^* j^*(\bar{\mathcal{B}})$ has the structure of a presheaf of bi-modules over $j_* \circ j_{U*} j_U^* j^*(\bar{\mathcal{B}})$. This shows that $Rf_!^{\#}(\mathcal{B}_U) = R\bar{f}_*(j_!^{\#} \circ j_{U!}^{\#} j_U^* j^*(\bar{\mathcal{B}}))$ has the structure of a presheaf of bi-modules over the presheaf of algebras $R(\bar{f}_* \circ j_* \circ j_{U*}) j_U^* j^*(\bar{\mathcal{B}}))$. The latter is a presheaf of algebras over $\mathcal{B}'$. Therefore, by taking sections over X' and using (3.0.2), it follows that both $\mathcal{RH}om_{\mathcal{B}'}(Rf_!^{\#}(\mathcal{B}_U), K)$ and $\mathcal{RH}om_{\mathcal{B}'}(Rf_!^{\#}(\mathcal{B}_U), \mathcal{G}^n QK)$ have the structure of a presheaf of left modules over the presheaf of algebras over $\mathcal{B}$. i.e.

So defined, $Rf_{\#}^!(K)$ *and* $Rf_{\#,n}^!(K) \varepsilon D(Mod_l(X, \mathcal{B}))$.

PROPOSITION 3.1. *Assume the above situation. Then*

(i) $R\Gamma(U, \mathcal{RH}om_{\mathcal{B}}(\mathcal{B}_U, Rf_{\#}^! K)) \simeq R\Gamma(U, RHom_{\mathcal{B}'}(Rf_!^{\#}(\mathcal{B}_U), K))$

(ii) If $K' \to K \to K'' \to \Sigma K'$ *is a triangle in* $D(Mod_l(X', \mathcal{B}'))$ *and* $j_U : U \to X$ *is in the site* $\mathfrak{S}$*, the diagram*

$$\Gamma(U, Rf_{\#}^! K') \to \Gamma(U, Rf_{\#}^! K) \to \Gamma(U, Rf_{\#}^! K'') \to \Gamma(U, Rf_{\#}^! \Sigma K')$$

is a triangle in $D(Mod_l(X, \mathcal{B}))$.

(iii) If $K' \to K$ *is a quasi-isomorphism in* $D(Mod_l(X', \mathcal{B}'))$*, the induced map* $\Gamma(U, Rf_{\#}^! K') \to \Gamma(U, Rf_{\#}^! K)$ *is a quasi-isomorphism for each* U *in the site* $\mathfrak{S}$.

(iv) If $f = j : U \to X$ *belongs to the site* $\mathfrak{S}$, $Rf_{\#}^! = j^*$.

The properties in (i) through (iii) also hold for the functors $Rf^!_{\#,n}$, $n \geq 0$.

PROOF. (i) is clear from the definition and Chapter II, (1.2.6). (ii) is clear from the definition of the functor $\mathcal{RH}om_{\mathcal{B}'}$ as in Chapter II, Definition (4.12), and the definition of $R\Gamma(X,-)$ as in Chapter II, (1.1.1). (iii) is also clear for the same reasons. In the case of (iv), the functor $j_!^{\#}$ has a right adjoint j^*; therefore $Rj^!_{\#}$ may be identified with j^*. $\square$

REMARK 3.2. The above proposition shows that the functor $Rf^!_{\#}$ induces a functor $Rf^!_{\#} : D(Mod_l(X', \mathcal{B}')) \to D(Mod_l(X, \mathcal{B}))$.

Next we proceed to prove a result that holds for the functor $Rf_!^{\#}$ as well as the functor $Rf^!_{\#}$. To handle both cases simultaneously we will consider the following abstract situation.

3.1. Let $(X, \mathcal{B})$, $(X', \mathcal{B}')$ denote two ringed sites as above. Let $\phi : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$ denote a functor with the following properties.

(i) ϕ preserves triangles and quasi-isomorphisms and each $\phi(F)$ has a filtration induced by the Cartan filtration on F.

(ii) there exist a sequence of functors $\{\phi^n : Mod_l(X, \mathcal{B})) \to Mod_l(X', \mathcal{B}')|n\}$ so that if $K \varepsilon D(Mod_l(X, \mathcal{B}))$, $\{\phi^n(K)|n\}$ forms a cosimplicial object in $Mod_l(X', \mathcal{B}')$ and $\phi(K) = \underset{\Delta}{\mathrm{holim}}\{\phi^n(K)|n\}$

(iii) for each $n \geq 0$, there exists a functor $\bar{\phi}^n : Mod_l(X, \mathcal{H}^*(\mathcal{B})) \to Mod_l(X', \mathcal{H}^*(\mathcal{B})')$ so that $Sp \circ \bar{\phi}^n$ is naturally quasi-isomorphic to $\phi^n \circ Sp$

(iv) There exists a spectral sequence $E_2^{u,v} = H^u(\{\phi^n(\mathcal{H}^v(F))|n\})\tilde{} \Rightarrow \mathcal{H}^{u+v}(\phi(F))\tilde{}$, $F \varepsilon D(Mod_l(X, \mathcal{B}))$ with $E_2^{u,v} = 0$ for $u < 0$. Moreover, there exists an integer $N >> 0$ so that $E_2^{u,v} = 0$ if $u > N$ independent of v and $F \varepsilon D(Mod_l(X, \mathcal{B}))$.

3.2. Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites as above. Now $Rf_!^{\#} : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$ as defined above clearly satisfies the above hypotheses. To see this just observe that the hypothesis in 2.2 implies the vanishing condition (iv) above, whereas the other conditions (i) through (iii) are clear. Moreover, one may readily see, in view of the definition of the functor $Rf^!_{\#}$ above, that the same hypotheses as above, imply the conditions in (iv) for the functor $Rf^!_{\#}$ at least if $\mathcal{H}^*(\mathcal{B}')$ is locally constant, $\mathcal{B} = f^{-1}(\mathcal{B}')$ and the sites are Noetherian. (See **??** for a proof.) See Proposition 5.12 for an application of this result.

LEMMA 3.3. *Let $L \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{B}))$ be of finite tor dimension in the sense of chapter III, Definition (3.1). If $P(L)_\bullet \to L$ is a simplicial resolution of L in the sense of Chapter II, Proposition (2.4) we obtain the quasi-isomorphisms:*

$$\underset{\Delta}{hocolim}\{\phi(P(L)_n)|n\} \simeq \phi(\underset{\Delta}{hocolim}\{P(L)_n|n\} \simeq \phi(L)$$

PROOF. The condition (iv) in 3.1 shows that if $M \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{A}))$, and n is a fixed integer, there are only finitely many $E_2^{u,v}$-terms whose sub-quotients appear as the associated graded terms of $\mathcal{H}^{-n}(\phi(M))\tilde{}$.

For such an M, we will define a *non-increasing filtration* by letting $F_m(M) = \tau_{\leq -m}M$ where $\tau_{\leq -m}$ is defined as in chapter I. We let $F_m(\phi(L)) = \phi(F_m L)$: by the hypotheses this defines a filtration of $\phi(L)$. Let n denote a fixed integer throughout the rest of the proof. By the definition of the filtration F_m on F and by the hypothesis (iv) above, for each fixed integer q,

$$\mathcal{H}^q(F_m(\phi(L)))\tilde{} \cong \mathcal{H}^q(\phi(L))\tilde{} \tag{3.2.1}$$

if m is sufficiently small.

The key-step in the proof will be to show that for each fixed integer q, the natural map $\underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_k))|k\} \to F_m(\phi(L))$ induces an isomorphism:

$$\mathcal{H}^q(\underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_k))|k\})\tilde{} \cong \mathcal{H}^q(F_m(\phi(L)))\tilde{} \tag{3.2.2}$$

if m is sufficiently small.

We will first complete the proof assuming (3.2.2). Observe that

$$\underset{m\to-\infty}{\text{colim}}\ \underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_n))|n\} \simeq \underset{\Delta}{\text{hocolim}}\{\underset{m\to-\infty}{\text{colim}}\ F_m(\phi(P(L)_n))|n\}$$

$$\simeq \underset{\Delta}{\text{hocolim}}\{\phi(P(L)_n)|n\}$$

(where the last $\simeq$ is by (3.2.1)) applied to each $P(L)_n$ instead of L; by (3.2.1) again

$$\underset{m\to-\infty}{\text{colim}}\ F_m(\phi(L)) \simeq \phi(L).$$

Therefore, it will follow $\underset{\Delta}{\text{hocolim}}\{\phi(P(L)_n)|n\} \simeq \phi(L)$.

Now we proceed to prove (3.2.2). First observe that $\mathcal{H}^v(F_m(L))\tilde{} \cong 0$ if $v > -m$ and for all F. Now consider the spectral sequence for the homotopy colimit in chapter I, section 1, (HCl):

$$E^2_{u,v} = H_u(\mathcal{H}^v(F_m(\{\phi(P(L)_n)|n\})))\tilde{} \Rightarrow \mathcal{H}^{-u+v}(\underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_n))|n\})\tilde{} \tag{3.2.3}$$

For a fixed integer q, the only $E^2_{u,v}$-terms whose sub-quotients appear as the associated graded terms of $\mathcal{H}^q(\underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_n))|n\})\tilde{}$ are those with $q \le v \le q+u$ and $v \le -m$ and hence in particular $m \le -q$. The same spectral sequence in (3.2.3) for F_mL also shows that $E_2^{u,v} \neq 0$ only for $m \le -q$. Therefore, for $m > -q$, $\mathcal{H}^q(\underset{\Delta}{\text{hocolim}}\{F_m(\phi(P(L)_n))|n\})\tilde{} = 0$ and $\mathcal{H}^q(F_m(\phi(L)))\tilde{} = 0$. Therefore, in order to prove (3.2.2), it suffices to show that the natural map

$$\underset{\Delta}{\text{hocolim}}\{(F_n\phi(P(L)_k)/F_m\phi(P(L)_k))|k\} \to F_n(\phi(L))/F_m(\phi(L)) \tag{3.2.4}$$

induces an isomorphism on $\mathcal{H}^q$ for all m and n with $n \le m$ and all q (in fact it suffices to consider $m \le -q$). Since both sides preserve triangles, one may use ascending induction on $m-n$ and reduce to the case where $m = n+1$. Now the left-hand-side (right-hand-side) of (3.2.4) may be identified with

$$\underset{\Delta}{\text{hocolim}}\{Gr_n(\phi(P(L)_k))|k\} \simeq \underset{\Delta}{\text{hocolim}}\{\phi(Gr_n(P(L)_k))|k\} \simeq \underset{\Delta}{\text{hocolim}}\{\phi(Sp(P(\bar{L})_k)_n|k\}$$

$$\simeq \underset{\Delta}{\text{hocolim}}\{Sp(\bar{\phi}(P(\bar{L})_k))_n|k\}$$

$(\phi(Gr_n(L)) \simeq \phi(Sp(\bar{L})_n) \simeq Sp(\bar{\phi}(\bar{L}))_n$, respectively).

Therefore, it suffices to show :

(3.2.5) $$\underset{\Delta}{\text{hocolim}}\{Sp(\bar{\phi}(P(\bar{L})_k))|k\} \simeq Sp(\bar{\phi}(\bar{L}))$$

Now one may identify the left-hand-side of (3.2.5) with

$$\underset{\Delta}{\text{hocolim}}\,\underset{\Delta}{\text{holim}}\{DN \circ GEM(\bar{\phi}^r(P(\bar{L})_s))|r\}$$

where the outer homotopy colimit is over all s and the inner homotopy limit is over all r. For a fixed s, $\{\bar{\phi}^r(P(\bar{L})_s)|r\}$ is a cosimplicial abelian presheaf. Therefore its normalization $N(\{\bar{\phi}^r(P(\bar{L})_s)|r\})$ is a co-chain complex. Now the hypothesis (3.1(iv)) shows we may replace this co-chain complex, by the bounded co-chain complex $\tau_{\leq N}N(\{\bar{\phi}^r(P(\bar{L})_s)|r\})$.

By axiom (ST9) of chapter I, we see that, for each fixed s:

(3.2.6)
$$\underset{\Delta}{\text{holim}}\, DN(GEM(\tau_{\leq N}N(\{\bar{\phi}^r(P(\bar{L})_s)|r\}))) \simeq \Omega^N \underset{\Delta}{\text{hocolim}} DNGEM(\tau_{\leq N}N(\{\bar{\phi}^r(P(\bar{L})_s)|r\}))[N_h]$$

Since two homotopy colimits commute, the left-hand-side of (3.2.5) may be identified with:

$$\Omega^N \underset{\Delta}{\text{hocolim}}\underset{\Delta}{\text{hocolim}}\{GEM(DN(GEM(\tau_{\leq N}N(\{\bar{\phi}^r(P(\bar{L})_s)|r\}))[N_h]))\}$$

where the inner (outer) $\underset{\Delta}{\text{hocolim}}$ is over the s (r, respectively). A direct computation using the spectral sequence for the homotopy colimit shows the latter may be identified with $\Omega^N \underset{\Delta}{\text{hocolim}}\{DN(GEM(\tau_{\leq N}(\bar{\phi}^r(\bar{L}))[N_h]))|r\}$. By the same argument as above, one may identify this with

$$\underset{\Delta}{\text{holim}}\{DNGEM(\tau_{\leq N}(\bar{\phi}^r(\bar{L})))|r\} \simeq \underset{\Delta}{\text{holim}}\{Sp(\tau_{\leq N}(\bar{\phi}^r(\bar{L})))|r\}$$

$$\simeq \underset{\Delta}{\text{holim}}\{Sp(\bar{\phi}^r(\bar{L}))|r\} \simeq \underset{\Delta}{\text{holim}}\{\phi^r Sp(\bar{L})|r\} \simeq \phi Sp(\bar{L}).$$

We have thereby shown that the map in (3.2.5) is a quasi-isomorphism. $\square$

REMARK 3.4. Observe that we have used the strong-t-structure in an essential manner. As pointed out in Chapter I this is needed mainly to be able make homotopy colimits and limits commute.

THEOREM 3.5. *Assume the above situation. Let* $K \varepsilon D(Mod_l(X', \mathcal{B}'))$ *and* $L \varepsilon D(Mod_l(X, \mathcal{B}))$. *Let* $j_U : U \to X'$ *be in the site* $\mathfrak{S}$ *and let* $V = U \underset{X'}{\times} X$. *Then one obtains:*

(i) $R\Gamma(V, \mathcal{RH}om_{\mathcal{B}}(L, Rf^!_{\#}(K))) \simeq R\Gamma(U, \mathcal{RH}om_{\mathcal{B}'}(Rf^{\#}_!(L), K))$ *(or equivalently*

$Rf_*(\mathcal{RH}om_{\mathcal{B}}(L, Rf^!_{\#}(K))) \simeq \mathcal{RH}om_{\mathcal{B}'}(Rf^{\#}_!(L), K)$*) and therefore*

(ii) $RMap_{\mathcal{B}}(L, Rf^!_{\#}(K)) \cong RMap_{\mathcal{B}'}(Rf^{\#}_!(L), K)$

PROOF. Choose a simplicial resolution $P(L)_\bullet \to L$ as in chapter II, Proposition 2.7. Recall each term $P(L)_n$ is of the form $\underset{s \varepsilon S}{\sqcup} j^{\#}_{U!} j^*_U(\Sigma^{n_s}\mathcal{B})$. Now

$$\mathcal{RH}om_{\mathcal{B}}(L, Rf^!_{\#}(K)) = \underset{\Delta}{\text{holim}}\{\mathcal{RH}om_{\mathcal{B}}(P(L)_n, Rf^!_{\#}K)|n\}$$

Next fix an integer n. Now

$$\mathcal{RH}om_{\mathcal{B}}(P(L)_n, Rf^!_\# K) = \mathcal{RH}om_{\mathcal{B}}(\underset{s\varepsilon S}{\sqcup} j^\#_{U!}j^*_U(\Sigma^{n_s}\mathcal{B}), Rf^!_\# K)$$

$$\simeq \prod_{s\varepsilon S}\mathcal{RH}om_{\mathcal{B}}(j^\#_{U!}j^*_U(\Sigma^{n_s}\mathcal{B}), Rf^!_\# K) \simeq \prod_{s\varepsilon S}\mathcal{RH}om_{\mathcal{B}}(Rf^\#_!(j^\#_{U!}j^*_U(\Sigma^{n_s}\mathcal{B})), K)$$

where the last $\simeq$ follows from Proposition 3.0.2 (i) and Chapter II, Proposition 2.1. The previous $\simeq$ follow from the definition of $\mathcal{H}om_{\mathcal{B}}$ as an equalizer in chapter II, (1.2.3) and(1.2.7). (See also chapter II, Propositions (3.5), (3.7) for the operadic case.) Now one may identify the last term with:

$$\mathcal{RH}om_{\mathcal{B}}(\underset{s\varepsilon S}{\sqcup} Rf^\#_!(j^\#_{U!}j^*_U(\Sigma^{n_s}\mathcal{B})), K) \simeq \mathcal{RH}om_{\mathcal{B}}(Rf^\#_!(\underset{s\varepsilon S}{\sqcup} j^*_U(\Sigma^{n_s}\mathcal{B})), K)$$

$$= \mathcal{RH}om_{\mathcal{B}'}(Rf^\#_!(P(L)_n), K)$$

The $\simeq$ follows from Theorem 2.8. Now

$$\mathcal{RH}om_{\mathcal{B}}(L, Rf^!_\#(K)) = \mathcal{RH}om_{\mathcal{B}}(\underset{\Delta}{\mathrm{hocolim}} P(L)_\bullet, Rf^!_\#(K))$$

$$\simeq \underset{\Delta}{\mathrm{holim}}\{\mathcal{RH}om_{\mathcal{B}}(P(L)_n, Rf^!_\#(K))|n\} \simeq \underset{\Delta}{\mathrm{holim}}\{\mathcal{RH}om_{\mathcal{B}'}(Rf^\#_!(P(L)_n), K)|n\}$$

$$\simeq \mathcal{RH}om_{\mathcal{B}'}(\underset{\Delta}{\mathrm{hocolim}}\{Rf^\#_!(P(L)_n)|n\}, K) \simeq \mathcal{RH}om_{\mathcal{B}'}(Rf^\#_!(L), K)$$

The last $\simeq$ follows from the observation that $\underset{\Delta}{\mathrm{hocolim}}\{Rf^\#_!(P(N))_n|n\} \simeq Rf^\#_! L$. This in turn follows from the previous lemma. This proves (i). The assertion (ii) follows from (i) by Chapter II, Proposition (2.8). $\square$

REMARK 3.6. Observe that the proof uses in an essential way the hypothesis in 2.2 as well as the axiom (ST9) from chapter I.

COROLLARY 3.7. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ and $g : (X', \mathcal{B}') \to (X'', \mathcal{B}'')$ denote two maps of ringed sites as before. Now there is a natural isomorphism $Rf^!_\# \circ Rg^!_\# \simeq R(g \circ f)^!_\# : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X'', \mathcal{B}''))$.*

PROOF. Observe from 2.10 that there is a natural isomorphism of the derived functors $Rg^\#_! \circ Rf^\#_! \simeq R(g \circ f)^\#_!$. Now theorem 3.5 provides the required isomorphism. $\square$

PROPOSITION 3.8. *Assume the above situation. Let $\bar{K}\varepsilon D(Mod_l(X', \mathcal{H}^*(\mathcal{B}'))$ and $K = Sp(\bar{K})$. Now $Rf^!_\#(Sp(\bar{K})) \simeq Sp(Rf^!_\#(K))$*

PROOF. Take $\mathcal{B} = Sp(\mathcal{H}^*(\mathcal{B}))$, $\mathcal{B}' = Sp(\mathcal{H}^*(\mathcal{B}'))$ and $L = j^\#_{U!}j^*_U Sp(\mathcal{H}^*(\mathcal{B}))$ in the above theorem. Let $\bar{L} = j^*_{U!}j^*_U(\mathcal{H}^*(\mathcal{B}))$. Now

$$\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(L, Rf^!_\#(Sp(\bar{K}))) \simeq Rf_*\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(Rf^\#_!(L), Sp(\bar{K}))$$

$$\simeq Rf_*\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(Rf^\#_!(Sp(\bar{L})), Sp(\bar{K})) \simeq Rf_*\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(Sp(Rf^\#_!(\bar{L})), Sp(\bar{K}))$$

$$\simeq Rf_*Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(Rf^\#_!(\bar{L}), \bar{K})) \simeq Sp(Rf_*\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(Rf^\#_!(\bar{L}), \bar{K}))$$

$$\simeq Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\bar{L}, Rf^!_\#(\bar{K}))) \simeq \mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(Sp(\bar{L}), Sp(Rf^!_\#(\bar{K})))$$

The third and fifth $\simeq$ are by Proposition 2.12 while the fourth and last $\simeq$ are by Chapter III, Proposition (2.13). Now take $R\Gamma(U, -)$ of both sides. The left-hand-side now becomes $R\Gamma(U, Rf^!_\#(Sp(\bar{K})))$ while the right-hand-side becomes $R\Gamma(U, Sp(Rf^!_\#(\bar{K})))$. $\square$

PROPOSITION 3.9. *(i) Let $i : Y \to X$ denote a closed imbedding with $j : U = X - Y \to X$ the corresponding open imbedding. Define a functor $i_*Ri^! : D(Mod_l(X,\mathcal{B})) \to D(Mod_l(X,\mathcal{B}))$ by*

$$i_*Ri^!F = \text{the canonical homotopy fiber of the map } F \to Rj_*j^*F,\ F\varepsilon D(X,\mathcal{B}).$$

Then there exists a natural quasi-isomorphism $i_*Ri^!F \simeq i_*Ri^!_\#F$, natural in F.

(ii) Consider a cartesian square:

$$\begin{tikzcd} (Y,\mathcal{C}) \arrow[r, "g'"] \arrow[d, "f'"'] & (X,\mathcal{B}) \arrow[d, "f"] \\ (Y',\mathcal{C}') \arrow[r, "g"] & (X',\mathcal{B}') \end{tikzcd}$$

where $Y = Y' \underset{X'}{\times} X$ and $\mathcal{C} = f'^{-1}(\mathcal{C}') \underset{(f'^{-1}(g^{-1}(\mathcal{B}')))}{\otimes} f^{-1}(\mathcal{B})$. Assume further that the base-change map $Lf^*Rg^\#_! \to Rg'^\#_! Lf'^*$ is a natural isomorphism of functors $D(Mod_l(Y',\mathcal{C}')) \to D(Mod_l(X,\mathcal{B}))$. Then one obtains the canonical isomorphism of functors $Rg^!_\# \circ Rf_* \simeq Rf'_* \circ Rg'^!_\#$ as functors $D(Mod_l(X,\mathcal{B})) \to D(Mod_l(Y',\mathcal{C}'))$ and $Lf'^* \circ Rg^!_\# \simeq Rg'^!_\# \circ Lf^*$ as functors $D(Mod_l(X',\mathcal{B}')) \to D(Mod_l(Y,\mathcal{C}))$.

(iii) Let $f : (X,\mathcal{B}) \to (X',\mathcal{B}')$ denote a map of ringed sites. Then there exists a natural transformation:

$$Rf^!_\#(-)\overset{L}{\underset{\mathcal{B}}{\otimes}}Lf^*(-) \to Rf^!_\#(-\overset{L}{\underset{\mathcal{B}'}{\otimes}}-)$$

PROOF. (i) Let $K\varepsilon D(Mod_l(X,\mathcal{B}))$. We begin with the triangle $j^\#_!j^*(K) \to K \to i_*i^{-1}K$. Taking $\mathcal{RH}om_{\mathcal{B}}$, we obtain the triangle:

$$\mathcal{RH}om_{\mathcal{B}}(i_*i^{-1}K,F) \to \mathcal{RH}om_{\mathcal{B}}(K,F) \to \mathcal{RH}om_{\mathcal{B}}(j^\#_!j^*K,F)$$

Now the first term may be identified with $i_*\mathcal{RH}om_{\mathcal{A}}(K, i_*Ri^!_\#F)$ while the last may be identified with
$\mathcal{RH}om_{\mathcal{B}}(K, Rj_*j^*F)$. Now take $K = \mathcal{B}$ to obtain the triangle: $i_*Ri^!_\#F \to F \to Rj_*j^*F$. The definition of $i_*Ri^!$ shows one may identify $i_*Ri^!F$ and $i_*Ri^!_\#F$. This proves (i).

(ii) Let $K\varepsilon D(Mod_l(X,\mathcal{B}))$ and $L\varepsilon D(Mod_l(X',\mathcal{B}'))$. Now

$$RMap_{\mathcal{C}'}(L, Rg^!_\# Rf_*K) \simeq RMap_{\mathcal{B}'}(Rg^\#_! L, Rf_*K) \simeq RMap_{\mathcal{B}}(Lf^*(Rg^\#_! L), K)$$

$$\simeq RMap_{\mathcal{B}}(Rg'^\#_! Lf'^*L, K) \simeq RMap_{\mathcal{C}}(Lf'^*L, Rg'^!_\# K) \simeq RMap_{\mathcal{C}'}(L, Rf'_*Rg'^!_\# K).$$

The first and fourth $\simeq$ are by Theorem 3.5, the second and last are by Proposition 2.6 and the third by our assumption. This proves the first assertion in (ii). The second is established similarly.

(iii) Let $\mathcal{S}$ denote the unit of category $Presh(\mathfrak{S})$ as in 1.0.3. Let $F\varepsilon D(\mathfrak{S}',\mathcal{S})$, $F_1\varepsilon D(Mod_l(X',\mathcal{B}'))$, $F_2\varepsilon D(Mod_r(X',\mathcal{B}'))$ respectively. We obtain:

$$RMap_{\mathcal{S}}(Rf^!_\# F_1\overset{L}{\underset{\mathcal{B}}{\otimes}}Lf^*(F_2), Rf^!_\# F) \simeq RMap_{\mathcal{S}}(Rf^\#_!(Rf^!_\# F_1\overset{L}{\underset{\mathcal{B}}{\otimes}}Lf^*(F_2)), F)$$

$$\simeq RMap_{\mathcal{S}}(Rf_!^{\#}Rf_{\#}^!(F_1)\overset{L}{\underset{\mathcal{B}'}{\otimes}}F_2, F)$$

The last $\simeq$ follows by the projection formula in section 1. Now take $F = F_1\overset{L}{\underset{\mathcal{B}'}{\otimes}}F_2$. The natural map $Rf_!^{\#}Rf_{\#}^!(F_1) \to F_1$ provided by Theorem 3.5 provides a map $Rf_!^{\#}Rf_{\#}^!(F_1)\overset{L}{\underset{\mathcal{B}}{\otimes}}F_2 \to F$. This provides the required map by the above quasi-isomorphisms. $\square$

We will conclude this section with a some-what different construction of the functor $Rf_{\#}^!$ making use of some recent results of Neeman. We begin by recalling the notion of *compact objects* from [Neem] p. 210. An object $K\varepsilon D(Mod_l(X;\, \mathcal{B}))$ is *compact* if for any collection $\{F_\alpha|\alpha\}$ of objects in $D(Mod_l(X;\, \mathcal{B}))$

$$Hom_{D(Mod_l(X;\mathcal{B}))}(K, \underset{\alpha}{\oplus}F_\alpha) \cong \underset{\alpha}{\oplus}Hom_{D(Mod_l(X;\mathcal{B}))}(K, F_\alpha) \tag{3.2.7}$$

PROPOSITION 3.10. *(i) Every object of the form $j_{U!}^{\#}j_U^*(\Sigma^n\mathcal{B})$ for $U\varepsilon\mathfrak{S}$ and n an integer is compact. (ii) The category $D(Mod_l(X;\, \mathcal{B}))$ is compactly generated by the above objects as U varies among a cofinal set of neighborhoods of all the points i.e. the above collection of objects is a small set T of compact objects in $D(Mod_l(X;\, \mathcal{B}))$, closed under suspension, so that $Hom_{D(Mod_l(X;\mathcal{B}))}(T, x) = 0$ for all T implies $x = 0$.*

PROOF. Once again we will let $\mathcal{S}$ denote the unit of $Presh(\mathfrak{S})$ as in 1.0.3.

(i) Observe that

$$Hom_{D(Mod_l(X;\mathcal{B}))}(j_{U!}^{\#}(j_U^*\Sigma^n\mathcal{B})),\, F) \cong Hom_{D(Mod_l(X;\mathcal{B}))}(j_U^*(\Sigma^n\mathcal{B}),\, j_U^*F)$$

$$\cong Hom_{D(Mod_l(X;\mathcal{S})}(\Sigma^n\mathcal{S}_{|U},\, j_{|U}^*(F)) \cong \mathcal{H}^{-n}(R\Gamma(U,\, F))$$

- see chapter II, Proposition 2.1 and chapter II, Proposition 3.7. Therefore, by Theorem 2.8, one now observes that

$$Hom_{D(Mod_l(X;\mathcal{B}))}(j_{U!}^{\#}(j_U^*(\Sigma^n\mathcal{B})),\, \underset{\alpha}{\oplus}F_\alpha) \cong \mathcal{H}^{-n}(R\Gamma(U,\, \underset{\alpha}{\oplus}F_\alpha))$$

$$\cong \underset{\alpha}{\oplus}\mathcal{H}^{-n}(R\Gamma(U,\, F_\alpha))$$

This proves (i). Suppose $\mathcal{H}^{-n}(R\Gamma(U,\, F)) = 0$ for all U that form a cofinal system of neighborhoods of all points in the site $\mathfrak{S}$ and all n. It follows immediately that F is *acyclic* and therefore is isomorphic to 0 in the derived category $D_l(\mathfrak{S};\, \mathcal{B})$. This proves (ii). $\square$

DEFINITION 3.11. (Compactly generated triangulated categories) Let $\mathbf{S}$ denote a triangulated category. Suppose all small co-products exist in $\mathbf{S}$. Suppose also that there exists a small set of objects S of $\mathbf{S}$ so that

(i) for every $s\varepsilon S$, $Hom_{\mathbf{S}}(s, -)$ commutes with co-products in the second argument and

(ii) if $y\varepsilon\mathbf{S}$ is an object so that $Hom_{\mathbf{S}}(s, y) = 0$ for all $s\varepsilon S$, then $y = 0$.

Such a triangulated category is said to be *compactly generated.* An object s in a triangulated category $\mathbf{S}$ is called *compact* if it satisfies the hypothesis (i) above.

THEOREM 3.12. *(Neeman: see [N] Theorems 4.1 and 5.1) Let $\mathbf{S}$ denote a compactly generated triangulated category and let $F : \mathbf{S} \to \mathbf{T}$ denote a functor of triangulated categories. Suppose F has the following property:*

if $\{s_\lambda|\lambda\}$ is a small set of objects in $\mathbf{S}$*, the co-product* $\underset{\lambda}{\sqcup}F(s_\lambda)$ *exists in* $\mathbf{T}$ *and the natural map* $\underset{\lambda}{\sqcup}F(s_\lambda) \to F(\underset{\lambda}{\sqcup}s_\lambda)$ *is an isomorphism.*

Then F has a right adjoint G. Moreover, the functor G preserves co-products (i.e. if $\{t_\alpha|\alpha\}$ is a small set of objects in $\mathbf{T}$ *whose sum exists in* $\mathbf{T}$*, $G(\underset{\alpha}{\sqcup}t_\alpha) = \underset{\alpha}{\sqcup}G(t_\alpha)$) if for every s in a generating set S for* $\mathbf{S}$*, $F(s)$ is a compact object in* $\mathbf{T}$*.*

We will apply the above theorem in the following manner.

THEOREM 3.13. *Let T denote a functor $T : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$. If T sends sums in $D(Mod_l(X, \mathcal{B}))$ to sums in $D(Mod_l(X', \mathcal{B}'))$, T has a right adjoint which we will denote by $T^!_\#$. Moreover, if $T(j^\#_{U!}(j^*_U(\Sigma^n\mathcal{B})))$ is a compact object in $D(Mod_l(X', \mathcal{B}'))$ for all objects $j_U : U \to X$ in the site $\mathfrak{S}$ and all integers n, then the functor $T^!_\#$ preserves sums.*

PROOF. Recall the derived categories $D(Mod_l(X, \mathcal{B}))$ and $D(Mod_l(X', \mathcal{B}'))$ are triangulated categories and that (see Proposition 3.10 above), that $\{j^\#_{U!}(j^*_U(\Sigma^n\mathcal{B}))|j_U \to \mathfrak{S} \quad in \quad \mathfrak{S}, \quad n\varepsilon\mathbb{Z}\}$ is a small set of compact objects that generate the category $D(Mod_l(X, \mathcal{B}))$. Therefore, if T preserves sums, Theorem 3.12 shows it has an adjoint $T^!_\#$. The functor $T^!_\#$ preserves sums, if $T(j^\#_{U!}(j^*_U(\Sigma^n\mathcal{B})))$ is a compact object in $D(Mod_l(X', \mathcal{B}'))$ for all objects $j_U : U \to X$ in the site $\mathfrak{S}$ and all integers n. □

THEOREM 3.14. *(Existence of a right adjoint to $Rf^\#_!$) Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites. Suppose the site $\mathfrak{S}$ is algebraic and $\mathfrak{S}'$ is locally coherent. Suppose the functor $Rf^\#_!$ is well-defined. Then the functor*

$$Rf^\#_! : D(Mod_l(X, \mathcal{B})) \to D(Mod_l(X', \mathcal{B}'))$$

has a right adjoint (which we denote by $Rf^!_\#$). Moreover, if $Rf^\#_!$ sends a compact generating set for $D(Mod_l(X, \mathcal{B}))$ to compact objects in $D(Mod_l(X', \mathcal{B}'))$, the functor $Rf^!_\#$ preserves sums.

PROOF. First observe from Theorem 2.8 that the functor $Rf^\#_!$ commutes with filtered direct limits of presheaves. Therefore, it follows that if $\{M_\alpha|\alpha\}$ is a collection of objects in $D(Mod_l(X, \mathcal{B}))$, the natural map:

$$\underset{\alpha}{\sqcup}Rf^\#_!(M_\alpha) \overset{\simeq}{\to} Rf^\#_!(\underset{\alpha}{\sqcup}M_\alpha)$$

is a quasi-isomorphism. It follows that the functor $Rf^\#_!$ preserves *sums*. Since the derived category $D(Mod_l(\mathfrak{S}, \mathcal{B}))$ is compactly generated as shown in Proposition 3.10, it follows that $Rf^\#_!$ has a right adjoint. The last assertion is now clear from the last assertion of Theorem 3.12. □

REMARK 3.15. Despite the elegance of the above construction, one looses the bi-module structure (see the remarks in 4.2 below) on $Rf^!_\#(\mathcal{B})$ by the above construction. This bi-module structure is essential in obtaining a bi-duality theorem and hence the full strength of Grothendieck-Verdier duality as in the next section.

4. The dualizing presheaves and the bi-duality theorem

We begin by defining dualizing presheaves both in the relative and absolute situation. We will assume throughout that all maps are compactifiable in the sense of 2.4. Furthermore we will assume that $(\mathbf{S}, \mathcal{A})$ is a base-ringed site and that all ringed sites we consider are defined over it. Let Z denote the terminal object in the site $\mathbf{S}$. Moreover, we will assume that the following hypothesis holds:

$\mathcal{A} \varepsilon Presh(\mathbf{S})$ and $\mathcal{B} \varepsilon Presh(\mathfrak{S})$ are *commutative* algebras.

DEFINITION 4.1. (i) Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites. Now we define the *relative dualizing presheaf* D_f to be $Rf^!_{\#}(\mathcal{B}')$.

(ii) If $p : (X, \mathcal{B}) \to (Z, \mathcal{A})$ is the structure map of the ringed space, we let $D_{\mathcal{B}} = Rp^!_{\#}(\mathcal{A})$ and call it the *dualizing presheaf* for the categories $D(Mod_l(X, \mathcal{B}))$ and $D(Mod_r(X, \mathcal{B}))$.

(iii) Assume the situation in (ii). We define a functor $\mathbb{D}_{\mathcal{B}} : D(Mod_l(X, \mathcal{B})) \to D(Mod_r(X, \mathcal{B}))$ (and similarly $\mathbb{D}_{\mathcal{B}} : D(Mod_r(X, \mathcal{B})) \to D(Mod_l(X, \mathcal{B}))$ by $D(F) = \mathcal{RH}om_{\mathcal{B}}(F, D_{\mathcal{B}})$.

REMARK 4.2. Recall that

$$\Gamma(U, D_{\mathcal{B}}) = R\Gamma(S, \mathcal{RH}om_{\mathcal{A}}(Rp^{\#}_!(\mathcal{B}_U), \mathcal{A})) \tag{4.0.8}$$

for each U in the site $\mathfrak{S}$. In taking the $\mathcal{RH}om_{\mathcal{A}}$, we use the structure of a presheaf of left $\mathcal{A}$-modules on $Rf^{\#}_!(\mathcal{B}_U)$. We already saw in (3.0.2) that $D_{\mathcal{B}}$ belongs to $D(Mod_l(X, \mathcal{B}))$. In fact one may show readily that $D_{\mathcal{B}}$ has the structure of a presheaf of *bi-modules* over $\mathcal{B}$: (4.0.8) in fact shows that the left -module- structure (right-module-structure) is induced from the structure of a presheaf of right $\mathcal{B}$-modules (right-$\mathcal{B}$-modules, respectively) on $Rp^{\#}_!(\mathcal{B}_U)$.) Observe that, in this case, the commutativity of the algebras implies the left $\mathcal{A}$-module structure on $Rp^{\#}_!(\mathcal{B}_U)$ commutes with the left $\mathcal{B}$-module structure. Therefore $D_{\mathcal{B}}$ has the structure of a presheaf of bi-modules and therefore defines functors $\mathbb{D}_{\mathcal{B}}$ as stated.

4.1. Let $\mathbb{D}_{\mathcal{B}}$ be filtered by the filtration induced from the Cartan filtration on $\mathcal{A}$ and $\mathcal{B}$. Now $Gr(D_{\mathcal{B}}) \simeq D_{Gr(\mathcal{B})} \simeq D_{Sp(\mathcal{H}^*(\mathcal{B}))}$. The first $\simeq$ follows from the definition of $D_{\mathcal{B}}$ along with Chapter III, Proposition 2.7. The second $\simeq$ now follows from Chapter III, Proposition 2.10 (ii) and Chapter III, Proposition 2.13.

PROPOSITION 4.3. *Assume the situation of 4.1. Then* $Rf_*\mathbb{D}_{\mathcal{B}}(F) \simeq \mathbb{D}_{\mathcal{B}'}(Rf^{\#}_!(F))$

PROOF. This follows readily from Theorem 3.5(i) and corollary 3.7. □

In order to prove the dualizing pre-sheaf is *reflexive* (see Theorem 4.7 below) one will have to further restrict to one of the following two situations:

4.2. (i) $(\mathbf{S}, \mathcal{A})$ is a ringed site so that $\mathcal{H}^*(\mathcal{A})$ is locally constant on the site $\mathbf{S}$, $\mathcal{B} = p^{-1}(\mathcal{A})$ and we restrict to the full sub-category $D(Mod^{c,f.t.d}(\mathfrak{S}, \mathcal{B}))$ of $D(Mod^{f.t.d}(\mathfrak{S}, \mathcal{B}))$ of objects that are constructible in the sense of the following definition *or*

(ii) with no further restriction on the ringed site $(\mathbf{S}, \mathcal{A})$, we restrict to $D(Mod^{perf}(\mathfrak{S}, \mathcal{B}))$.

DEFINITION 4.4. Assume the first situation above. (i) Let $F \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{B}))$. We say F *is locally constant* on $\mathfrak{S}$ if $\mathcal{H}^*(F)\tilde{}$ is locally constant as a sheaf of graded left modules over the sheaf $\mathcal{H}^*(\mathcal{B})\tilde{}$.

(ii) F *is constructible* if $\mathcal{H}^*(F)\tilde{}$ is *constructible* as a sheaf of graded left-modules over the sheaf $\mathcal{H}^*(\mathcal{B})\tilde{}$. Recall this means: there exists a finite filtration

$$X_0 \overset{i_0}{\subseteq} X_1 \overset{i_1}{\subseteq} \dots \overset{i_n}{\subseteq} X_n$$

by locally closed subspaces of X, so that $\mathcal{H}^*(F)\tilde{}_{|X_i - X_{i-1}}$ is locally constant (and finitely generated) as a sheaf of $\mathcal{H}^*(\mathcal{B})\tilde{}_{X_i - X_{i-1}}$-modules for each i. (Recall that $\mathcal{H}^*(\mathcal{B})$ is assumed to be a sheaf of Noetherian rings so that there is no distinction between finitely generated and finitely presented objects.)

Proposition 4.5. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites as before. Suppose that $Rf_!^{\#}$ (Rf_*) sends constructible sheaves of $\mathcal{H}^*(\mathcal{B})\tilde{}$-modules to constructible sheaves of $\mathcal{H}^*(\mathcal{B}')\tilde{}$-modules on X'. Then $Rf_!^{\#}$ (Rf_*, respectively) sends constructible presheaves of modules over $\mathcal{B}$ to constructible presheaves of modules over $\mathcal{B}'$.*

Proof. Recall the spectral sequence:

$$E_2^{s,t} = R^s f_!^{\#} \mathcal{H}^t t(F)\tilde{} \to \mathcal{H}^{s+t}(Rf_!^{\#} F)\tilde{},\ F \varepsilon D(Mod_l(\mathfrak{S}, \mathcal{A}))$$

Now the hypothesis of 2.2 shows that it suffices to prove $\underset{s}{\oplus} R^s f_!^{\#} \mathcal{H}^t(F)\tilde{}$ is constructible as a sheaf of modules over $\mathcal{H}^*(\mathcal{B}')\tilde{}$. This is clear from the hypothesis. The assertion about Rf_* is established similarly. □

Definition 4.6. We say f is constructible if Rf_* and $Rf_!^{\#}$ send constructible sheaves of $\mathcal{H}^*(\mathcal{B})$-modules to constructible sheaves of $\mathcal{H}^*(\mathcal{B}')$-modules.

4.3. Terminology and conventions for the rest of the chapter. *For the rest of this chapter, we will adopt the following terminology.* With no further restrictions on the site, $D(Mod^?(X, \mathcal{B})) = D(Mod_l^{perf}(X, \mathcal{B}))$. In case the ringed site $(X, \mathcal{B})$ satisfies the hypotheses in 4.2 (i), $D(Mod^?(X, \mathcal{B}))$ will denote $D(Mod_l^{c,f.t.d}(X, \mathcal{B}))$ as in definition 4.4. In the former case $D^?(X, \mathcal{H}^*(\mathcal{B})))$ will denote the derived category $D(Mod_l^{perf}(X, \mathcal{H}^*(\mathcal{B})))$ of perfect complexes of sheaves of $\mathcal{H}^*(\mathcal{B})$-modules. Moreover, any map $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ of ringed sites in the above sense will be automatically assumed to be perfect in the first case and of finite tor dimension and constructible in the second case in the sense of definition 2.13 and the definition **??** above. In either case $D_{\mathcal{H}^*(\mathcal{B})}$ will denote the dualizing complex in the derived category $D^?(X, \mathcal{H}^*(\mathcal{B}))$.

Theorem 4.7. *(Bi-duality) Assume in addition to the above situation that the natural map*

$$\bar{F} \to D_{\mathcal{H}^*(\mathcal{B})}(D_{\mathcal{H}^*(\mathcal{B})}(\bar{F}))$$

is a quasi-isomorphism for every $\bar{F} \varepsilon D^?(X, \mathcal{H}^*(\mathcal{B})\tilde{})$. Let $F \varepsilon D(Mod^?(X, \mathcal{B}))$. Then the natural map $F \to \mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(F))$ is a quasi-isomorphism. The same conclusions hold if $D_{\mathcal{B}} \varepsilon D^?(Mod_{bi}(X, \mathcal{B}))$ so that the hypotheses in 4.1 are satisfied.

Proof. The second spectral sequence in Chapter III, Theorem 2.18 plays a key-role in the proof. Observe next that the given filtration on F and the canonical Cartan filtration on $\mathcal{B}$ induce a filtration on $\mathbb{D}_{\mathcal{B}}(F)$ as well as on $\mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(F))$. The natural map $F \to \mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(F))$ is compatible with the above filtrations. Now the filtrations provide us with spectral sequences; since the above map is compatible with the filtrations, we also obtain a map of these spectral sequences.

Next recall that for F, $Gr(F) = Sp(\bar{F})$, where $\bar{F} \varepsilon D^?(X, \mathcal{H}^*(\mathcal{B}))$ is a bounded complex of finite tor dimension (or is a perfect complex). Recall $\bar{F} = \underset{i}{\Pi} \bar{F}(i)$. The spectral sequence for F now is given by:

$$(4.3.1)\qquad \begin{aligned} E_2^{s,t} = \mathcal{H}^{s+t}(Gr_t(F))\,\tilde{}\cong \mathcal{H}^{s+t}(Gr_t(Sp(\bar{F})))\,\tilde{}\cong \mathcal{H}^{s+t}(Gr_t(\underset{\Delta}{\text{holim}}\,\prod_i EM_i(\bar{F}(i))))\,\tilde{} \\ \cong \mathcal{H}^{sf+t}(\underset{\Delta}{\text{holim}}\,EM_t(\bar{F}(t)))\,\tilde{}\Rightarrow \mathcal{H}^{s+t}(F)\,\tilde{} \end{aligned}$$

One may compute $\mathcal{H}^{s+t}(\underset{\Delta}{\text{holim}}\,EM_t(\bar{F}(t)))\,\tilde{}$ by means of the spectral sequence:

$$E_2^{u,v} = H^u(\mathcal{H}^v(EM_t(\bar{F}(t))))\,\tilde{}\Rightarrow \mathcal{H}^{u+v}(\underset{\Delta}{\text{holim}}\,EM_t(\bar{F}(t)))\,\tilde{}.$$

This spectral sequence degenerates since $E_2^{u,v} = 0$ unless $v = t$ and $E_2^{u,v} = H^u(\bar{F}(t))$ if $v = t$. i.e.

$\mathcal{H}^{s+t}(\underset{\Delta}{\text{holim}}\,EM_t(\bar{F}(t)))\,\tilde{}\cong H^s(\bar{F}(t))$. Since $\bar{F}$ is a bounded complex, there exists an integer $N >> 0$, independent of t so that $H^s(\bar{F}(t)) = 0$ if $s > N$ or if $s << 0$. It follows that $\mathcal{H}^{s+t}(Gr_t(F))\,\tilde{}\cong 0$ if $s > N$ or if $s << 0$. Thus the spectral sequence in (4.3.1) converges strongly.

Now we consider the spectral sequence for $\mathbb{D}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(F)) = \mathcal{RH}om_{\mathcal{A}}(\mathcal{RH}om_{\mathcal{A}}(F, D_{\mathcal{A}}), D_{\mathcal{A}})$. The $E_2^{s,t}$-terms are given by

$$(4.3.2)\qquad E_2^{s,t} = \mathcal{H}^{s+t}(Gr_t(\mathcal{RH}om_{\mathcal{A}}(\mathcal{RH}om_{\mathcal{A}}(F, D_{\mathcal{A}}), D_{\mathcal{A}})))\,\tilde{}$$

By Chapter III, Proposition 2.7 applied twice we see that $Gr(\mathcal{RH}om_{\mathcal{B}}(\mathcal{RH}om_{\mathcal{B}}(F, D_{\mathcal{B}}), D_{\mathcal{B}}))$ $\simeq \mathcal{RH}om_{Gr(\mathcal{B})}(Gr(\mathcal{RH}om_{\mathcal{B}}(F, D_{\mathcal{B}})), Gr(D_{\mathcal{B}}))$

$$\simeq \mathcal{RH}om_{Gr(\mathcal{B})}(\mathcal{RH}om_{Gr(Q\mathcal{B})}(Gr(F), Gr(D_{\mathcal{B}})), Gr(D_{\mathcal{B}})).$$

By Chapter III, Proposition 2.10(ii) and Proposition 2.13 this may be identified with

$$\mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(\mathcal{RH}om_{Sp(\mathcal{H}^*((\mathcal{B})))}(Sp(\bar{F}), D_{Sp(\mathcal{H}^*(\mathcal{B}))}), D_{Sp(\mathcal{H}^*(\mathcal{B}))})$$

$$\simeq \mathcal{RH}om_{Sp(\mathcal{H}^*(\mathcal{B}))}(Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\bar{F}, D_{\mathcal{H}^*(\mathcal{B})})), Sp(D_{\mathcal{H}^*(\mathcal{B})}))$$

$$\simeq Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\bar{F}, D_{\mathcal{H}^*(\mathcal{B})}), D_{\mathcal{H}^*(\mathcal{B})})).$$

Now one may apply the computation in Chapter III, Proposition 2.17(ii) to identify the $E_2^{s,t}$ -terms in (4.3.2) with

$$\mathcal{E}xt^{s,t}_{\mathcal{H}^*(\mathcal{B})}(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\bar{F}, D_{\mathcal{H}^*(\mathcal{B})}), D_{\mathcal{H}^*(\mathcal{B})}).$$

Under the hypothesis of the theorem, we see that natural map

$$\bar{F} \to \mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{B})}(\bar{F}, D_{\mathcal{H}^*(\mathcal{B})}), D_{\mathcal{H}^*(\mathcal{B})})$$

is a quasi-isomorphism. Therefore we obtain an isomorphism of the $E_2^{s,t}$-terms in (4.3.1) and (4.3.2). (In particular the second spectral sequence also converges strongly.) Since both the spectral sequences converge strongly (recall the hypothesis of finite tor dimension or perfection on F), it follows that the map $F \to \mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(F))$ is a quasi-isomorphism. □

Situations where the theorem applies.

4.4. Consider schemes or algebraic spaces of finite type over a base scheme $\mathbf{S}$. Assume all the schemes and algebraic spaces are provided with the étale topology and L is a non-empty set of primes different from the residue characteristics. Let $\mathcal{A}$ denote a presheaf of commutative algebras on $\mathbf{S}$ so that, $\mathcal{H}^*(\mathcal{A}) = \underset{n}{\oplus}\mathcal{H}^n(\mathcal{A})$ is locally constant on the étale topology of $\mathbf{S}$ and has L-primary torsion. Now the hypotheses in the Bi-duality theorem are satisfied by any $\bar{F}\varepsilon D(Mod_l^{c,f.t.d}(\mathfrak{S}, \mathcal{H}^*(\mathcal{A})))$. (See [SGA]$4_{1/2}$ p. 250.) The bi-duality theorem also holds for suitable L-completions of a presheaf of algebras $\mathcal{A}$. See 6.1 for a detailed discussion of this application.

4.5. Next assume $\mathcal{B}$ is a commutative algebra on X, $\bar{F}\varepsilon D^{perf}(Mod(X, \mathcal{H}^*(\mathcal{B})))$ and that (modulo a globally determined shift) $D_{\mathcal{H}^*(\mathcal{B})}$ is locally quasi-isomorphic to $\mathcal{H}^*(\mathcal{B})$. In this case the conclusion of the theorem holds for any $F\varepsilon D(Mod^{perf}(X, \mathcal{B}))$ so that $Gr(F) \simeq Sp(\bar{F})$.

4.6. Consider locally compact Hausdorff topological spaces over a base space $\mathbf{S}$ of the same type. Assume that L is a (possibly empty) set of primes for which all the spaces are of finite L-cohomological dimension. (Recall that if L is empty, this means all the spaces are of finite cohomological dimension.) Now let $\mathcal{A}$ denote a presheaf of commutative algebras on $\mathbf{S}$ so that $\mathcal{H}^*(\mathcal{A}) = \underset{n}{\oplus}\mathcal{H}^n(\mathcal{A})$ is locally constant and of L-primary torsion. Let X denote a topological space as above and let $p : X \to \mathbf{S}$ denote the obvious structure map. Now the hypotheses in the bi-duality theorem are satisfied by any $\bar{F}\varepsilon D^{c,f.t.d}(X, \mathcal{H}^*(\mathcal{A}))$. (See [K-S-2] chapter III.)

We conclude this section by defining the homology with compact supports. Assume one of the above situations. Let $F\varepsilon D(Mod_l(X, \mathcal{B}))$. We let

DEFINITION 4.8. $H_\bullet(X, F) = \mathbf{H}(X; \mathbb{D}_{\mathcal{B}}(F))$ and $H_*(X, F) = \mathcal{H}^{-*}(H_\bullet(X, F))$. We call this *the homology of X with compact supports with respect to F.*

5. The extra-ordinary derived functors and the formalism of Grothendieck-Verdier duality

In this section we complete formalism of Grothendieck-Verdier duality. Throughout we will assume all the hypotheses in 4.3.

PROPOSITION 5.1. *Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites.*

(i) Now there exists a natural isomorphism of derived functors: $Rf_* \circ \mathbb{D}_{\mathcal{B}} \cong \mathbb{D}_{\mathcal{B}'} \circ Rf_!^{\#}$: $D(Mod_l(X, \mathcal{B})) \to D(Mod_r(X', \mathcal{B}'))$

(ii) There exists also a natural isomorphism of derived functors: $Rf_{\#}^! \circ \mathbb{D}_{\mathcal{B}'} \cong \mathbb{D}_{\mathcal{B}} \circ Lf^*$: $D(Mod_l(X', \mathcal{B}')) \to D(Mod_r(X, \mathcal{B}))$. More generally, if L, $K\varepsilon D(Mod_l(X', \mathcal{B}'))$, there exists a quasi-isomorphism
$Rf_{\#}^!(\mathcal{RH}om_{\mathcal{B}'}(L, K)) \simeq \mathcal{RH}om_{\mathcal{B}}(Lf^*(L), Rf_{\#}^! K)$

PROOF. We will let $\mathcal{S}$ denote the unit of $Presh(\mathfrak{S})$ as in 1.0.3. The first assertion follows readily from Theorem 3.5 and Corollary 3.7 by taking $K = D_{\mathcal{B}}$. Let $P\varepsilon D(Mod_l(X', \mathcal{S})$. Then one obtains the following quasi-isomorphisms:

$$RMap_{\mathcal{S}}(P, Rf_{\#}^!(\mathcal{RH}om_{\mathcal{B}'}(L, K))) \simeq RMap_{\mathcal{S}}(Rf_!^{\#}(P), \mathcal{RH}om_{\mathcal{B}'}(L, K))$$

$$\simeq RMap_{\mathcal{B}'}(Rf_!^{\#}(P)\overset{L}{\underset{\mathcal{S}}{\otimes}}L, K) \simeq RMap_{\mathcal{B}'}(Rf_!^{\#}(P\overset{L}{\underset{\mathcal{S}}{\otimes}}Lf^*(L)), K)$$

$$\simeq RMap_{\mathcal{B}}(P\overset{L}{\underset{\underline{S}}{\otimes}}Lf^*(L), Rf_{\#}^!(K)) \simeq RMap_{\underline{S}}(P, \mathcal{RH}om_{\mathcal{B}}(Lf^*(L), Rf_{\#}^!(K)))$$

The first and fourth quasi-isomorphisms are by Theorem 3.5, the second and last quasi-isomorphisms are by Chapter II, 2.0.15 and the third by the projection formula. This proves the second assertion in (ii). The first assertion in (ii) follows by taking $K = D_{\mathcal{B}'}$. Observe in view of corollary 3.7 that $Rf^!_{\#}(D_{\mathcal{B}'}) \simeq D_{\mathcal{B}}$.) □

5.1. Next we define functors

(5.1.1) $Rf_! : D(Mod^?(X, \mathcal{B})) \to D(Mod^?(X, \mathcal{B}'))$ *by* $Rf_!(F) = \mathbb{D}_{\mathcal{B}'}(Rf_*(\mathbb{D}_{\mathcal{B}}(F)))$ *and*

(5.1.2) $Rf^! : D(Mod^?(X', \mathcal{B}')) \to D(Mod^?(X, \mathcal{B}))$ *by* $Rf^!(K) = \mathbb{D}_{\mathcal{B}}(Lf^*(\mathbb{D}_{\mathcal{B}'}(K)))$

PROPOSITION 5.2. *(i) Let $f : (X, \mathcal{B}) \to (X', \mathcal{B}')$ denote a map of ringed sites as above. Then $Rf^!_{\#}(D_{\mathcal{B}'}) \simeq D_{\mathcal{B}}$.*

(ii) There exists a natural isomorphism of functors $Rf_! \cong Rf^{\#}_! : D(Mod^?(X, \mathcal{B})) \to D(Mod^?(X', \mathcal{B}'))$

(iii) There exists a natural isomorphism of functors $Rf^! \cong Rf^!_{\#} : D(Mod^?(X', \mathcal{B}')) \to D(Mod^?(X, \mathcal{B}))$.

PROOF. (i) Let $p : (X, \mathcal{B}) \to (\mathbf{S}, \mathcal{A})$ ($p' : (X', \mathcal{B}') \to (\mathbf{S}, \mathcal{A})$) denote the structure map of the ringed site $(X, \mathcal{B})$ ($(X', \mathcal{B}')$, respectively). Let $K \varepsilon D(Mod^?(X, \mathcal{B}))$. Now one obtains the quasi-isomorphisms:

$$RHom_{\mathcal{B}'}(Rf^{\#}_!(K), D_{\mathcal{B}'}) \simeq RHom_{\mathcal{B}'}(Rf^{\#}_!(K), Rp'^!_{\#}(\mathcal{A})) \simeq RHom_{\mathcal{A}}(Rp'^{\#}_! Rf^{\#}_!(K), \mathcal{A})$$

$$\simeq RHom_{\mathcal{A}}(R(p' \circ f)^{\#}_!(K), \mathcal{A}) \simeq RHom_{\mathcal{B}}(K, R(p' \circ f)^!_{\#}(\mathcal{A})) \simeq RHom_{\mathcal{B}}(K, D_{\mathcal{B}}).$$

By Theorem 3.5, the first term above may also be identified with $RHom_{\mathcal{B}}(K, Rf^!_{\#}\mathbb{D}_{\mathcal{B}})$. Since this holds for all $K \varepsilon D(Mod^?(X, \mathcal{B}))$, we see that there exists a quasi-isomorphism $Rf^!_{\#}(D_{\mathcal{B}'}) \simeq D_{\mathcal{B}}$. This completes the proof of (i).

Let $K \varepsilon D(Mod^?(X, \mathcal{B}))$. By the definition of $Rf_!$, $Rf_! \circ \mathbb{D}_{\mathcal{B}}(K) = \mathbb{D}_{\mathcal{B}'} Rf_* \mathbb{D}_{\mathcal{B}}(\mathbb{D}_{\mathcal{B}}(K)) \simeq \mathbb{D}_{\mathcal{B}'} Rf_*(K)$. Moreover, by our hypotheses f is perfect (or of finite tor dimension and constructible as the case may be), so that $Rf_*(K) \varepsilon D(Mod^?(X', \mathcal{B}'))$. Therefore, $Rf_*(K) \simeq \mathbb{D}_{\mathcal{B}'}(\mathbb{D}_{\mathcal{B}'}(Rf_*(K))) \simeq \mathbb{D}_{\mathcal{B}'}(Rf_!(\mathbb{D}_{\mathcal{B}}(K)))$. Finally replace K, by $\mathbb{D}_{\mathcal{B}}(K)$ to obtain: $Rf_*(\mathbb{D}_{\mathcal{B}}(K)) \simeq \mathbb{D}_{\mathcal{B}'}(Rf_!(K))$. Next recall from Proposition 5.1 above that $Rf_* \circ \mathbb{D}_{\mathcal{B}} \cong \mathbb{D}_{\mathcal{B}'} \circ Rf^{\#}_!$. It follows that there is a natural quasi-isomorphism $\mathbb{D}_{\mathcal{B}'}(Rf^{\#}_!(K)) \simeq \mathbb{D}_{\mathcal{B}'}(Rf_!(K))$. Take the dual with respect to $D_{\mathcal{B}'}$ once more to obtain (ii).

In view of (ii), it suffices to show that the functor $Rf^!$ is right adjoint to $Rf_!$. This may be established as follows. Let $P \varepsilon D(Mod^?(X, \mathcal{B}))$ and $K \varepsilon D(Mod^?(X', \mathcal{B}'))$. Then

$$RHom_{\mathcal{B}}(P, Rf^! K) = RHom_{\mathcal{B}}(P, \mathbb{D}_{\mathcal{B}}(Lf^* \mathbb{D}_{\mathcal{B}'}(K))) = RHom_{\mathcal{B}}(P, \mathcal{RH}om_{\mathcal{B}}(Lf^* \mathbb{D}_{\mathcal{B}'}(K), D_{\mathcal{B}}))$$

$$\simeq RHom_{\mathcal{B}}(Lf^* \mathbb{D}_{\mathcal{B}'}(K) \overset{L}{\underset{\mathcal{B}}{\otimes}} P, D_{\mathcal{B}}) = RHom_{\mathcal{B}}(Lf^* \mathbb{D}_{\mathcal{B}'}(K) \overset{L}{\underset{\mathcal{B}}{\otimes}} P, Rf^!_{\#}(D_{\mathcal{B}'}))$$

$$\simeq RHom_{\mathcal{B}'}(Rf^{\#}_!(Lf^* \mathbb{D}_{\mathcal{B}'}(K) \overset{L}{\underset{\mathcal{B}}{\otimes}} P), D_{\mathcal{B}'}) \simeq RHom_{\mathcal{B}'}(\mathbb{D}_{\mathcal{B}'}(K) \overset{L}{\underset{\mathcal{B}}{\otimes}} Rf^{\#}_!(P), D_{\mathcal{B}'})$$

$$\simeq RHom_{\mathcal{B}'}(Rf^{\#}_!(P), \mathbb{D}_{\mathcal{B}'}(\mathbb{D}_{\mathcal{B}'}(K))) \simeq RHom_{\mathcal{B}'}(Rf^{\#}_!(P), K)$$

This completes the proof of (iii). □

DEFINITION 5.3. Henceforth $Rf_!$ will denote either $Rf_!^{\#} : D(Mod^?(X,\mathcal{B})) \to D(Mod^?(X',\mathcal{B}'))$ or the functor $D(Mod^?(X,\mathcal{B})) \to D(Mod^?(X',\mathcal{B}'))$ defined above. Similarly $Rf^!$ will denote either the functor $Rf^!_{\#} : D(Mod^?(X',\mathcal{B}')) \to D(Mod^?(X,\mathcal{B}))$ or the functor $Rf^! : D(Mod^?(X',\mathcal{B}')) \to D(Mod^?(X,\mathcal{B}))$ defined above. The *trace map* will be the natural transformation $Rf_! \circ Rf^! \to id$ adjoint to the identity $Rf^! \to Rf^!$.

COROLLARY 5.4. *Let $f : (X,\mathcal{B}) \to (X',\mathcal{B}')$. Then there exists a natural isomorphism of derived functors: $\mathbb{D}_{\mathcal{B}'} \circ Rf_* \cong Rf_! \circ \mathbb{D}_{\mathcal{B}} : D(Mod^?(X,\mathcal{B})) \to D(Mod^?(X',\mathcal{B}'))$.*

(ii) There exists also a natural isomorphism of derived functors: $\mathbb{D}_{\mathcal{B}} \circ Rf^! \cong Lf^* \circ \mathbb{D}_{\mathcal{B}'} : D(Mod^?(X',\mathcal{B}')) \to D(Mod^?(X,\mathcal{B}))$

PROOF. (i) It suffices to apply $\mathbb{D}_{\mathcal{B}'}$ to both sides of the isomorphism $Rf_* \circ \mathbb{D}_{\mathcal{B}} \cong \mathbb{D}_{\mathcal{B}'} \circ Rf_!^{\#}$ followed by the observation that $\mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}} \cong id$ on $D(Mod^?(X,\mathcal{B}))$ and $\mathbb{D}_{\mathcal{B}'} \circ \mathbb{D}_{\mathcal{B}'} \cong id$ on $D(Mod^?(X',\mathcal{B}'))$. This proves (i). Apply $\mathbb{D}_{\mathcal{B}}$ to both sides of the isomorphism $Rf^! \circ \mathbb{D}_{\mathcal{B}'} \cong \mathbb{D}_{\mathcal{B}} \circ Lf^*$ followed by the observation that $\mathbb{D}_{\mathcal{B}} \circ \mathbb{D}_{\mathcal{B}} \cong id$ on $D(Mod^?(X,\mathcal{B}))$ and $\mathbb{D}_{\mathcal{B}'} \circ \mathbb{D}_{\mathcal{B}'} \cong id$ on $D(Mod^?(X',\mathcal{B}'))$. This proves (ii). □

5.2. Cohomological correspondence. Consider the following cartesian squares:

$$\begin{tikzcd}
& X_1 \underset{S}{\times} X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 \arrow[dr, "g"'] & & X_2 \arrow[dl, "h"] \\
& S &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y_1 \underset{S}{\times} Y_2 \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
Y_1 \arrow[dr, "g'"'] & & Y_2 \arrow[dl, "h'"] \\
& S &
\end{tikzcd}$$

We will let $\mathcal{A}$ denote a sheaf of algebras on the base site $\mathbf{S}$ so that $\mathcal{H}^*(\mathcal{A})$ is locally constant. The corresponding inverse image of this presheaf of algebras on all the spaces considered above will also be denoted $\mathcal{A}$. Let M_1, $N_1 \varepsilon D(Mod^?_{bi}(X_1;\mathcal{A}))$, $N_2 \varepsilon D(Mod^?_l(X_2;\mathcal{A}))$ and $M_2 \varepsilon D(Mod^?_l(X_2;\mathcal{A}))$. Let $M = M_1 \overset{L}{\underset{\mathcal{A}}{\boxtimes}} M_2 = p_1^*(M_1) \overset{L}{\underset{\mathcal{A}}{\otimes}} p_2^*(M_2)$, $N = N_1 \boxtimes N_2 = p_1^*(N_1) \overset{L}{\underset{\mathcal{A}}{\otimes}} p_2^*(N_2)$. There exists a natural map

$$\mathcal{RH}om_{\mathcal{A}}(M_1, N_1) \overset{L}{\underset{\mathcal{A}}{\boxtimes}} \mathcal{RH}om_{\mathcal{A}}(M_2, N_2) \to \mathcal{RH}om_{\mathcal{A}}(M, N) \tag{5.2.1}$$

as in [SGA]5 Exposé III, (2.2.4).

PROPOSITION 5.5. *The above map is a quasi-isomorphism.*

PROOF. We will provide all the presheaves with the Cartan filtration; now Chapter III, 2.1 and Chapter III, Proposition 2.7 show that the associated graded terms for the induced filtrations on the terms above are given by:

$$\begin{aligned}
&Gr(\mathcal{RH}om_{\mathcal{A}}(M_1,\, N_1) \overset{L}{\underset{\mathcal{A}}{\boxtimes}} \mathcal{RH}om_{\mathcal{A}}(M_2,\, N_2)) \\
&\simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M_1),\, Gr(N_1)) \overset{L}{\underset{Gr(\mathcal{A})}{\boxtimes}} \mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M_2),\, Gr(N_2))
\end{aligned}$$

while $Gr(\mathcal{RH}om_{\mathcal{A}}(M,\, N)) \simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M),\, Gr(N))$. Now $Gr(M_i) \simeq Sp(\mathcal{H}^*(M_i))$, and clearly

$\oplus_n \mathcal{H}^n(M_i) \varepsilon Mod_{\mathcal{H}^*(\mathcal{A}),r}(X_i)$, $i = 1, 2$. Therefore Chapter III, Proposition 2.10(ii) provides the identification:

$$\mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M_i),\, Gr(N_i)) \simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Sp(\mathcal{H}^*(M_i)),\, Sp(\mathcal{H}^*(N)))$$

$$\simeq Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_i),\, \mathcal{H}^*(N_i)))$$

Chapter III, 2.1 and Proposition 2.10 again show

$$Gr(\mathcal{RH}om_{\mathcal{A}}(M_1, N_1) \overset{L}{\underset{\mathcal{A}}{\boxtimes}} \mathcal{RH}om_{\mathcal{A}}(M_2, N_2)) \tag{5.2.2}$$

$$\simeq Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_1),\, \mathcal{H}^*(N_1)) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_2),\, \mathcal{H}^*(N_2)))$$

Clearly the last term is quasi-isomorphic to

$$Sp(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_1) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(M_2), \quad \mathcal{H}^*(N_1)) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(N_2)) \tag{5.2.3}$$

(See [SGA] Exposé III, Proposition (2.3).) Moreover,

$$\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_1) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(M_2),\, \mathcal{H}^*(N_1)) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(N_2))$$

$$\cong \mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_1),\, \mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M_2),\, \mathcal{H}^*(N_1)) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(N_2))$$

Therefore two applications of Chapter III, Propositions 2.10(2.11), 2.12 and 2.13 show the term in (5.2.2) is quasi-isomorphic to:

$$\mathcal{RH}om_{Gr(\mathcal{A})}(Sp(\mathcal{H}^*(M_1)),\, \mathcal{RH}om_{Gr(\mathcal{A})}(Sp(\mathcal{H}^*(M_2)),\, Sp(\mathcal{H}^*(N_1)) \overset{L}{\underset{\mathcal{H}^*(\mathcal{A})}{\otimes}} \mathcal{H}^*(N_2)))$$

$$\simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Sp(\mathcal{H}^*(M_1) \overset{L}{\underset{Gr(\mathcal{A})}{\boxtimes}} Sp(\mathcal{H}^*(M_2)),\, Sp(\mathcal{H}^*(N_1)) \overset{L}{\underset{Gr(\mathcal{A})}{\otimes}} Sp(\mathcal{H}^*(N_2)))$$

$$\simeq \mathcal{RH}om_{Gr(\mathcal{A})}(Gr(M),\, Gr(N))$$

These also show that the spectral sequence (obtained from the induced filtrations) for both sides of (5.2.1) are strongly convergent and that therefore it suffices to obtain a quasi-isomorphism at the associated graded terms. We have therefore completed the proof that the map of (5.2.1) is a quasi-isomorphism. □

PROPOSITION 5.6. *(Kunneth formulae). Consider the situation in 5.2. Assume further that base-change as in (2.7.1) holds.*

Let $L_1 \varepsilon D(Mod_r^?((X_1, \mathcal{A}),\, L_2 \varepsilon D(Mod_l^?((X_2, \mathcal{A})),\, N_1 \varepsilon D(Mod_r^?(Y_1, \mathcal{A})),\, N_2 \varepsilon D(Mod_l^?(X_2, \mathcal{A}))$*; let* $f_i : X_i \to Y_i$ *denote maps over* S *and let* $f = f_1 \underset{S}{\times} f_2 : X \to Y$ *denote the induced map. Now there exists a natural quasi-isomorphism:*

$$Rf_{1*}L_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}Rf_{2*}L_2 \simeq Rf_*(L_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}L_2) \quad and \quad Rf_{1!}L_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}Rf_{2!}L_2 \simeq Rf_!(L_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}L_2) \tag{5.2.4}$$

$$f_1^*N_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}f_2^*N_2 \cong f^*(N_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}N_2) \quad and \tag{5.2.5}$$

$$Rf_1^!N_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}Rf_2^!N_2 \simeq Rf^!(N_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}N_2). \tag{5.2.6}$$

PROOF. We consider the proof of the second quasi-isomorphism in (5.2.4). First observe that left-hand-side and the right-hand-side are related by maps natural in the arguments. Now it suffices to prove these maps are quasi-isomorphisms at each point of $Y_1\underset{S}{\times}Y_2$. By base-change we reduce to the case where q_1, q_2, g' and h' are all isomorphisms and that S is a point (i.e the corresponding site is trivial). Observe that $f = f_1\underset{S}{\times}f_2 = f_1 \circ p_1 = f_2 \circ p_2$ where the maps $p_i : X_1\underset{S}{\times}X_2 \to X_i$, $i = 1, 2$, are the two projections. Now

$$Rf_{1!}L_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}Rf_{2!}L_2 \overset{\simeq}{\to} Rf_{1!}L_1\underset{\mathcal{A}}{\overset{L}{\otimes}}Rf_{2!}L_2 \overset{\simeq}{\to} Rf_{1!}(L_1\underset{\mathcal{A}}{\overset{L}{\otimes}}Lf_1^*(Rf_{2!}(L_2))) \overset{\simeq}{\to} Rf_{1!}(L_1\underset{\mathcal{A}}{\overset{L}{\otimes}}Rp_{1!}Lp_2^*(L_2))$$

$$\overset{\simeq}{\to} Rf_{1!}Rp_{1!}(Lp_1^*(L_1)\underset{\mathcal{A}}{\overset{L}{\otimes}}Lp_2^*(L_2)) = Rf_!(p_1^*(L_1)\underset{\mathcal{A}}{\overset{L}{\otimes}}Lp_2^*(L_2))$$

where the last $=$ is obvious from the definition, the second and the fourth are by the projection formula 2.17 while the first $\simeq$ is by the hypotheses which reduce to the case where g' and h' are the identity maps. This proves the second quasi-isomorphism in (5.2.4). The first is established similarly.

One may readily establish (5.2.5) using the observations that $q_i \circ f = f_i \circ p_i$, $i = 1, 2$.

Now we consider the proof of (5.2.6). First observe that $Rf_i^!(N_i) = \mathbb{D}_{\mathcal{A}}Lf_i^*\mathbb{D}_{\mathcal{A}}(N_i)$, $Rf^!(N_1\underset{\mathcal{A}}{\overset{L}{\boxtimes}}N_2) = \mathbb{D}_{\mathcal{A}}Lf^*(\mathbb{D}_{\mathcal{A}}(N_1)\underset{\mathcal{A}}{\overset{L}{\boxtimes}}\mathbb{D}_{\mathcal{A}}(N_2))$. Therefore, by (5.2.5) and Proposition 5.5 it suffices to consider the case where N_1 and N_2 are both $\mathcal{A}$ on the respective sites. i.e. it suffices to show that

$$D_{\mathcal{A}}\underset{\mathcal{A}}{\overset{L}{\boxtimes}}D_{\mathcal{A}} \simeq D_{\mathcal{A}} \tag{5.2.7}$$

where the $D_{\mathcal{A}}$ on the right is the dualizing presheaf for $X_1\underset{S}{\times}X_2$. One may easily show that there exists a natural from the left-hand-side to the right-hand-side which is compatible with the induced filtrations on each. Therefore we reduce, as in the proof of Proposition 5.5, to the case where $\mathcal{A}$ is replaced by $\mathcal{H}^*(\mathcal{A})$. This is clear, for example, by [SGA] 5, Exposé III, (1.7.3). □

5.3. Consider the situation of 5.2.1. Assume that S is a point. Let $M_1 = L_1$, $N_2 = L_2$, $M_2 =$ the constant pre-sheaf $\mathcal{S}$ on X_2 and $N_1 = Rg^!(\mathcal{S}) \simeq D_{\mathcal{S}}$ in $D(Mod_l^{c,f.t.d}(X_1;\ \mathcal{S}))$. Moreover, $\mathbb{D}_{\mathcal{S}}(L_1) = \mathcal{R}Hom_{\mathcal{S}}(L_1,\ Rg^!(\mathcal{S}))$ and $L_2 \simeq \mathcal{R}Hom_{\mathcal{S}}(\mathcal{S},\ L_2)$. Therefore one obtains a natural quasi-isomorphism:

$$\mathbb{D}_{\mathcal{S}}(L_1) \boxtimes L_2 = \mathcal{R}Hom_{\mathcal{S}}(L_1,\ Rg^!(\mathcal{S}))\underset{\mathcal{S}}{\overset{L}{\boxtimes}}\mathcal{R}Hom_{\mathcal{S}}(\mathcal{S},\ L_2)$$

$$\overset{\simeq}{\to}\mathcal{R}Hom_{\mathcal{S}}(L_1\underset{\mathcal{S}}{\overset{L}{\boxtimes}}\mathcal{S},\ Rg^!(\mathcal{S})\underset{\mathcal{S}}{\overset{L}{\boxtimes}}L_2) \overset{\simeq}{\to} \mathcal{R}Hom_{\mathcal{S}}(p_1^*(L_1),\ p_1^*Rg^!(\mathcal{S})\underset{\mathcal{S}}{\overset{L}{\otimes}}p_2^*(L_2))$$

Next let $Y_1 = S$, $Y_2 = X_2$, $g' : Y_1 \to S =$ the identity and $f_2 : X_2 \to Y_2 =$ the identity. Now (5.2.6) provides the quasi-isomorphism:

$$Rg^!(\mathcal{S}) \underset{\mathcal{S}}{\overset{L}{\boxtimes}} L_2 \overset{\simeq}{\to} Rp_2^!(\mathcal{S} \underset{\mathcal{S}}{\overset{L}{\boxtimes}} L_2) \simeq Rp_2^!(L_2)$$

Combining this with the above quasi-isomorphisms, in the above case, one obtains a natural quasi-isomorphism:

$$\mathbb{D}_{\mathcal{S}}(L_1) \boxtimes L_2 \overset{\simeq}{\to} \mathcal{R}Hom_{\mathcal{S}}(p_1^*(L_1), Rp_2^!(L_2)) \tag{5.3.1}$$

5.4. *Poincaré-Verdier duality.* Let $\mathcal{A}$ denote a commutative presheaf of algebras on $\mathbf{S}$ so that $\mathcal{H}^*(\mathcal{A})$ is locally constant. Let X denote an object over $\mathbf{S}$ with the structure map $p : X \to \mathbf{S}$. Let $F \varepsilon D(Mod_l(\mathbf{S}; \mathcal{A}))$. Now one obtains a pairing:

$$Rp^!(\mathcal{A}) \otimes p^*(F) \to Rp^!(F) \tag{5.4.1}$$

This is adjoint to a map $Rp_!(Rp^!(\mathcal{A}) \otimes p^*(F)) \simeq Rp_!(Rp^!(\mathcal{A})) \otimes F \overset{tr_{\mathcal{A}} \otimes id}{\to} \mathcal{A} \otimes F \to F$ where the last map is given by the structure of a presheaf of left-$\mathcal{A}$-modules on F. It follows that taking hypercohomology, we obtain a pairing:

$$\mathbf{H}(X; Rp^!(\mathcal{A})) \otimes \mathbf{H}(X; p^*(F)) \to \mathbf{H}(X; Rp^!(F)) \tag{5.4.2}$$

In particular it follows that if $\alpha \varepsilon H_n(X; \mathcal{A})$ is a class, we obtain a pairing:

$$\alpha \cap - : \mathcal{H}^k\mathbf{H}(X; p^*(F)) \to H_{n-k}(X; F) \tag{5.4.3}$$

For the following discussion, we will assume that $\mathbf{S}$ is a scheme and that we are considering schemes or algebraic spaces of finite type over $\mathbf{S}$.

DEFINITION 5.7. (Poincaré duality property) Suppose the L-cohomological dimension of X over $\mathbf{S}$ is n. We say that X has *the Poincaré-Verdier duality property for* X, if there exists a class $[X] \varepsilon H_n(X; \mathcal{A})$ so that $[X] \cap -$ is an isomorphism with $F = p^*(\mathcal{A})$ and for all k. We call $[X]$ *a fundamental class of* X in the homology with compact supports of X with respect to $\mathcal{A}$. We say that the algebra $\mathcal{A}$ has the *Poincaré-Verdier duality property over* $\mathbf{S}$ provided all smooth schemes (or algebraic spaces) X over $\mathbf{S}$ have the Poincaré-Verdier duality property.

PROPOSITION 5.8. *Assume that* $p : X \to \mathbf{S}$ *is smooth and that* $\mathcal{A}$ *is a presheaf of algebras on* $\mathbf{S}$ *having the Poincaré-Verdier duality property. If* $D^X_{\mathcal{A}} = Rp^!(\mathcal{A})$, $D^X_{\mathcal{A}} \simeq \Sigma^n Lp^*(\mathcal{A})$ *where* n *denotes the* l*-cohomological dimension of* X *over* $\mathbf{S}$.

PROOF. Fix a (geometric) point p of X. Now the fundamental class of X restricts to fundamental classes $[U] \varepsilon H_n(U; \mathcal{A})$ for each open neighborhood U of p. Each such $[U]$ defines an isomorphism $[U] \cap - : \mathcal{H}^k(\mathbf{H}(U; \mathcal{A})) \cong \mathcal{H}^{k-n}(\mathbf{H}(U; \Sigma^n \mathcal{A})) \to H_{n-k}(U; \mathcal{A}) \cong \mathcal{H}^{k-n}(\mathbf{H}(U; Rp^! \mathcal{A}))$. Taking the colimit over all open neighborhoods of the point p, we obtain a quasi-isomorphism: $\Sigma^n p^*(\mathcal{A})_p \simeq Rp^!(\mathcal{A})_p$. Since this holds for all points p, we obtain $\Sigma^n p^*(\mathcal{A}) \simeq Rp^!(\mathcal{A}) = D^X_{\mathcal{A}}$. $\square$

The following proposition is considered solely for the applications in Chapter V. We let pt denote a *point* in the usual sense provided with the trivial topology. Let $\mathcal{A} = \underline{KU}$ = the obvious constant sheaf on pt, where KU is the spectrum representing complex K-theory. (Recall from Appendix A that this has the structure of a ring object in the category of symmetric spectra.)

PROPOSITION 5.9. *Let $p : \mathbb{R}^1 \to pt$ denote the obvious projection of $\mathbb{R}^1$ to a point. Now $Rp^! \underline{KU} \simeq \Sigma p^{-1} \underline{KU}$.*

PROOF. It suffices to show that if $K_c(\mathbb{R}^1)$ denotes the spectrum of complex K-homology with compact supports for $\mathbb{R}^1$, there exists a fundamental class $[\mathbb{R}^1] \varepsilon \pi_1(K_c(\mathbb{R}^1))$. One may compute the above group using an Atiyah-Hirzebruch spectral sequence and the observation that the integral Borel-Moore homology of $\mathbb{R}^1$ is trivial in all degrees except 1 where it is $\mathbb{Z}$. □

5.5. For the rest of the discussion we will restrict to the category of algebraic spaces (or schemes) of finite type over a base scheme $\mathbf{S}$ provided with the big étale topology. In either case we will assume that there exists a set L of primes so that all the spaces we consider are of finite L-cohomological dimension. Let $Presh(\mathbf{S})$ denote a category of presheaves on the big étale site of $\mathbf{S}$. Let $\mathcal{A}$ denote a presheaf of algebras on $\mathbf{S}$ and let $\mathcal{A}$ denote their inverse images on X, X', Y and Y'. Let $f : (X, \mathcal{A}) \to (Y, \mathcal{A})$ denote a map of ringed sites as before. We will say *smooth base-change* holds if the following condition is satisfied: let

$$\begin{tikzcd} (X', \mathcal{A}) \arrow[r, "f'"] \arrow[d, "g'"] & (X, \mathcal{A}) \arrow[d, "g"] \\ (Y', \mathcal{A}) \arrow[r, "f"] & (Y, \mathcal{A}) \end{tikzcd} \tag{5.5.1}$$

denote a cartesian square with f smooth. Now the natural map $f^*(Rg_*(F)) \to Rg'_* f'^*(F)$ is a quasi-isomorphism for all $F \varepsilon D^{c,f.t.d}(Mod_l(X, \mathcal{A}))$.

LEMMA 5.10. *Smooth-base change holds in the following situations. We are considering algebraic spaces provided with the étale topology and for each ringed space $(X, \mathcal{A})$ as above each $\mathcal{H}^{-n}(\mathcal{A})$ is torsion. Moreover, the base-scheme $\mathbf{S}$ has finite L-cohomological dimension for some non-empty set L of primes different from the residue characteristics and each $\mathcal{H}^{-n}(\mathcal{A})$ is L-torsion.*

PROOF. The proof is entirely similar to that of Proposition 2.10. □

PROPOSITION 5.11. *Assume the above situation. Then the functor*

$$Rf^!_{\#} : D^{c,f.t.d}(Mod_l(Y, \mathcal{A})) \to D^{c,f.t.d}(Mod_l(X, \mathcal{A}))$$

satisfies the hypotheses of 3.1(iv).

PROOF. Recall $\Gamma(U, Rf^!_{\#}(K)) = R\Gamma(Y, \mathcal{RH}om_{\mathcal{A}}(Rf^{\#}_!(\mathcal{A}_U), K))$. Moreover one has the spectral sequence

$$E_2^{u,v} = H^u(R\Gamma(Y, \quad) \circ Gr_v(\mathcal{RH}om_{\mathcal{H}^*(\mathcal{A})}(Rf^{\#}_1(\mathcal{H}^*(\mathcal{A}_U)), \mathcal{H}^*(K)))) \Rightarrow \mathcal{H}^{u+v}(\Gamma(U, Rf^!_{\#}(K))$$

In view of the hypotheses on uniform finite cohomological dimension and the hypothesis that f_* is constructible, one may now readily verify that there exists an $N >> 0$ independent of K and v so that $E_2^{u,v} = 0$ for $u > N$. □

PROPOSITION 5.12. *Assume the situation of 5.5. Let $f : X \to Y$ denote a smooth map as above and let the L-cohomological dimension of the fiber be n. Then there exists a natural isomorphism of functors:*

$$Rf^! \cong f^*\Sigma^n : D(Mod_l^?(Y,\mathcal{A})) \to D(Mod_l^?(X,\mathcal{A})).$$

PROOF. *Step 1.* We first consider a presheaf of the form $j_{U!}^{\#}j_U^*(\mathcal{A})$, where $j_U : U \to Y$ belongs to the big étale site of Y. Then $Gr(j_{U!}^{\#}j_U^*(\mathcal{A})) \simeq j_{U!}^{\#}j_U^*(Gr_{\mathcal{C}}(\mathcal{A}) \simeq j_{U!}^{\#}j_U^*(Sp(\mathcal{H}^*(\mathcal{A})))$ and therefore, by the hypotheses, $j_{U!}^{\#}j_U^*(\mathcal{A})$ has finite tor dimension . Let $V = X \underset{Y}{\times} U = f^{-1}(U)$. We will show there exists a quasi-isomorphism

$$Rf^!(j_{U!}^{\#}j_U^*(\mathcal{A})) \simeq j_{V!}^{\#}j_V^*f^*\Sigma^n(\mathcal{A}) \tag{5.5.2}$$

natural in U. First observe that if $f_U : V \to U$ is the induced map, there exists a natural map

$$j_{V!}^{\#}Rf_U^!j_U^*(\mathcal{A}) \to Rf^!(j_{U!}^{\#}j_U^*(\mathcal{A})) \tag{5.5.3}$$

By Proposition 5.8, $Rf_U^!(j_U^*(\mathcal{A})) \simeq f_U^*\Sigma^n(\mathcal{A})$. Therefore the left-hand-side identifies with $j_{V!}^{\#}f_U^*\Sigma^n(j_U^*(\mathcal{A}))$
$\simeq f^*\Sigma^n(j_{U!}^{\#}j_U^*(\mathcal{A})) \simeq j_{V!}^{\#}j_V^*f_U^*\Sigma^n(\mathcal{A})$. Thus it suffices to show the map in (2.1.4) is a quasi-isomorphism. Recall $j_{V!}^{\#}Rf_U^!j_U^*(\mathcal{A}) \simeq \mathbb{D}_{\mathcal{A}}Rj_{U*}\mathbb{D}_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}f_U^*(\mathbb{D}_{\mathcal{A}}j_U^*(\mathcal{A})) \simeq \mathbb{D}_{\mathcal{A}}Rj_{U*}f_U^*j_U^*(D_{\mathcal{A}})$ and $Rf^!(j_{U!}^{\#}j_U^*(\mathcal{A}))$
$\simeq \mathbb{D}_{\mathcal{A}}f^*\mathbb{D}_{\mathcal{A}}\mathbb{D}_{\mathcal{A}}Rj_{U*}\mathbb{D}_{\mathcal{A}}j_U^*(\mathcal{A}) \simeq \mathbb{D}_{\mathcal{A}}f^*Rj_{U*}j_U^*(D_{\mathcal{A}})$. The last two are quasi-isomorphic by the smooth-base change in (5.5.1). This completes step 1.

Step 2. Next consider an $L\varepsilon D(Mod_l(Y,\mathcal{A}))$ and let $P(L)_\bullet \to L$ denote a simplicial resolution as in Chapter II, Proposition (2.4). Recall each term $P(L)_k = \underset{U,k}{\sqcup} j_{U!}^{\#}j_U^*\Sigma^n(\mathcal{A})$. Therefore, by step 1, there exists a quasi-isomorphism

$$Rf^!(P(L)_k) \overset{\simeq}{\to} f^*\Sigma^n(P(L)_k)$$

natural in k. Now take the homotopy colimit $\underset{\Delta}{\text{hocolim}}\{Rf^!(P(L)_k)|k\}$. By Lemma 3.3 with $\phi = Rf^! = Rf_{\#}^!$ this is quasi-isomorphic to $Rf^!\underset{\Delta}{\text{hocolim}}\{P(L)_k|k\} \simeq Rf^!L$. (See the proposition above which shows that $Rf^!$ in fact satisfies the hypotheses there.) On the other hand taking homotopy colimits preserve quasi-isomorphisms and commute with the functor f^*. It follows that $\underset{\Delta}{\text{hocolim}}\{f^*\Sigma^n(P(L)_k)|k\} \simeq f^*\Sigma^n(\underset{\Delta}{\text{hocolim}}\{(P(L)_k)|k\}) \simeq f^*\Sigma^n(L)$. □

COROLLARY 5.13. *Let $f : X \to Y$ denote a smooth between complex quasi-projective varieties of relative dimension n. Now the functor*

$$Rf^! : D(Mod_l^{c,f.t.d}(Y,\underline{KU}) \to D(Mod_l^{c,f.t.d}(X,\underline{KU}))$$

identifies naturally with the functor $f^*\Sigma^{2n}$. In particular, there exists a fundamental class in $H_*(X,\underline{KU})$ which is the homology of X with compact supports with respect to $\underline{KU}$ as defined earlier.

PROOF. This is similar to the proof of the last proposition. □

6. Examples

With a view towards further applications (see for example, the next chapter), we will presently discuss in detail the two examples considered in (i) and (iii) after the statement of the bi-duality theorem in section 1.

6.1. The Étale topoi. Here S will denote a base Noetherian scheme. $(schemes/S)$ will denote the category of schemes locally of finite type over S. $((smt.schemes/S)$ will denote the full-subcategory of smooth schemes over S.) We will provide this category with the big étale topology: this site will be denoted $(schemes/S)_{Et}$. (One may similarly consider the big étale site of all algebraic spaces locally of finite type over S, though for simplicity we will restrict to schemes.) $Presh((schemes/S)_{Et})$ will denote a category of presheaves on this site as before. Let l denote a prime different from the residue characteristics of S. Presently we will discuss an l-adic variant of the basic theory developed so far, under some mild additional assumptions.

Let l denote a fixed prime. We let $Presh((schemes/S)_{Et}; l)$ denote the full sub-category of $Presh((schemes/S)_{Et})$ consisting of objects P so that each $\mathcal{H}^n(P)$ is l-primary torsion as a presheaf.

6.2. Existence of completions. We will assume that the obvious inclusion of $Presh((schemes/S)_{Et}; l)$ into $Presh((schemes/S)_{Et})$ has a left-adjoint which we call *the l-completion* functor.

Given an object $P \varepsilon Presh((schemes/S)_{Et})$, its l-completion will be denoted $\hat{P_l}$. We will assume that this functor preserves the structure of strongly triangulated categories and that any pairing $M_1 \otimes M_2 \otimes ... \otimes M_n \to N$ will induce a pairing $(M_1)\hat{}_l \otimes (M_2)\hat{}_l \otimes ... \otimes (M_n)\hat{}_l \to (N)\hat{}_l$. It follows readily that if $\mathcal{A}$ is an algebra in $Presh((schemes/S)_{Et})$, then its l-completion $\hat{\mathcal{A}_l}$ will also be an algebra in $Presh((schemes/S)_{Et})$.

EXAMPLES 6.1. (i) As examples of this one may consider the following. Let $Presh((schemes/S)_{Et})$ denote a category of presheaves of spectra; in this case the Bousfield-Kan completion functor for simplicial sets extends to such a completion functor.

(ii) In case $Presh((schemes/S)_{Et}) = C(Mod((schemes/S)_{Et}, \mathcal{R}))$ where $\mathcal{R}$ is the constant sheaf associated to a commutative ring with unit, the completion at l will have the usual meaning.

Let $\mathcal{A}$ denote a commutative algebra in $Presh((schemes/S)_{Et})$. We will put the following assumption on $\mathcal{A}$:

6.3. (i) for each $\nu \geq 0$, there is natural map $l^\nu : \widehat{\mathcal{A}}_l \to \widehat{\mathcal{A}}_l$ which on $\mathcal{H}^*$ induces multiplication by l^ν.

(ii) the natural map $\widehat{\mathcal{A}}_l \to \underset{\nu}{\text{holim}}(\widehat{\mathcal{A}}_l/l^\nu)$ is a quasi-isomorphism and

(ii) the natural map $\mathcal{H}^n(\underset{\nu}{\text{holim}}(\widehat{\mathcal{A}}_l/l^\nu)) \to \underset{\nu}{\lim}\mathcal{H}^n((\widehat{\mathcal{A}}_l))/l^\nu$ is an isomorphism for each n.

EXAMPLES 6.2. (i) Let $Presh((schemes/S)_{Et})$ denote the category of all presheaves of symmetric spectra on $(schemes/S)_{Et}$. We let KU denote the spectrum representing complex (topological) K-theory. Let $\widehat{\underline{KU}}_l$ denote the l-adic completion of the constant presheaf of spectra representing complex K-theory. In this case $\widehat{\underline{KU}}_l/l^\nu$ has the usual meaning. If KU denotes the spectrum representing complex K-theory, recall that $\pi_n(KU) \cong \mathbb{Z}$ if n is even and trivial otherwise. Therefore, the above hypotheses are met by $\widehat{\underline{KU}}_l$. (The next chapter will consider a detailed application of these ideas to the construction of Euler-classes.)

(ii) Let $S = Spec \quad k$ denote the spectrum of an algebraically closed field of characteristic p. We will consider the big étale site of all quasi-projective smooth schemes over k: this will be denoted $((qp.sm.schemes/k))_{Et}$. Let $Presh((qp.sm.schemes/Spec \quad k)_{Et}) = C(Mod((schemes/Spec \quad k)_{Et}, \mathbb{Z}[1/p]))$. We let $\mathcal{A}$ denote the sheaf of E^∞ differential graded algebras associated to the motivic complex as in [J-6]: we will denote this by $\underset{n}{\oplus}\mathbb{Z}_{mot}(n)$. In this case, the rigidity property of mod-l^ν motivic complexes shows that $\mathbb{Z}_{mot}/l^\nu(n)$ is simply the pull-back of the complex $(\mathbb{Z}_{mot}/l^\nu)_{|Spec\quad k}$. The latter is computed in [FSV] to be given by $\mathcal{H}^i((\mathbb{Z}_{mot}/l^\nu)_{|Spec\quad k}(n)) = \underline{\mathbb{Z}}/l^\nu$, $i = 2n \geq 0$ and trivial otherwise. Therefore the hypotheses in 6.3 are satisfied by $\hat{\mathcal{A}_l}$.

DEFINITION 6.3. Let $\mathfrak{S}$ denote one of the sites considered above. We will define $M\varepsilon Mod_l^{filt}(\mathfrak{S}, \widehat{\mathcal{A}}_l)$ to be *constructible* if the following hold:

(i) there exists an $\bar{M}\varepsilon D_b(Mod_l(\mathfrak{S}, \mathcal{H}^*(\widehat{\mathcal{A}}_l)))$ so that $Gr(M) \simeq Sp(\bar{M})$ and

(ii) $\bar{M} \simeq \lim_{\nu} \bar{M}/l^\nu$, $\bar{M}/l^\nu \varepsilon D_b^{c,f.t.d}(Mod_l(\mathfrak{S}, \mathcal{H}^*(\widehat{\mathcal{A}}_l)/l^\nu))$, for each $\nu \geq 0$.

M will be *of finite tor dimension* if $\bar{M}$ is of finite tor dimension.

THEOREM 6.4. *With the above definition, the formalism of Grothendieck-Verdier duality as in the earlier sections carries over to* $D(Mod_l^{c,f.t.d}(\mathfrak{S}, \widehat{\mathcal{A}}_l))$.

PROOF. It suffices to observe that the derived functors of the direct and inverse image functors preserve the derived category of objects with constructible cohomology sheaves. □

REMARK 6.5. Assume that, in addition, $\mathcal{H}^*(\widehat{\mathcal{A}}_l)$ is a sheaf of regular rings. In this, case every sheaf of modules over it is of finite tor dimension. Therefore the formalism of Grothendieck-Verdier duality will hold for all objects that are constructible; i.e. every object is automatically of finite tor dimension. In particular, this applies to the two examples considered in 6.2.

6.4. Locally compact Hausdorff spaces of finite cohomological dimension. One may consider a big site where the objects are locally compact Hausdorff spaces of finite cohomological dimension. The coverings will given by coverings in the usual sense for the given topology on each space. Let $\mathfrak{S}$ denote this big site and $Presh(\mathfrak{S})$ denote the corresponding category of presheaves as before. Let $\mathcal{A}$ denote a commutative algebra in $Presh(\mathfrak{S})$ so that each $\mathcal{H}^n(\mathcal{A})$ is constant. (For example $\mathcal{A}$ itself is constant.) It is clear that in this case the entire formalism of Grothendieck-Verdier duality as in the earlier sections applies. Suppose, in addition, that $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded regular rings. In this case, every sheaf of modules over $\mathcal{H}^*(\mathcal{A})$ is of finite tor dimension, so that the entire formalism of Grothendieck-Verdier duality applies to all objects that are constructible.

As an example of this one may let $Presh(\mathfrak{S})$ denote the category of presheaves of (symmetric) spectra on $\mathfrak{S}$. If $\underline{KU}\varepsilon Presh(\mathfrak{S})$ denotes the constant presheaf associated to the spectrum representing complex K-theory, the above hypotheses are satisfied. The next chapter will consider a detailed application of these ideas.

CHAPTER V

Character cycles in K-theory for constructible sheaves

1. Introduction

In this chapter we provide a concrete application of the theory developed so far to define an additive map from the Grothendieck group of constructible sheaves on a space to its K-homology. There are various *avatars* of the basic technique: if X is a suitably nice topological space and F is a constructible sheaf of $\mathbb{Z}$-modules on X, we associate to F a class in the complex K-homology of X. The same technique applies in the étale setting to constructible l-adic sheaves on the étale topology of a variety in positive characteristic $p \neq l$ and provides classes in its étale K-homology (completed at l). These are Euler classes and generalize the cycle classes in K-homology associated to closed smooth subvarieties of smooth varieties. Finally we also obtain a micro-local version of these classes; we also show that these are K-theoretic versions of the *character cycles* with values in homology with locally compact supports as defined by Kashiwara and Schapira.

In the second section we will provide definitions of Fourier transformation, specialization and microlocaliztion for presheaves of spectra. (All the spectra we consider in this section may be assumed to be symmetric spectra (as in [H-S-S]) and may in fact be replaced by presheaves of Γ-spaces if one is willing to consider connected spectra.) These will be related by strongly convergent spectral sequences whose $E-2$-terms will be the corresponding operations applied to the homotopy sheaves of the above presheaves of spectra. In the next section we define and study the properties of a trace-map (and an associated *Euler-class*) for constructible presheaves of KU-module spectra on complex varieties as well as for constructible presheaves of $\widehat{KU}_\ell$-module spectra on the étale site of varieties in positive characteristic. (Here l is assumed to be different from the characteristic of k and $\widehat{KU}_\ell$ is the completion at l in the sense of [B-K] and [T-1] of the symmetric ring spectrum KU. See Appendix A and the end of the last chapter for some details on this.)

In the fourth section we show how to associate functorially a constructible presheaf of KU- ($\widehat{KU}_\ell$-) module spectra to any constructible sheaf of $\mathbb{Z}$-modules (any constructible l-adic sheaf, respectively). In the fifth section we explore the relationship between our classes in K-homology and the corresponding classes in homology with locally compact supports as defined by Kashiwara and Schapira. (See [K-S-2]). Topological K-homology will mean the homology with compact supports with respect to the constant sheaf of spectra $\underline{KU}$ (or with respect to $\underline{\widehat{KU}_l}$ in positive characteristic $p \neq l$) in the sense of Chapter IV, Definition 4.8. For a space X, $H_0(X, \underline{KU})$ ($H_0(X, \underline{\widehat{KU}_l})$) will be denoted $K_0^{top}(X)$ ($\widehat{K_0^{top}(X)}_l$, respectively). In the final section we combine the results of the earlier sections to define an Euler-class with values in topological K-homology as an additive homomorphism from the Grothendieck group of constructible sheaves to topological K-homology commuting with direct images under suitable restrictions. We also obtain such a micro-local Euler class for $\mathbb{Z}$-constructible sheaves on complex varieties. One may state the main theorem as follows. If X is a complex variety, we will let $Const(X; \mathbb{Z})$ denote the category of all constructible sheaves of $\mathbb{Z}$-modules on X. If X is a variety defined over a field k of positive characteristic p (satisfying the conditions in (1.1.1)), l is a prime different from p and ν is a positive integer, $Const^{f.t.d}(X; l-adic)$ will denote the full sub-category of constructible l-adic sheaves that are of finite tor dimension. We will let $K(Const(X;\mathbb{Z}))$ $(K(Const^{f.t.d}(X; l-adic)))$ denote the Grothendieck group of the corresponding category.

THEOREM 1.1. *(See Theorem (6.1)) (i) If X is a complex algebraic variety, there exists an additive homomorphism:*

$$Eu : K(Const(X;\mathbb{Z})) \to K_0^{top}(X).$$

(ii) If X is, in addition, a smooth quasi-projective variety, there exists another additive homomorphism:

$$Eu_\mu : K(Const(X;\mathbb{Z})) \to K_0^{top}(T^*X)$$

which factors through the obvious map $K_0^{top}(\Lambda_F) \to K_0^{top}(T^*X)$ where Λ_F is the micro-support of F. The Todd homomorphism sends these classes to the corresponding Euler-classes in Borel-Moore homology.

(iii). If X is a variety defined over a field k as in (0.1) of characteristic p and l is a prime different from p, there exists an additive homomorphism

$$Eu : K(Const^{f.t.d}(X; l-adic)) \to \widehat{K_0^{top}(X)}_l$$

The map from K-homology to étale homology (as in (5.0.7)) sends these classes to the corresponding Euler-classes, at least, in the case of projective varieties.

(iv). The maps in (i) and (iii) commute with direct-images for proper maps. The map in (ii) commutes with direct images for proper and smooth maps of complex varieties.

Our interest in these problems was awakened by a question of Pierre Schapira about the possibility of defining such classes directly (i.e. without the intermediary machinery of $\mathcal{D}$-modules) whom we thank warmly. One may also observe that the theory of $\mathcal{D}$-modules can provide such classes for $\mathbb{C}$- constructible sheaves, while our constructions apply also to constructible sheaves of $\mathbb{Z}$-modules and also to varieties in positive characteristic.

1.1. Throughout the chapter, we will follow most of the conventions and terminology of the earlier chapters; any exception to this will be stated explicitly below. Topological K-homology will mean topological K-homology with locally compact supports for locally compact Hausdorff spaces and étale K-homology with locally compact supports as defined in chapter IV for varieties in positive characteristics. These are defined by ring spectra in the sense of appendix A, section 2: the ring spectrum representing complex K-theory will be denoted KU. In positive characteristic p, we will restrict to schemes of finite type defined over a field k with finite cohomological dimension and so that for each prime l different from the characteristic of k, each $H^n(Gal(\bar{k}/k);\ \mathbb{Z}/l^\nu)$ is finite for all n, $\nu \geq 1$. Here $\bar{k}$ is the separable closure of k. (For example k could be a finite field or the separable closure of one.) If X is a locally compact Hausdorff space with finite cohomological dimension (a scheme of finite type over a field k as above) $\mathcal{C}_X$ will denote the usual site (the small étale site, respectively) associated to X. We will use the generic term *space* to denote a topological space or a scheme as above.

1.2. We will adopt the basic terminology as in [K-S-1] or [K-S-2] for various aspects of the micro-local theory. The functor $\tilde{\ }$ will denote the sheafifying functor; we will apply this only to abelian presheaves. If R is a noetherian ring $D_+(\mathcal{C}_X;\, R)$ will denote the derived category of bounded-below complexes of R-modules on the site $\mathcal{C}_X$. (R will denote $\mathbb{Z}$ if X is a topological space as above while it will denote $\widehat{\mathbb{Z}}_l$, $\nu >> 0$ and l different from char (k) if X is a scheme in positive characteristic.) $D^c_+(\mathcal{C}_X;\, R)$ ($D^{c,f.t.d}_+(\mathcal{C}_X;\, R)$) will denote its full subcategory of complexes with constructible cohomology sheaves (that are also of finite tor dimension, respectively).

1.3. Let E denote a ring spectrum in the sense of appendix A. Now $Mod_l(\mathcal{C}_X, E)$ will denote the category of presheaves of left-module-spectra over E on the site $\mathcal{C}_X$. The derived category $D^c(X, E)$ will denote the derived category of presheaves of E-modules that are constructible as in earlier chapters, i.e. $D(Mod^c_l(\mathcal{C}_X, E))$ while $D^{c,f.t.d}(X, E)$ will denote the associated full sub-category of all objects of finite tor dimension. (Recall this was denoted $D(Mod^{c,f.t.d}_l(X, E))$ in earlier chapters.)

REMARK 1.2. In positive characteristics, we will assume that the given spectrum E is the l-completion of another spectrum E' so that the homotopy groups of E are in fact the l-completion of the homotopy groups of E'. (See Appendix A, (2.2).)

2. Fourier transformation, specialization and micro-localization for presheaves of spectra

In this section we will restrict to complex varieties or often to locally compact topological spaces. (All our results should carry over to carry over to positive characteristics (at least in principle) using the étale site using the appropriate variations of the Fourier transform, specialization and micro-localization. With such an extension, it would be possible to obtain micro-local classes in étale K-theory for constructible sheaves on varieties in positive characteristics. However the details seem to be a bit involved - for example, the appropriate notion of micro-localization would be that of Gabber and Laumon and the appropriate notion of Fourier transformation would be that of Deligne and Laumon (see [Lau]). We hope to discuss this more fully elsewhere.)

2.1. The Fourier-transformation. Let $q_1 : \mathcal{E} \rightarrow Z$ denote a locally trivial *real* vector bundle on a locally compact space Z with finite cohomological dimension. If R denotes a graded ring, we will let $D_+(\mathcal{E};\, R)$ denote the derived category of bounded-below complexes of sheaves of graded R-modules on $\mathcal{E}$. Let $D_{+,conic}(\mathcal{E};\, R)$ denote the full-subcategory of $D_+(\mathcal{E};\, R)$ of complexes whose cohomology sheaves are locally constant on half-lines of $\mathcal{E}$.

2.1.1. Let $D^c_{+,conic}(\mathcal{E};\, E)$ will denote the full sub-category of the category $D^{c,f.t.d}(\mathcal{E};\, E)$ consisting of presheaves F so that $\pi_*(F)\tilde{} = \underset{i}{\oplus}\pi_i(F)\tilde{}$ belongs to $D^c_{+,conic}(\mathcal{E};\, \pi_*(E))$.

Let $i_1 : Z \rightarrow \mathcal{E}$ ($i_2 : Z \rightarrow \mathcal{E}^*$) denote the closed imbedding provided by the zero-section. Let $F' \varepsilon D^c_{+,conic}(\mathcal{E};\, E)$. Now the map $q_1^{-1}Rq_{1*}(F') \rightarrow F'$ defines a map

$$Rq_{1*}(F') \overset{\simeq}{\rightarrow} i_1^{-1}q_1^{-1}Rq_{1*}(F') \rightarrow i^{-1}(F') \tag{2.1.2}$$

natural in F'.

Similarly the map $i_{1!}Ri^!_1(F') \rightarrow F'$ defines a map

$$Ri^!_1(F') \overset{\simeq}{\rightarrow} Rq_{1!} \circ i_{1!} \circ Ri^!_1(F') \rightarrow Rq_{1!}(F'), \tag{2.1.3}$$

again natural in F'. The Cartan-filtration on F' is compatible with the above maps (by naturality); this filtration provides spectral sequences that converge to the sheaves of homotopy groups of the above and whose $E_2^{s,t}$-terms are given by the corresponding s-th derived functor applied to the sheaf $\mathcal{H}^{-t}(F')\tilde{} = \pi_t(F')\tilde{}$ - see (7.1.2). Since the above spectral sequences converge strongly and we obtain an isomorphism at the E_2-terms (see [K-S-2] p. 170), we observe that the above maps are *quasi-isomorphisms in general.*

Let $q_2 : \mathcal{E}^* \rightarrow Z$ denote the dual bundle to $\mathcal{E}$. Let

$P = \{(x,y)\varepsilon\mathcal{E}\underset{Z}{\times}\mathcal{E}^* | < x, y >\geq 0\}$, $P' = \{(x,y)\varepsilon\mathcal{E}\underset{Z}{\times}\mathcal{E}^* | < x, y >\leq 0\}$,

Let $i : P \rightarrow \mathcal{E}\underset{Z}{\times}\mathcal{E}^*$ and $i' : P' \rightarrow \mathcal{E}\underset{Z}{\times}\mathcal{E}^*$ denote the obvious closed imbeddings. Let $R\Gamma_P = i_*Ri^!$. Let $p_1 : \mathcal{E}\underset{Z}{\times}\mathcal{E}^* \rightarrow \mathcal{E}$ ($p_2 : \mathcal{E}\underset{Z}{\times}\mathcal{E}^* \rightarrow \mathcal{E}^*$) denote the obvious projection. Let $z_1 : \mathcal{E}\underset{Z}{\times}Z \rightarrow \mathcal{E}\underset{Z}{\times}\mathcal{E}^*$ and $z_2 : Z\underset{Z}{\times}\mathcal{E}^* \rightarrow \mathcal{E}\underset{Z}{\times}\mathcal{E}^*$ denote the obvious zero-sections.

Now one defines the *Fourier-Sato transform* (see [K-S-2] chapter III and [Bryl-2]) of $F\varepsilon D^c_{+,conic}(\mathcal{E};\,E)$ to be

$$\widehat{F} = Rp_{2*}R\Gamma_P(p_1^{-1}(F)) \simeq R(p_{2*}\circ\Gamma_P)(p_1^{-1}(F)) \tag{2.1.4}$$

Observe from (7.1.2), that, so defined, there exists a spectral sequence

$$E_2^{s,t} = \mathcal{H}^s(\widehat{\pi_{-t}(F)}\,) = R^s(p_{2*}\circ\Gamma_P)(p_1^{-1}(\widehat{pi_{-t}(F)})) \Rightarrow \pi_{-s-t}(\widehat{F}\ \ ) \tag{2.1.5}$$

which is strongly convergent since all the spaces are assumed to have finite cohomological dimension. Observe that *the E_2-terms are the cohomology sheaves of the Fourier transforms of the abelian sheaf* $\pi_{-t}(F)\check{}$. Using this spectral sequence, and various basic results from Chapter IV, one can easily recover all the usual properties (see [K-S-2] chapter III or [Bryl-2]) of the Fourier transform. For example, one may show readily that

(2.F.1) $\widehat{F}\ \varepsilon D^c_{+,conic}(\mathcal{E}^*;\,E)$, if $F\varepsilon D^c_{+,conic}(\mathcal{E};\,E)$.

There exists a natural quasi-isomorphism:

(2.F.2) $\widehat{F} \simeq Rp_{2!}i'_*i'^*(p_1^{-1}F)$, $F\varepsilon D^c_{+,conic}(\mathcal{E};\,E)$

To see this, first observe the existence of the following natural maps:

$$Rp_{2*}R\Gamma_P(p_1^{-1}F) \rightarrow Rp_{2*}R\Gamma_P i'_*i'^*(p_1^{-1}F) \leftarrow Rp_{2!}R\Gamma_P i'_*i'^*(p_1^{-1}F) \rightarrow Rp_{2!}i'_*i'^*(p_1^{-1}F)$$

These maps are compatible with the Cartan-filtration on F and hence they induce maps of spectral sequences that converge to the respective sheaves of homotopy groups. The $E_2^{s,t}$-terms are the corresponding s-th derived functor applied to $\pi_t(F)\check{}$. Therefore we obtain an isomorphism of the corresponding E_2-terms as shown in in [K-S-2]p.171. This suffices to prove the maps in (2.F.2) are quasi-isomorphisms in general - see (7.1.2).

Similarly one also obtains a quasi-isomorphism:

(2.F.3) $Rq_{2*}(\widehat{F}) \simeq Rq_{1!}(F)$.

One may deduce this from 2.1.2 and 2.1.3 as follows. First observe the existence of natural maps:

$$Rq_{2*}(\widehat{F}) = Rq_{2*}(Rp_{2*}R\Gamma_P(p_1^{-1}(F))) \rightarrow Rq_{2*}(Rp_{2*}R\Gamma_P(i'_*i'^*p_1^{-1}(F)))$$

$$\longleftarrow Rq_{2!}(Rp_{2!}R\Gamma_P(i'_*i'^*p_1^{-1}(F)))$$

By [K-S-2] p. 171, and by (7.1) the above maps may be seen to be quasi-isomorphisms. Now we apply 2.1.3 to p_2 and q_2; this provides the quasi-isomorphism of the last term above with $Ri_1^! Rz_1^! R\Gamma_P(i'_* i'^* p_1^{-1}(F))$.

It is shown in [K-S-2] p.171 that the support of ${i'}_* {i'}^{-1} R\Gamma_P(p_1^{-1}(\pi_*(F))\tilde{}$ is contained in $Z \underset{Z}{\times} \mathcal{E}^*$. Therefore we may replace the $Rz_1^!$ with z_1^{-1}. Now there exists a natural map

$$Ri_1^! z_1^{-1} {i'}_* {i'}^{-1} R\Gamma_P(p_1^{-1}F) \to Ri_1^! z_1^{-1} {i'}_* {i'}^{-1}(p_1^{-1}F) = Ri_1^!(F) \simeq Rq_{1!}(F)$$

where the last quasi-isomorphism follows from 2.1.3 applied to q_1. Now (7.1) shows the composition:

$$Rq_{2*}(\widehat{F}) \to Rq_{1!}(F)$$

is a *quasi-isomorphism.*

Assume E has the Poincaré-duality property. Let $L \varepsilon D^{c,f.t.d}(Z;\, E)$. Now we also obtain the natural quasi-isomorphism:

(2.F.4) $\widehat{Rq_1^!(L)} \simeq q_2^{-1}(L)$.

To see this, observe $\widehat{Rq_1^!(L)} \simeq Rp_{2!} {i'}_* {i'}^{-1}(p_1^{-1}(Rq_1^!(L))) \simeq Rz_2^! {i'}_* {i'}^{-1}(Rp_2^! q_2^{-1}(F))$ since $Rq_i^! \simeq q_i^{-1}[2d]$ and $Rp_i^! \simeq p_i^{-1}[2d']$ for suitable d and d' by the Poincaré duality property of E. Now the last term has a natural map from $Rz_2^! Rp_2^!(q_2^{-1}(L)) = q_2^{-1}(L)$. By (7.1) we reduce to showing the above maps are quasi-isomorphisms when $L \varepsilon D_b^{c,f.t.d}(Z;\, \pi_*(E))$. This is clear by [K-S-2] p. 175.

Finally one may also observe that

(2.F.5) $Sp(\widehat{F}) \simeq (\widehat{Sp(F)})$, $F \varepsilon D^c_{+,conic}(\mathcal{E};\, \pi_*(E))$.

where $Sp : D^c_{+,conic}(\mathcal{E};\, \pi_*(E)) \to D^c_{+,conic}(\mathcal{E};\, Gr(E))$ is the functor defined in Chapter I, Definition (4.6). This follows readily from Chapter IV, Proof of Proposition 2.12 and the discussion following it, where it is shown that all the functors involved in the definition of the Fourier transformation commute with the functor Sp.

2.2. Specialization. Let X denote a real manifold of class C^α, $\alpha \geq 2$ and let $f : M \subseteq X$ denote the imbedding of a submanifold in X. We let $\tilde{X}_M$ denote the blow-up of $X \times \mathbb{R}$ along $M \times 0$. We let $p : \tilde{X}_M \to X$ and $t : \tilde{X}_M \to \mathbb{R}$ denote the obvious maps. The fibers $t^{-1}(c)$ are isomorphic to X for $c \neq 0$, while for $c = 0$, $t^{-1}(0) \simeq T_M X =$ the normal bundle to the imbedding of M in X. Let $\Omega = t^{-1}(\{x \varepsilon \mathbb{R} | x > 0\})$, $j : \Omega \to \tilde{X}_M$ the obvious open imbedding and $s : T_M X \to \tilde{X}_M$ the obvious closed imbedding. If $F \varepsilon D_c(X;\, E)$, one lets

$$\nu_M(F) = (Rj_* j^{-1} p^{-1} F)|_{T_M X} = s^{-1} Rj_* j^{-1}(p^{-1}F)$$

and call it *the specialization* of F along M. In this context one obtains a strongly convergent spectral sequence (as in (7.1.2)):

$$E_2^{s,t} = \mathcal{H}^s(\nu_M(\pi_{-t}(F)\tilde{}\ )) \Rightarrow \pi_{-s-t}(\nu_M(F))\tilde{}$$

using which one recovers the usual properties (see [K-S-1] chapter 2) of specialization. For example, one obtains a natural quasi-isomorphism:

(2.S.1) $\nu_M(F) \simeq Rs^! j_! j^! Rp^!(F)$

To see this, one begins with the fibration sequence:

$$s_* Rs^!(j_! j^{-1}(p^{-1}F)) \to j_! j^{-1}(p^{-1}F) \to Rj_* j^{-1} p^{-1}(F)$$

On applying s^{-1} to it, one observes that $s^{-1} \circ j_! \simeq *$ and hence

$$s^{-1} Rj_* j^{-1} p^{-1}(F) \simeq \Sigma(Rs^! j_! j^{-1} p^{-1} F) \simeq Rs^! j_! \Sigma(\tilde{p}^{-1} F)$$

where $\tilde{p} = p \circ j : \Omega \to X$. Now $\Omega \simeq X \times \{r \varepsilon \mathbb{R} | r > 0\}$ and $\tilde{p}$ = the projection to the first factor. One may therefore conclude readily that $\Sigma(\tilde{p}^{-1}F) \simeq R\tilde{p}^!(F)$ - see Chapter IV, Proposition 5.12. This gives (2.S.1). As a corollary to (2.S.1) one obtains the natural quasi-isomorphism

(2.S.2) $\nu_M(D_X^E(F)) \simeq D_{T_M X}^E(\nu_M(F))$.

One also obtains the following property. Let $f : M \to X$ denote the closed imbedding of M in X, $\tau : T_M X \to M$ and $\pi : T_M^* X \to M$ the obvious projections. Let $z : M \to T_M X$ denote the zero-section. Now one verifies readily that there is a natural map:

(2.S.3) $f^{-1}(F) \to R\tau_*(\nu_M(F))$

(To see this, observe $f^{-1}(L) = z^{-1}s^{-1}p^{-1}(F) \to z^{-1}s^{-1}Rj_* j^{-1}p^{-1}(F) = z^{-1}(\nu_Y(F))$. The last term may be identified with $R\tau_*(\nu_M(F))$ by 2.1.2.) Using the spectral sequence above (which converges strongly) along with the identification of its E_2-terms, one may show readily that this map is a *quasi-isomorphisms* stalkwise. Now apply the above map to $D_X(F)$ instead of F. Using the theory of generalized Verdier duality as in chapter IV, (2.S.2) and (2.S.3) one now obtains a natural quasi-isomorphism:

(2.S.4) $f^!(F) \simeq D_M(f^{-1}D_X(F)) \leftarrow D_M(R\tau_* D_{T_M X}(\nu_M(F)) \simeq R\tau_!(\nu_M(F))$

Next assume $L \varepsilon D^{c,f.t.d}(M;\, E)$. Applying (2.S.4) to $F = f_!(L) = f_*(L)$, one obtains a natural map

$$L \simeq Rf^! f_!(L) \overset{\simeq}{\leftarrow} R\tau_!(\nu_M(f_!(L)))$$

Applying $R\tau^!$ to this map, one obtains a natural map

(2.S.5) $R\tau^!(L) \overset{\simeq}{\leftarrow} R\tau^! R\tau_!(\nu_M(f_! L)) \leftarrow \nu_M(f_! L)$

Finally one may also show that one has a natural quasi-isomorphism:

(2.S.6) $Sp(\nu_M(\bar{F})) \simeq \nu_M(Sp(\bar{F}))$, $\bar{F} \varepsilon D_b^c(M;\, \pi_*(E))$. This follows readily from Chapter IV, Proof of Proposition 2.12 and the discussion following it, where it is shown that all the functors involved in the definition of the functors involved in ν_M commute with the functor Sp.

2.3. Micro-localization. Assume the situation of 1.1 through 1.3. If $F \varepsilon D^{c,f.t.d}(X;\, E)$, we define *the micro-localization* of F along M to be

$$\mu_M(F) = (\widehat{\nu_M(F)}).$$

In this context one obtains a strongly convergent spectral sequence:

$$E_2^{s,t} = \mathcal{H}^s(\mu_M(\pi_{-t}(F)\check{\ }\ )) \Rightarrow \pi_{-s-t}(\mu_M(F))$$

using which one may recover all the usual properties (see [K-S-1] chapter 2) of micro-localization. (Observe once again that the E_2-terms are now the micro-localizations of the abelian sheaf $\pi_{-t}(F)\tilde{}$.) For example let $f : Y \rightarrow X$ denote a closed imbedding of manifolds as above. Now one obtains a natural quasi-isomorphism

$$f^!(F) \overset{\simeq}{\rightarrow} R\tau_!(\nu_Y(F)) \simeq R\pi_*(\mu_Y(F)) \tag{2.3.1}$$

where the first quasi-isomorphism is from (2.S.4) while the second one is from (2.F.3) with $q_1 : \mathcal{E} \rightarrow Z$ ($q_2 : \mathcal{E}^* \rightarrow Z$) being $\tau : T_M X \rightarrow M$ ($\pi : T^*_M X \rightarrow M$, respectively). Using this one may readily obtain the following result as well. Let $f : Y \rightarrow X$ denote a map between manifolds. Let q_j, $j = 1, 2$ denote the $j - th$ projection on $X \times Y$ and let Δ denote the graph of f in $X \times Y$. Let $\pi : T^*_\Delta(X \times Y) \rightarrow \Delta \cong Y$ denote the obvious projection. (Here $T^*_\Delta(X \times Y)$ is the conormal bundle associated to the imbedding $\delta : \Delta \rightarrow X \times Y$.) Let $F \varepsilon D^{c,f.t.d}(X;\, E)$ and let $G \varepsilon D^{c,f.t.d}(Y;\, E)$. Then one obtains natural quasi-isomorphisms:

$$\begin{aligned} R\pi_*\mu_\Delta R\mathcal{H}om_E(q_2^{-1}(G), q_1^!(F)) &\simeq \delta^!(R\mathcal{H}om_E(q_2^{-1}(G), q_1^!(F))) \\ &\simeq R\mathcal{H}om_E(\delta^{-1} \circ q_2^{-1}G, \delta^! \circ q_1^!(F)) \simeq R\mathcal{H}om_E(G, Rf^!F) \end{aligned} \tag{2.3.2}$$

where the first quasi-isomorphism follows from 2.3.1 with F (f) there replaced by $R\mathcal{H}om_E(q_2^{-1}(G), q_1^!(F))$ (δ, respectively) and the second quasi-isomorphism follows from Chapter IV, Proposition 5.1(ii). Moreover, Chapter IV (5.6.6), provides the quasi-isomorphism:

$$R\mathcal{H}om_E(q_2^{-1}(G), q_1^!(F)) \simeq D_X(G) \boxtimes F. \tag{2.3.3}$$

Assume the situation of (1.1); let $F \varepsilon D^{c,b}(X;\, E)$ and let $\delta : X \rightarrow X \times X$ denote the obvious diagonal imbedding. Now one obtains a natural map (observe that δ is a closed imbedding) making use of (2.F.4) and (2.S.5):

$$\mu_\Delta(\delta_! F) = (\nu_\Delta\widehat{(\delta_!(F)}))) \rightarrow (\widehat{R\tau^!(F)})) \simeq \pi^{-1}(F) \tag{2.3.4}$$

Let $F \varepsilon D^{c,f.t.d}(X;\, E)$ so that there are only finitely many distinct $\pi_i(F)$. (i.e. either they are nontrivial in all but finitely many degrees or they are periodic. For example if F is a constructible presheaf of KU-module spectra, the sheaf of homotopy groups of F are Bott-periodic of period 2.) If $i_1, \ldots\ i_n$ are these distinct values, one may define *the micro-support* of F to be the smallest closed conic subspace of T^*X containing the micro-supports of $SS(\pi_{i_j}(F)\tilde{})$, $j = 1, \ldots, n$.

3. The Trace map and the Euler-class

We will assume the basic situation in 1.1 through 1.3. In this section we will define a trace-map

$$Tr^F : R\Gamma(X, \mathcal{RH}om_E(F, F)) \rightarrow \mathbb{H}(X; D_X^E)), F \varepsilon D^{c,f.t.d}(X; E) \tag{3.0.5}$$

where it is assumed that there are only finitely many *distinct* nontrivial $\pi_i(F)$.

If X is a smooth variety over the complex numbers or the reals and $F \varepsilon D^{c,b}(X;\, E)$ has only finitely many *distinct* $\pi_i(F)$ (so that its micro-support may be defined), we will also define a micro-local trace-map

$$Tr^F_\mu : R\Gamma(X, \mathcal{RH}om_E(F, F)) \rightarrow \mathbb{H}(\Lambda_F; D_X^E) \tag{3.0.6}$$

where Λ_F is the micro-support of F.

Let $F \varepsilon D^{c,f.t.d}(X;\, E)$ be globally of finite tor dimension. Now one observes the natural quasi-isomorphism:

$\mathcal{RH}om_E(F,\, F) \simeq D_X^E(F) \boxtimes F$. (Take $X_1 = X_2 = S$ in chapter IV (5.6.5) and $\mathcal{S} = E$.)

Therefore one obtains a natural map $\epsilon : D_E^X(F) \boxtimes F \to D_E^X$. This is simply the *evaluation map.* Taking hypercohomology we obtain the *trace-map*

$$Tr^F : R\Gamma(X, \mathcal{RH}om_E(F, F)) \to \mathbb{H}(X; D_E^X) \tag{3.0.7}$$

(Observe that Tr^F (above) may also be viewed as the composition of the following maps:

$$\begin{aligned} &R\Gamma(X, \mathcal{RH}om_E(F \underset{E}{\otimes} E, F \underset{E}{\otimes} E)) \overset{\simeq}{\to} R\Gamma(X, \mathcal{RH}om_E(E, RHom_E(F, F \underset{E}{\otimes} E))) \\ &\overset{\simeq}{\leftarrow} R\Gamma(X, \mathcal{RH}om_E(E, F \underset{E}{\otimes} D_E(F))) \overset{\epsilon}{\to} R\Gamma(X, \mathcal{RH}om_E(E, D_E^X)) \end{aligned} \tag{3.0.8}$$

One may compare this with the definition of the trace-map adopted in [Ill].)

Next assume that $F \varepsilon D^{c,f.t.d}(X;\, E)$ has only finitely many distinct $\pi_i(F)\check{}$ so that the micro-support Λ_F of F is defined as a conic subset of T^*X. Now we observe the quasi-isomorphisms:

$$\mathcal{RH}om_E((F,\, F) \overset{\simeq}{\to} R\pi_* R\Gamma_{\Lambda_F} \mu_\Delta \mathcal{RH}om_E(q_2^{-1}F,\, q_1^! F) \overset{\simeq}{\to} R\pi_* R\Gamma_{\Lambda_F} \mu_\Delta (D_E^X(F) \boxtimes F)$$

Clearly there is a natural map $D_E^X(F) \boxtimes F \to \delta_* \delta^*(D_E^X(F) \boxtimes F) \to \delta_*(D_E^X(F) \otimes F) \to \delta_*(D_E^X)$ where the last map is ϵ. Now (2.3.4) provides a natural map :

$$R\pi_* R\Gamma_{\Lambda_F} \mu_\Delta(\delta_*(D_E^X)) \to R\pi_* R\Gamma_{\Lambda_F}(\pi^{-1}(D_E^X)).$$

On taking the hypercohomology spectrum on X, therefore one obtains a map:

$$R\Gamma(X, \mathcal{RH}om_E(F, F)) \to \mathbb{H}(X;\, R\pi_* R\Gamma_{\Lambda_F}(\pi^{-1} D_E^X))) \simeq \mathbb{H}(\Lambda_F;\, D_E)$$

3.1. The composition of the above maps will be called the *micro-local trace* and will be denoted Tr_μ^F. (One may compare this with [K-S-2] p. 377.) We proceed to establish the main properties of these trace maps.

(3.Tr.1) On taking the homotopy groups, the trace-map induces an additive homomorphism $Tr^F : \pi_n(RHom_E(F,\, F)) \to \pi_n(\mathbb{H}(X;\, D_E^X))$ for each n. Similarly the micro-local trace induces an additive homomorphism $Tr_\mu : \pi_n(R\Gamma(X, \mathcal{RH}om_E(F, F))) \to \pi_n(\mathbb{H}(\Lambda_F;$ $D_E)) \to \pi_n(\mathbb{H}(T^*X;\, D_E))$.

(3.Tr.2) $Gr(Tr^F) \simeq Tr^{Gr(F)}$ and similarly $Gr(Tr_\mu^F) \simeq Tr_\mu^{Gr(F)}$, where Gr denotes the associated graded terms with respect to the Cartan filtration. If $\bar{M}^{\cdot} \varepsilon D^{c,b}(X;\, \pi_*(E))$, one also obtains natural quasi-isomorphisms: $Tr^{Sp(\bar{M}^{\cdot})} \simeq Sp(Tr^{\bar{M}^{\cdot}})$ and $Tr_\mu^{Sp(\bar{M}^{\cdot})} \simeq Sp(Tr_\mu^{\bar{M}^{\cdot}})$. For the trace-map, this follows readily from Chapter III, Propositions 2.7, 2.10(ii), 2.13 and Chapter IV, (4.2.1). For the micro-local trace this follows from the same and (2.F.5) along-with (2.S.6).

3.2. In order to establish *the additivity* of the trace-map, it is convenient to consider *the filtered category* $Presh^{fil,f.t.d}(\mathcal{C}_X; E)$. (See [Ill] chapitre V .) The objects of this category are presheaves $F \varepsilon Presh^{f.t.d}(\mathcal{C}_X; E)$ provided with a *finite increasing* filtration by sub-objects in the same category and indexed by the integers so that there exist integers m and M such that $F_i = *$ if $i < m$ and $F_i = F_M$ for all $i > M$. i.e. One obtains

$$* \subseteq F_m \subseteq F_{i+1} \subseteq .... \subseteq F_M = F$$

We also require that each map $\Gamma(U, F_i) \to \Gamma(U, F_{i+1})$ is a stable cofibration for each $U \varepsilon \mathcal{C}_X$ and so is $(F_i)_{\bar{x}} \to (F_{i+1})_{\bar{x}}$ for each point $\bar{x}$ in the site $\mathcal{C}_X$. The corresponding derived category will be denoted $D^{fil,f.t.d}(\mathcal{C}_X;\, E)$.

3.3. One may also define *the graded category* $Presh_E^{gr,f.t.d}(\mathcal{C}_X)$. The objects of this category are presheaves $F \varepsilon Presh_E^{f.t.d}(\mathcal{C}_X)$ provided with a grading by the integers so that all but finitely many terms are trivial. (i.e. There exist $F_m,\ldots,\ F_M\ \varepsilon Presh_E(\mathcal{C}_X)$ along with an isomorphism $F \cong \bigvee_{m \le i \le M} F_i$. The corresponding derived category will be denoted $D_{gr,f.t.d}(\mathcal{C}_X;\, E)$

3.4. Given $F,\, F' \varepsilon Presh_E^{fil,f.t.d}(\mathcal{C}_X)$ one puts the obvious filtration on $RHom_E(F,\, F')$. i.e. $RHom_E(F,\, F')_n = \{f \varepsilon RHom_E(F, F' | f(F_i) \subseteq F'_{i+n}\}$. Given $F \varepsilon Presh_E^{fil,f.t.d}(\mathcal{C}_X)$ and $F' \varepsilon Presh_E^{fil,f.t.d}(\mathcal{C}_X)$, we let $F \underset{E}{\otimes} F'$ be filtered by $(F \underset{E}{\otimes} F')_n =$ the image of $\bigvee_{i+j=n} F_i \underset{E}{\otimes} F'_j \to F \underset{E}{\otimes} F'$

3.5. Let $F = \{* \subseteq F_0 \subseteq F \ldots \subseteq F_p = F\}$ be an object in $Presh_E^{fil,f.t.d}(\mathcal{C}_X)$. Since the filtration is finite, it is automatically exhaustive and complete. Moreover, by re-indexing the filtration, one may also assume it is decreasing.) Therefore, Chapter III, Proposition 2.7 (where E is provided with the trivial filtration) provides a natural quasi-isomorphism:

$$gr(R\Gamma(X, \mathcal{RH}om_E(F, F))) \simeq R\Gamma(X, \mathcal{RH}om_E(gr(F),\, gr(F)))$$

Here gr denotes taking the associated graded terms with respect to the given filtration.

3.6. Let $Gr(E)$ denote the associated graded spectrum obtained from E using the Cartan filtration. Observe that an object $F \varepsilon Presh_E^{fil,f.t.d}(\mathcal{C}_X)$ consists of a filtered object so that the filtration as above is compatible with another decreasing filtration $\{F^n | n\}$ so that

(i) $Gr(F) = \{F^n / F^{n+1} | n\} \varepsilon Presh_{Gr(E)}^{fil}(\mathcal{C}_X)$ and

(ii) there exists an object $\tilde{P}_0^{\cdot} \to \ldots \to \tilde{P}_p^{\cdot}\ \varepsilon D_{\pi_*(E),r}^{fil}(\mathcal{C}_X)$ which is globally of finite tor dimension so that one obtains a homotopy commutative diagram:

$$\begin{array}{ccccccc}
Gr(F_0) & \longrightarrow & Gr(F_1) & \longrightarrow \cdots\cdots \longrightarrow & Gr(F_p) \simeq Gr(F) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
Sp(\tilde{P}_0^{\cdot}) & \longrightarrow & SP(\tilde{P}_1^{\cdot}) & \longrightarrow \cdots\cdots \longrightarrow & Sp(\tilde{P}_p^{\cdot}) \simeq Sp(\tilde{P}^{\cdot})
\end{array} \quad)$$

3.7. An object $F \varepsilon Presh_{E,r}^{gr,f.t.d}(\mathcal{C}_X)$ is a graded object $F = \bigvee_i F_i \varepsilon Presh_{E,r}^{gr}(\mathcal{C}_X)$ so that the gradation above is compatible with a decreasing filtration $\{F^n | n\}$ so that

(i) $Gr(F) = \{F^n / F^{n+1} | n\} \varepsilon Presh_{Gr(E)}^{gr}(\mathcal{C}_X)$ and

(ii) there exists an object $\tilde{P}^{\cdot} = \underset{0 \le i \le p}{\oplus} \tilde{P}_i^{\cdot}\ \varepsilon D_{\pi_*(E),r}^{gr}(\mathcal{C}_X)$ which is globally of finite tor dimension so that one has a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_i Sp(\tilde{P}_i) & \xrightarrow{\simeq} & Sp(\tilde{P}.) \\ \simeq\downarrow & & \simeq\downarrow \\ \bigvee_i Gr(F_i) & \xrightarrow{\simeq} & Gr(F) \end{array}$$

)

Notation. Let $F\varepsilon Presh(\mathcal{C}_X;E)$. $Gr(F)$ will denote the associated graded object obtained from a decreasing filtration on F compatible with the Cartan filtration on E. If $F\varepsilon Presh^{fil}(\mathcal{C}_X)$, $gr(F)$ will denote the associated graded object with respect to the given filtration on F. This belongs to $Presh^{gr}_E(\mathcal{C}_X)$.

Observe that the same definition of the trace Tr^F applies to the filtered and graded cases. Observe (from 3.4) that the dualizing presheaf D^X_E has only the trivial filtration (with D^X_E in degree 0) and that

$$(D_X(F)\boxtimes F)_0 = D_X(U(F))\boxtimes U(F)$$

where F is a filtered object as in 3.5, $U(F)\varepsilon Presh_{E,r}(\mathcal{C}_X)$ is the object obtained by forgetting the filtration and $D_X(F)\boxtimes F$ is provided with the obvious induced filtration. It follows that

$$\pi_*(Tr^F(f)) = \pi_*(Tr^{U(F)}(U(f))), f\varepsilon R\Gamma(X,\mathcal{RH}om_E(F,F)) \tag{3.7.1}$$

3.8. Next assume $F\cong \bigvee_i F_i\varepsilon Presh^{gr,f.t.d}_{E,r}(\mathcal{C}_X)$. Observe that now an $f\varepsilon R\Gamma(X,\mathcal{RH}om_E(F,F))$ of grade 0 is given by a collection $\{f_i\varepsilon RHom_E(F_i,\ F_i)|i\}$. Now one may readily show that

$$\pi_*(Tr^F(f)) = \bigoplus_i \pi_*(Tr^{F_i}(f_i))$$

Let $F\varepsilon Presh^{fil,f.t.d}_{E,r}(\mathcal{C}_X)$ denote a filtered object as in 3.5. In view of the quasi-isomorphism there, the functor gr induces a map

$\pi_*(R\Gamma(X,\mathcal{RH}om_E(F,F))) \to \pi_*(gr(R\Gamma(X,\mathcal{RH}om_E(F,F)))) \cong \pi_*(R\Gamma(X,\mathcal{RH}om_E(gr(F), gr(F))))$.

Now one obtains the commutative square

$$\begin{array}{ccc} R\Gamma(X,\mathcal{RH}om_E(F,F))_0 & \xrightarrow{Tr^F} & R\Gamma(X,\mathcal{RH}om_E(E,D^X_E))_0 \\ gr\downarrow & & \downarrow gr=id \\ R\Gamma(X,\mathcal{RH}om_E(gr(F),gr(F)))_0 & \xrightarrow{Tr^{gr(F)}} & R\Gamma(X,\mathcal{RH}om_E(E,D^X_E))_0 \end{array}$$

by taking the associated graded terms of degree 0. (Recall that $\mathcal{RH}om_E(E,D^X_E)$ is provided with the trivial filtration.) Therefore, one obtains the commutative square:

$$\begin{array}{ccc} \pi_*(R\Gamma(X,\mathcal{RH}om_E(F,F))) & \xrightarrow{\pi_*(Tr^F)} & \pi_*(R\Gamma(X,\mathcal{RH}om_E(E,D^X_E))) \\ \downarrow gr & & \downarrow gr \\ \pi_*(RHom_E(gr(F),gr(F))) & \xrightarrow{\pi_*(Tr^{gr(F)})} & \pi_*(R\Gamma(X,\mathcal{RH}om_E(E;D^X_E))) \end{array} \tag{3.8.1}$$

Since $\pi_*(Tr^F(f)) = \pi_*(Tr^{U(F)}(f))$ (see (3.7.1), 3.8) the above square shows that, if $f \varepsilon R\Gamma(X, \mathcal{RH}om_E(F,F))$ is of degree 0:

$$\pi_*(Tr^{U(F)}(U(f))) = \sum_i \pi_*(Tr^{gr_i(F)}(gr_i(f))) \tag{3.8.2}$$

PROPOSITION 3.1. *Let*

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "f'"] & F \arrow[r] \arrow[d, "f"] & F'' \arrow[d, "f''"] \\ F' \arrow[r] & F \arrow[r] & F'' \end{tikzcd} \tag{3.8.3}$$

denote a commutative diagram in $Presh_E^{f.t.d}(\mathcal{C}_X)$ *so that the two rows are cofibration sequences. Then*

$Tr^F(f) = Tr^{F'}(f') + Tr^{F''}(f'')$ *as classes in* $\pi_*(\mathbb{H}(X;\, D_E))$

If X *is a smooth variety over the real or complex numbers and* F, F' *and* F'' *all have only finitely many distinct sheaves of homotopy groups one also obtains:*

$Tr_\mu^F(f) = Tr_\mu^{F'}(f') + Tr_\mu^{F''}(f'')$ *as classes in* $\pi_*(\mathbb{H}(\Lambda;\, D_E))$

where Λ *is the smallest conic subspace of* T^*X *containing the micro-supports of all the sheaves* $\pi_i(F')\tilde{}$, $\pi_i(F)\tilde{}$ *and* $\pi_i(F'')\tilde{}$ *for all* i.

PROOF. We will only prove this for the trace-map since the proof for the micro-local trace will be similar. It suffices to interpret the diagram in (3.8.3) as a map f of filtered objects: we let F be filtered by $F_0 = F'$ and $F_1 = F$. We proceed to verify that there is an object $\bar{P}^\cdot \varepsilon D^{fil,c}_{\pi_*(E),r}(\mathcal{C}_X)$ filtered by $\bar{P}^\cdot_0 \subseteq \bar{P}^\cdot_1 = \bar{P}^\cdot$ so that we obtain a homotopy commutative diagram:

$$\begin{tikzcd} Gr(F_0) \arrow[r] \arrow[d, "\simeq"] & Gr(F_1) = Gr(F) \arrow[d, "\simeq"] \\ Sp(\bar{P}^\cdot_0) \arrow[r] & Sp(\bar{P}^\cdot_1) = Sp(\bar{P}^\cdot) \end{tikzcd} \tag{3.8.4}$$

Let $\bar{F}'^\cdot$, $\bar{F}^\cdot$ and $\bar{F}''^\cdot$ $\varepsilon D^c_r(\mathcal{C}_X;\pi_*(E))$ be globally of finite tor dimension so that $Sp(\bar{F}'^\cdot) \simeq Gr(F')$, $Sp(\bar{F}^\cdot) \simeq Gr(F)$ and $Sp(\bar{F}''^\cdot) \simeq Gr(F'')$. On taking the homotopy groups, the commutative diagram (3.8.3) provides the commutative diagram:

$$\begin{tikzcd} \bar{F}'^\cdot \arrow[r, "\bar{\alpha}"'] \arrow[d, "\bar{f}'"'] & \bar{F}^\cdot \arrow[r] \arrow[d, "\bar{f}"'] & \bar{F}''^\cdot \arrow[d, "\bar{f}''"'] \\ \bar{F}'^\cdot \arrow[r, "\bar{\alpha}"'] & \bar{F}^\cdot \arrow[r] & \bar{F}''^\cdot \end{tikzcd}$$

in the category of complexes of sheaves of graded modules over $\pi_*(E)$. Let $\tilde{P}^\cdot = Cyl(\tilde{\alpha})$ denote the mapping cylinder of the map α of complexes. There is a monomorphism, $\bar{F}'^\cdot \to Cyl(\alpha)$ which is split degree-wise. Therefore we may define a filtration on $Cyl(\alpha)$ by $Cyl(\alpha)_0 = \bar{F}'^\cdot$, $Cyl(\alpha)_i = *$ if $i < 0$ and $Cyl(\alpha)_1 = Cyl(\alpha)$. It follows that, with the above filtration, $\bar{P}^\cdot \varepsilon D^{fil,c}(\mathcal{C}_X;\pi_*(E))$ is globally of finite tor dimension and provides a diagram as in 3.8.4. The hypotheses guarantee that we now obtain a map f in $Presh^{fil,f.t.d}(\mathcal{C}_X;E)$. This completes the proof of the above proposition. $\square$

3.9. Assume the above situation. We define the *Euler class* of F (denoted *eu(F)*) to be $Tr^F(id_F)\varepsilon\pi_0(\mathbb{H}(X;\, D_E))$

3.10. Next assume the situation of 3.1. Now we will define the *micro-local Euler class* of F (denoted $eu_\mu(F)$) to be the image of $Tr^F_\mu(id_F)\varepsilon\pi_0(\mathbb{H}(\Lambda_F;\, D_E))$ in $\pi_0(\mathbb{H}(T^*X;\, D_E))$.

PROPOSITION 3.2. *Assume the hypotheses of (3.Tr.3). Then* $eu(F) = eu(F') + eu(F'')$ *and* $eu_\mu(F) = eu_\mu(F') + eu_\mu(F'')$.

PROOF. This is clear from (3.8.3) and (3.0.7), 3.1. □

Let $i : Y \to X$ denote the closed immersion of a smooth sub-variety into a smooth variety, both being defined over an algebraically closed field k of characteristic $p \geq 0$. Let $E = KU$ if $p = 0$ and $\widehat{KU}_l$, $l \neq p$, $\nu >> 0$ if $p > 0$. Let $F = i_*i^*(\underline{E})$, $\underline{E}$ being the constant sheaf on $\mathcal{C}_X$. Consider the following quasi-isomorphisms:

$$\mathbb{H}(X,\, \mathcal{RH}om_E(i_*i^*(\underline{E}),\, i_*i^*(\underline{E})))\overset{\simeq}{\to}\mathbb{H}(X,\, D_E(i_*i^*(\underline{E}))\underset{E}{\otimes}i_*i^*(\underline{E}))$$

$$\overset{\simeq}{\to}\mathbb{H}(X,\, i_*Ri^!(D^X_E)\underset{E}{\otimes}i_*i^*(\underline{E}))\overset{\simeq}{\to}\mathbb{H}(X,\, i_*Ri^!D^X_E)\overset{\simeq}{\to}\mathbb{H}(Y,\, D^Y_E)\to\mathbb{H}(X;\, D^X_E)$$

(The last term is the presheaf-hypercohomology of X with respect to E.) The trace-map sends id_F to the image of the fundamental class of Y in $H_0(X;\, E)$. By Poincaré-Lefschetz-duality this class identifies with the cycle class $cl(Y)\varepsilon\mathcal{H}^0(\mathbb{H}(X;\, E))$. Now observe that the cycle class $cl(Y)$ = the Euler-class of the normal-bundle to the imbedding of Y in X. This justifies calling the classes in 3.9 Euler-classes.

4. Passage from constructible sheaves of $\mathbb{Z}$-modules to constructible presheaves of KU-module spectra

In this section we will show how to functorially associate to any constructible sheaf of $\mathbb{Z}$-modules ($\widehat{\mathbb{Z}}_l$-modules) on a suitable space a constructible presheaf of KU-module spectra ($\widehat{KU}_l$-module spectra, respectivelyif $\nu >> 0$).

4.1. Let X denote a space as before and let $\mathcal{C}_X$ denote its associated site. If X is a real or complex variety with $\mathcal{C}_X$ its usual site, we consider the ring spectrum KU (the ring $\mathbb{Z}$, respectively). If X is a scheme of finite type over a field k with characteristic $p > 0$ and l is a prime number $\neq p$, $\nu >> 0$, we will instead consider the ring spectrum $\widehat{KU}_l$ (the ring $\widehat{\mathbb{Z}}_l$, respectively). $\underline{KU}$ ($\underline{\mathbb{Z}}$, $\underline{\widehat{KU}_l}$, $\underline{\widehat{\mathbb{Z}}_l}$) will denote the obvious constant sheaves. One key observation is that $\pi_i(KU) \cong \mathbb{Z}$ ($\pi_i(\widehat{KU}_l) \cong \widehat{\mathbb{Z}}_l$) if i is *even* and trivial otherwise.

We will first consider the case where X is a real or complex variety. Let $\bar{F}$ denote a constructible sheaf of $\mathbb{Z}$-modules on $\mathcal{C}_X$. Let $\mathcal{R}(\bar{F}) \to \bar{F}$ denote a resolution by a chain complex of *flat* sheaves of $\mathbb{Z}$-modules. Let Σ denote the sphere spectrum and let $\underline{\Sigma}$ denote the associated constant sheaf: now form $\pi_*(\Sigma)\underset{\mathbb{Z}}{\otimes}\mathcal{R}(\bar{F}) = \pi_*(\Sigma)\overset{L}{\underset{\mathbb{Z}}{\otimes}}\bar{F}$. Apply the functor GEM from Chapter I, section 1 (ST4) (see also Chapter I, section 4, Proposition 4.4 to this object in each degree to obtain a chain-complex of presheaves of generalized Eilenberg-Maclane spectra. Next we de-normalize this to obtain the corresponding simplicial object of presheaves of Eilenberg-Maclane spectra. Now we consider:

$$K_\bullet(\bar{F}) = \underline{KU}\overset{L}{\underset{\Sigma}{\otimes}}DN(GEM(\pi_*(\Sigma)\overset{L}{\underset{\mathbb{Z}}{\otimes}}(\bar{F}))) \tag{4.1.1}$$

This is a simplicial object of presheaves of KU-module spectra. Next one takes its homotopy colimit to obtain a presheaf of spectra which will be denoted

$$K(\bar{F}) \tag{4.1.2}$$

By taking a fixed flat resolution of $\pi_*(\Sigma)$, one may observe that the functor $\bar{F} \mapsto K(\bar{F})$ is an *exact* functor in the following sense:

4.2. if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of sheaves of $\mathbb{Z}$-modules on the site $\mathcal{C}_X$, the corresponding diagram $K(F') \to K(F) \to K(F'')$ is a distinguished triangle in $Presh(\mathcal{C}_X; KU)$.

PROPOSITION 4.1. *Assume the above situation. Now $\pi_i(K(F))\tilde{\ } \cong \bar{F}$ if i is an even integer and trivial otherwise.*

PROOF. We compute the homotopy groups of each term, $K_n(\bar{F})$, of the simplicial object using the spectral sequence in Chapter III, Proposition 1.2. Here

$$E_2^{s,t} = Tor_{s,t}^{\pi_*(\Sigma)}(\pi_*(KU), \pi_*([DN(GEM(\pi_*(\Sigma)))\underset{\mathbb{Z}}{\overset{L}{\otimes}}(\bar{F})]_n))$$

$$\Rightarrow \pi_{s+t}(K_n(\bar{F}))$$

One may identify $\pi_*(DN(GEM(\pi_*(\Sigma))\underset{\mathbb{Z}}{\overset{L}{\otimes}}(\bar{F})))$ with $DN(\pi_*(\Sigma)\underset{\mathbb{Z}}{\overset{L}{\otimes}}(\bar{F}))$. The latter is a flat module over $\pi_*(\Sigma)$ in each simplicial degree. Therefore

$E^2_{s,t} = 0$ if $s > 0$ and

$E^2_{o,t} = (\pi_*(KU)\underset{\mathbb{Z}}{\otimes}\mathcal{R}_n(\bar{F})))_t \cong \mathcal{R}_n(\bar{F})$ if t is even and trivial otherwise.

(Recall that $\pi_t(KU) = \mathbb{Z}$ if t is even and trivial otherwise.) It follows that $\pi_t(K_n(\bar{F})) \cong \mathcal{R}_n(\bar{F})$ if t is even and $\cong 0$ otherwise. Therefore, when we compute the homotopy groups of $\pi_*(K(\bar{F}))$ using the spectral sequence for the homotopy colimit of a simplicial object as in Chapter I, section 1, (HCl), we obtain the isomorphism as stated in the proposition. $\square$

4.3. In positive characteristic p, we will need to modify the definition of $K_\bullet(\bar{F})$ as follows. We replace $\mathbb{Z}$ (KU) everywhere by its l-adic completion $\widehat{\mathbb{Z}}_l$ (the l-completion $\widehat{KU}_l$, respectively). One also needs to replace Σ by its l-completion $\widehat{\Sigma}_l$; $\mathcal{R}_*(\bar{F}) \to \bar{F}$ will be a resolution by a complex of sheaves of flat $\widehat{\mathbb{Z}}_l$-modules. Then the same computations show that $\pi_i(K(\bar{F})) \cong \bar{F}$ if i is even and $\cong 0$ if i is odd.

In case $F = \{F_\nu|\nu\}$ is an inverse system of sheaves of l-adic sheaves one applies the functor K to each term of the inverse system to obtain the inverse system $\{K(F_\nu)|\nu\}$. Now one takes the homotopy inverse limit of the $\{K(F_\nu)|\nu\}$ to obtain $K(F)$.

PROPOSITION 4.2. *In characteristic* 0, *the assignment $\bar{F} \to K(\bar{F})$ sends short exact sequences of sheaves of $\mathbb{Z}$-modules to fibration sequences of presheaves of spectra. In positive characteristics, the corresponding statement also holds for l-adic sheaves.*

PROOF. This should be clear from the definition of the functor $\bar{F} \to K(\bar{F})$. $\square$

DEFINITION 4.3. If $\bar{F}$ is a constructible sheaf as above, we define $Eu(\bar{F})\varepsilon K_0(X)$ ($\widehat{K_0(X)_l}$) as $eu(K(\bar{F}))$. If X is a smooth complex variety, and $\bar{F}$ is a $\mathbb{Z}$-constructible sheaf, we let $Eu_\mu(\bar{F}) = eu_\mu(K(\bar{F}))$.

5. Relations with the Euler class in homology with locally compact supports

In this section we will relate the above Euler classes to the ones taking values in homology with locally compact supports. *Throughout this section we will restrict to complex projective varieties.* We also let KU denote the ring spectrum representing complex K-theory and let $Gr(KU)$ denote the associated graded object defined with respect to the Cartan filtration; now $Gr(KU) = \bigvee_i Gr_i(KU)$ and $Gr_0(KU)$ is also a ring spectrum and the obvious map $Gr_0(KU) \to Gr(KU)$ is a map of ring spectra.

PROPOSITION 5.1. $D_{Gr(KU)} \simeq D_{Gr_0(KU)} \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU})$

PROOF. We begin with the following observations:

5.0.1. If $f : X \to Y$ is a map of spaces, then $Rf_!(Gr(\underline{K}U)) \simeq Rf_!(Gr_0(\underline{K}U)) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU})$. This follows from the projection formula in Chapter IV, Proposition 2.17.

5.0.2. If $K \varepsilon Presh_{KU}(X)$ so that $Gr(K) = Gr_0(K) \underset{Gr_0(KU)}{\otimes} Gr(\underline{KU})$, then

$$\mathcal{RH}om_{Gr(KU)}(Gr(\underline{KU}), Gr(K)) \simeq \mathcal{RH}om_{Gr_0(KU)}(Gr_0(\underline{KU}), Gr_0(K)) \underset{Gr_0(KU)}{\otimes} Gr(\underline{KU}).$$

This follows from Chapter II, (2.0.11).

Now

$$\mathcal{RH}om_{Gr(KU)}(Rf_!(Gr(\underline{KU})), Gr(K))$$

$$\simeq \mathcal{RH}om_{Gr(KU)}(Rf_!(Gr_0(\underline{KU})) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU}), Gr(K))$$

$$\simeq \mathcal{RH}om_{Gr_0(KU)}(Rf_!(Gr_0(\underline{KU})), Gr(K))$$

The first $\simeq$ follows from the observation 5.0.1 above, while the second $\simeq$ follows from Chapter II, (2.0.11). In view of the hypothesis on K, one may identify the last term with

$$\mathcal{RH}om_{Gr_0(KU)}(Rf_!(Gr_0(\underline{KU})), Gr_0(K) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU})).$$

By replacing $Rf_!(Gr_0(\underline{KU}))$ with a resolution as in Chapter II, Proposition 2.4 and making use of (5.1.2), we may now identify the latter with

$$\mathcal{RH}om_{Gr_0(KU)}(Rf_!(Gr_0(\underline{KU})), Gr_0(K)) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU}).$$

Finally this identifies with

$$\mathcal{RH}om_{Gr_0(KU)}(Gr_0(\underline{KU}), Rf^!(Gr_0(K))) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU}) \simeq Rf^!(Gr_0(K)) \underset{Gr_0(\underline{KU})}{\otimes} Gr(\underline{KU}).$$

$\square$

It follows by applying the projection formula (see Chapter IV (2.17)) to the structure map $p : X \to pt$, (when p is proper) that

$$(5.0.3) \qquad \begin{gathered} \mathbb{H}(X; D^X_{Gr(KU)}) \simeq \mathbb{H}(X; D^X_{Gr_0(KU)}) \overset{L}{\underset{Gr_0(KU)}{\otimes}} Gr(KU) \quad \textit{and} \\ \pi_i(\mathbb{H}(X; D^X_{Gr(KU)})) \cong \bigoplus_n H_{2n-i}(X; \pi_0(KU)), \quad i = 0 \quad \textit{or} \quad i = 1 \end{gathered}$$

which is the sum of all the integral homology groups (with locally compact supports) of X.

Let $\bar{F}$ denote a constructible sheaf of $\mathbb{Z}$-modules on X and let $F = K(\bar{F})$ denote the presheaf of KU-module-spectra defined as in 4.2. Recall from Proposition 4.1 that $\pi_i(F)\tilde{} = \bar{F}$ for all i *even* and trivial otherwise. Let $Gr(F) = \bigvee_i Gr(F)_i \varepsilon Presh^c(\mathcal{C}_X; Gr(KU))$. Then $Gr(F) \simeq Gr_0(F) \underset{Gr_0(KU)}{\otimes} Gr(KU)$. Now one obtains the natural quasi-isomorphisms:

(5.0.4)

$$\begin{aligned} R\Gamma(X, \mathcal{RH}om_{Gr(KU)}(Gr(F), Gr(F))) &\simeq \mathbb{H}(X; \mathcal{RH}om_{Gr_0(KU)}(Gr_0(F), Gr_0(F)) \overset{L}{\underset{Gr_0(\underline{KU})}{\otimes}} Gr(\underline{KU})) \\ &\simeq \mathbb{H}(X; (D^X_{Gr_0(KU)}(Gr_0(F)) \boxtimes Gr_0(F)) \overset{L}{\underset{Gr_0(\underline{KU})}{\otimes}} Gr(\underline{KU})) \\ &\simeq \mathbb{H}(X; (D^X_{Gr_0(KU)}(Gr_0(F)) \boxtimes Gr_0(F))) \overset{L}{\underset{Gr_0(KU)}{\otimes}} Gr(KU)) \\ &\simeq R\Gamma(X, \mathcal{RH}om_{Gr_0(KU)}(Gr_0(F), Gr_0(F))) \overset{L}{\underset{Gr_0(KU)}{\otimes}} Gr(KU) \end{aligned}$$

The last-but-one quasi-isomorphism follows from the projection formula in Chapter IV, (2.17) applied to the obvious map $p : X \to pt$.

Moreover the spectral sequences in Chapter III, Proposition 1.2 applied to the above tensor-products degenerate identifying

$$\begin{aligned} &\pi_i(R\Gamma(X, \mathcal{RH}om_{Gr(KU)}(Gr(F), Gr(F))))\tilde{} \\ \cong\ &\pi_i(R\Gamma(X, \mathcal{RH}om_{Gr(KU)_0}(Gr_0(F), Gr_0(F))) \underset{Gr_0(KU)}{\otimes} Gr(KU))\tilde{} \qquad (5.0.5) \\ \cong\ &\bigoplus_n \pi_{2n-i}(R\Gamma(X, \mathcal{RH}om_{Gr(KU)_0}(Gr_0(F), Gr_0(F))))\tilde{} \quad \textit{and} \end{aligned}$$

$$\pi_i(\mathbb{H}(X; D^X_{Gr(KU)})) \cong \bigoplus_n \pi_{2n-i}(\mathbb{H}(X; D^X_{Gr_0(KU)}))\tilde{} \qquad (5.0.6)$$

Moreover $\mathcal{RH}om_{Gr_0(KU)}(Gr_0(F),\ Gr_0(F)) \simeq Sp(RHom_{\mathbb{Z}}(\bar{F},\ \bar{F}))$ according to Chapter III, Proposition 2.13 The spectral sequence in Chapter III, Theorem 2.18(ii) degenerates and provides the identifications:

$\pi_*(R\Gamma(X, RHom_{Gr_0(KU)}(Gr_0(F), Gr_0(F))))\tilde{} \cong R\Gamma(X, \mathcal{RH}om_{\mathbb{Z}}(\bar{F}, \bar{F}))$ and

$\pi_*(\mathbb{H}(X; D^X_{Gr_0(KU)}))\tilde{} \cong \mathbb{H}^*(X;\ D_{\mathbb{Z}}) = H_*(X;\ \mathbb{Z})$

Next we define a homomorphism

$$Gr : K^{top}(X) \simeq \mathbb{H}(X; D_{KU}) \to \mathbb{H}(X; D_{Gr(KU)}) \qquad (5.0.7)$$

as follows. Observe the left-hand side may be identified with

$RMap(\underline{\Sigma}^0, D_{KU}) \overset{\simeq}{\to} RHom_{KU}(\underline{KU}_X;\ D_{KU})$.

Now the functor Gr defines a map from the above term to $RHom_{Gr(KU)}(Gr\underline{KU}_X;\ D_{Gr(KU)}) \simeq RMap(\underline{\Sigma}^0,\ D_{Gr(KU)}) \simeq \mathbb{H}(X;\ D_{Gr(KU)})$. Thus the map in (5.0.7) induces a map $Gr : K^{top}(X) \to \mathbb{H}_*(X;\ \mathbb{Z})$. Observe that the same definition applies in positive characteristics and defines a map $Gr : \widehat{K^{top}(X)}_l \simeq \mathbb{H}(X, D_{\widehat{KU}_l}) \to \mathbb{H}(X, D_{Gr(\widehat{KU}_l)})$.

PROPOSITION 5.2. *Assume the above situation. Then* $\underset{i=0,1}{\oplus} \pi_i(Gr(Tr^F(f))) = Tr^{\bar{F}}(f)$ *and* $\underset{i=0,1}{\oplus} \pi_i(Gr(Eu(\bar{F}))) =$ *the Euler class of* $\bar{F}$ *with values in* $H_*(X;\ \mathbb{Z})$. *Similarly if* X *is* *a smooth complex quasi-projective variety,* $\underset{i=0,1}{\oplus}\pi_i(Eu_\mu(\bar{F}))$ = *the micro-local Euler class of* $\bar{F}$ *with values in* $H_*(T^*X;\mathbb{Z})$.

PROOF. (5.0.4), (5.0.5) and the above discussion provide us with the commutative diagram

$$\begin{tikzcd} R\Gamma(X,\mathcal{RH}om_{Gr(KU)}(Gr(F),Gr(F))) \arrow[r, "Tr^{Gr(F)}_{Gr(KU)}"] \arrow[d, "\simeq"] & \mathbb{H}(X;D^X_{Gr(KU)}) \arrow[d, "\simeq"] \\ R\Gamma(X,\mathcal{RH}om_{Gr_0(KU)}(Gr_0(F),Gr_0(F)))\underset{Gr_0(KU)}{\otimes}Gr(KU) \arrow[r, "Tr^{Gr_0(F)}_{Gr_0(KU)}\underset{Gr_0(KU)}{\otimes}Gr(KU)"] & \mathbb{H}(X;D^X_{Gr_0(KU)})\underset{Gr_0(KU)}{\otimes}Gr(KU) \end{tikzcd}$$

where we have used Tr^K_A to denote the trace-map defined for the presheaf K of module-spectra over A as in section 2. Sending F to its associated graded object $Gr(F)$ defines a quasi-isomorphism: $Gr(RHom_{KU}(F,\,F))\overset{\simeq}{\to}RHom_{Gr(KU)}(Gr(F),\,Gr(F))$. One may therefore extend the above diagram to:

$$\begin{tikzcd} R\Gamma(X,\mathcal{RH}om_{KU}(F,F)) \arrow[r, "Tr^F_{KU}"] \arrow[d, "Gr"] & \mathbb{H}(X;D^X_{KU}) \arrow[d, "Gr"] \\ R\Gamma(X,\mathcal{RH}om_{Gr(KU)}(Gr(F),Gr(F))) \arrow[r, "Tr^{Gr(F)}_{Gr(KU)}"] \arrow[d, "\simeq"] & \mathbb{H}(X;D^X_{Gr(KU)}) \arrow[d, "\simeq"] \\ R\Gamma(X,\mathcal{RH}om_{Gr_0(KU)}(Gr_0(F),Gr_0(F)))\underset{Gr_0(KU)}{\otimes}Gr(KU) \arrow[r, "Tr^{Gr_0(F)}_{Gr_0(KU)}\underset{Gr_0(KU)}{\otimes}Gr(KU)"] & \mathbb{H}(X;D^X_{Gr_0(KU)})\underset{Gr_0(KU)}{\otimes}Gr(KU) \end{tikzcd}$$

We have thereby shown:

$$Gr(Tr^F_{KU}) = Tr^{Gr_0(F)}_{Gr_0(KU)}\underset{Gr_0(KU)}{\otimes}Gr(KU)$$

One may similarly show that

$$Gr((Tr^F_\mu)_{KU}) = (Tr^{Gr_0(F)}_\mu)_{Gr_0(KU)}\underset{Gr_0(KU)}{\otimes}Gr(KU)$$

Now consider the map

$$\pi_*(RHom_{Gr_0(KU)}(Gr_0(F),Gr_0(F))\underset{Gr_0(KU)}{\otimes}Gr(KU))$$

$$\xrightarrow{\pi_*(Tr^{Gr_0(F)}_{Gr_0(KU)}\underset{Gr_0(KU)}{\otimes}Gr(KU))}\pi_*(\mathbb{H}(X;D^X_{Gr_0(KU)})\underset{Gr_)(KU)}{\otimes}Gr(KU))$$

On taking π_*, the spectral sequences in Chapter III, Proposition 1.2 degenerates. Observe that on taking the sum $\underset{i=0,1}{\oplus}\pi_i$, the term in (5.0.5) identifies with $H^*(RHom_{\mathbb{Z}}(\bar{F},\bar{F}))$ while the term in (5.0.6) identifies with $H_*(X,\mathbb{Z})$. Therefore we obtain the first assertion. Considering $\underset{i=0,1}{\oplus}\pi_i(Tr^F_{KU}(id_F))$ and $\underset{i=0,1}{\oplus}\pi_i((Tr_\mu)^F_{KU})$, one obtains the remaining two assertions as well. $\square$

If X denotes a complex variety $K_*^{top}(X)$ will denote the (complex) topological K-theory of X. (Recall this is represented by the spectrum KU.) By arguments as in [J-3], one may identify this with $\underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X, \underline{KU}))$. One may now identify the Chern-character map with the map:

$$(5.0.8)\qquad \begin{aligned} ch' : \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; \underline{KU})) &\overset{Gr}{\to} \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; Gr(\underline{KU}))) \\ &\to \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; Gr(\underline{KU})_{\mathbb{Q}}))) \cong H^*(X, \mathbb{Q}) \end{aligned}$$

(In fact, the above map is induced by the universal chern-character $ch : KU \to KU_{\mathbb{Q}} \simeq \underset{i}{\Pi} K(\mathbb{Q}, 2i)$.) Next consider the map

$$(5.0.9)\qquad \begin{aligned} \tau' : \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; D_{KU})) &\overset{Gr}{\to} \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; D_{Gr(KU)})) \\ &\to \underset{i=0,1}{\oplus} \pi_i(\mathbb{H}(X; D_{Gr(KU)_{\mathbb{Q}}})) \cong H_*(X, \mathbb{Q}) \end{aligned}$$

One may observe readily that the map τ' is a module map over the multiplicative map ch' and that it is a natural transformation of functors that are covariant with respect to proper maps between quasi-projective complex varieties. Moreover, one may see readily that, if pt denotes a point and $[pt]_K$ ($[pt]$) denotes the fundamental class in K-homology (in homology) $\tau'([pt]_K) = [pt]$. (See Chapter IV, Corollary (5.12) that provides fundamental classes in complex K-homology.) These two properties show, as in [BFM] p. 129 and [F-2] p. 166, that the transformation τ' must be the Todd homomorphism. Therefore, we obtain the following theorem.

THEOREM 5.3. *Let X denote a complex projective variety. Let $\bar{F}$ denote a constructible sheaf of $\mathbb{Z}$-modules on X.*

(i) Then $\tau'(Eu(\bar{F})) = Eu(\bar{F}) \varepsilon H_*(X; \mathbb{Q})$ if X is projective.

(ii) If X is a smooth projective complex variety, $\tau'(Eu_\mu(\bar{F}^\cdot)) = Eu_\mu(\bar{F}^\cdot) \varepsilon H_*(T^*X; \mathbb{Q})$

where the Euler-class in rational homology (the micro-local Euler class in rational homology) is the one defined as in [K-S-2] p.377.

PROOF. This is clear from the above discussion. □

6. The main Theorem

We will adopt the terminology of section 3 for the rest of the paper. If X is a complex variety, we will let $Const(X; \mathbb{Z})$ denote the category of all constructible sheaves of $\mathbb{Z}$-modules on X. If X is a variety defined over a field k of positive characteristic p (satisfying the conditions in 1.1), l is a prime different from p and ν is a positive integer, $Const^{f.t.d}(X; l-adic)$ will denote the full sub-category of constructible l-adic sheaves that are also of finite tor dimension. We will let $K(Const(X; \mathbb{Z}))$ ($K(Const^{f.t.d}(X; l-adic))$) denote the Grothendieck group of the corresponding category.

THEOREM 6.1. *(i) If X is a complex variety, there exist an additive homomorphism:*

$Eu : K(Const_{\mathbb{Z}}(X)) \to K_0^{top}(X)$.

(ii) If X is, in addition, a smooth quasi-projective variety, there exists another additive homomorphism:

$$Eu_\mu : K(Const_{\mathbb{Z}}(X)) \to K_0^{top}(T^*X)$$

which factors through the obvious map $K_0^{top}(\Lambda_F) \to K_0^{top}(T^*X)$ where Λ_F is the micro-support of F. The Todd homomorphism sends these classes to the corresponding Euler-classes in rational homology at least for projective varieties.

(iii). If X is a variety defined over a field k as in 1.1 of characteristic p, there exists an additive homomorphism

$$Eu : K(Const^{f.t.d}(X; \widehat{\mathbb{Z}}_l)) \to \widehat{K_0^{top}(X)}_l$$

The map from K-homology to étale homology (as in 5.0.7) sends these classes to the corresponding Euler-classes at least for projective varieties.

(iv). The maps in (i) and (iii) commute with direct-images for proper maps. The map in (ii) commutes with direct images for proper and smooth maps of complex varieties.

PROOF. Clearly it suffices to prove the last assertion. Let $F \varepsilon D(Mod_l(\mathcal{C}_X, E))$ where E denotes KU and $\mathcal{C}_X$ is the usual site in characteristic 0 ($\widehat{KU}_l$ and $\mathcal{C}_X$ denotes the étale site in positive characteristic p, respectively, with $l \neq p$). The proof that the maps in (i) and (iii) commute with the direct image functor for proper maps will follow from the commutativity of the following diagram:

$$\begin{tikzcd}
Rf_*(D_E^X(F)) \underset{E}{\otimes} Rf_*(F) \arrow[r] \arrow[d] & Rf_*(D_E^X(F) \underset{E}{\otimes} F) \arrow[r] & Rf_*(D_E^X) \arrow[d] \\
D_E^Y(Rf_*(F)) \underset{E}{\otimes} Rf_*(F) \arrow[rr] & & D_E^Y
\end{tikzcd}$$

The left-most vertical map exists because $Rf_* D_E^X(F) \simeq Rf_! D_E^X(F)$. One observes that the above diagram is the same as:

$$\begin{tikzcd}
Rf_* \mathcal{RH}om_E(F, Rf^! D_E^Y) \underset{E}{\otimes} Rf_*(F) \arrow[r] \arrow[d] & Rf_* Rf^! D_E^Y \arrow[d, "id"] \\
\mathcal{RH}om_E(Rf_*(F), Rf_* Rf^! D_E^Y) \underset{E}{\otimes} Rf_*(F) \arrow[r] \arrow[d] & Rf_* Rf^! D_E^Y \arrow[d] \\
\mathcal{RH}om_E(Rf_* F, D_E^Y) \underset{E}{\otimes} Rf_*(F) \arrow[r] & D_E^Y
\end{tikzcd}$$

where the map $Rf_* Rf^! D_E^Y \simeq Rf_! Rf^! D_E^Y \to D_E^Y$ is the trace defined in chapter IV. The commutativity of the above diagram is clear and this proves (iv) for the maps in (i) and (iii).

Next we consider the proof of (iv) for the map in (ii). First one observes the commutativity of the diagram:

$$\begin{tikzcd}
Rf_*(D_E^X(F)) \boxtimes Rf_*(F) \arrow[r] \arrow[d] & Rf_*(D_E^X(F) \boxtimes F) \arrow[r] & Rf_*(\delta_{Y*} D_E^X) \arrow[d] \\
D_E^Y(Rf_*(F)) \boxtimes Rf_*(F) \arrow[rr] & & \delta_{Y*} D_E^Y
\end{tikzcd}$$

This is established by the same argument as above. (Here $\delta_Y : Y \to Y \times Y$ is the diagonal immersion.) It suffices to observe a natural quasi-isomorphism:

$$Rf_* R\pi_{X*} R\Gamma_{\Lambda_M} \mu_{\Delta_X}(F) \simeq R\pi_{Y*} R\Gamma_{\Lambda_{Rf_*F}} \mu_{\Delta_Y} Rf_*(F)$$

where $\pi_X : T^*X \to X$ ($\pi_Y : T^*Y \to Y$) is the obvious projections, Λ_F (Λ_{Rf_*F}) is the micro-support of F, μ_{Δ_X} (μ_{Δ_Y}) is the micro-localization along $\Delta_X : X \to X \times X$ ($\Delta_Y : Y \to Y \times Y$, respectively). The hypothesis that f be proper and smooth implies f is transverse to $X \to X \times X$ and also proper on the support of F. The above quasi-isomorphism follows from [K-S-2] Proposition 4.2.4 along with the spectral sequence in (7.1.2). $\square$

7. A general technique

We will use the following general technique for extending results from abelian sheaves to presheaves of spectra.

THEOREM 7.1. *Let $\mathfrak{S}$ and $\mathfrak{S}'$ denote two sites as before. (In particular they have finite cohomological dimension (finite l-cohomological dimension in positive characteristic, respectively). Let $T, T' : Presh_{KU}(\mathfrak{S}) \to Presh_{KU}(\mathfrak{S}')$ denote two covariant (or two contravariant) functors that preserve fibration sequences and quasi-isomorphisms for any ring spectrum in the sense of Chapter I. Let $\phi : T \to T'$ denote a natural transformation. Assume there exists functors T_{ab}, ${T'}_{ab} : D_b(\mathfrak{S}, \pi_*(KU)) \to D_b(\mathfrak{S}'; \pi_*(KU))$ provided with natural quasi-isomorphisms $T(Sp(\bar{F}^\cdot)) \simeq Sp(T_{ab}(\bar{F}^\cdot))$ and similarly for T'. Assume further that T_{ab} and T'_{ab} have finite cohomological dimension.*

(7.1.1) Suppose in addition that there exists a natural map $\phi_{\bar{F}^\cdot} : T_{ab}(\bar{F}^\cdot) \to {T'}_{ab}(\bar{F}^\cdot)$ so that $\phi_{Sp(\bar{F}^\cdot)} = Sp(\phi_{\bar{F}^\cdot})$. Then ϕ_F is a quasi-isomorphism for all $F \varepsilon Presh_{KU}(\mathfrak{S})$ if and only if $\phi_{\bar{F}^\cdot}$ is a quasi-isomorphism for each $\bar{F}^\cdot \varepsilon D_b(\mathcal{C}; \pi_*(KU))$.

(7.1.2) Moreover there exist strongly convergent spectral sequences:

$$E_2^{s,t} = H^s(T_{ab}(\pi_{-t}(F)\tilde{})) \Rightarrow \pi_{-s-t}(T(F))\tilde{} \text{ and}$$

$$E_2^{s,t} = H^s({T'}_{ab}(\pi_{-t}(F)\tilde{})) \Rightarrow \pi_{-s-t}(T'(F))\tilde{}.$$

PROOF. It is enough to consider the canonical Cartan filtration on any $F \varepsilon Presh(\mathcal{C})$. Since both T and T' preserve fibration sequences, they send the above filtration to fibration-sequences. These provide long-exact sequences on taking the homotopy groups. The spectral sequences arise this way. The hypotheses on T and T' ensure the spectral sequences are strongly convergent.

Moreover the hypotheses ensure that there exists a natural map from the former to the latter. Therefore an isomorphism of the E_2-terms provides an isomorphism of the abutments. This proves the sufficiency of the hypothesis in (7.1.1). For presheaves of spectra of the form $Sp(\bar{F}^\cdot)$, $\bar{F}^\cdot \varepsilon D_b(\mathcal{C}; \pi_*(KU))$ a quasi-isomorphism $T(Sp(\bar{F}^\cdot)) \simeq T'(Sp(\bar{F}^\cdot))$ is equivalent to a quasi-isomorphism $T_{ab}(\bar{F}^\cdot) \simeq {T'}_{ab}(\bar{F}^\cdot)$. This proves the necessity of the hypothesis in (7.1.1). $\square$

CHAPTER VI

Survey of other applications

In this chapter we will provide a survey of various applications and potential applications of the theory developed so far.

1. Filtered Derived categories

In this section, we will show how to provide an extension of the basic theory to include algebras $\mathcal{A}$ that are provided with a non-decreasing filtration (i.e. in addition to the canonical Cartan filtration). We will assume that $Presh(\mathfrak{S})$ is as in Chapter III, (1.1) and (1.2) and that $\mathcal{A}\varepsilon Presh(\mathfrak{S})$ is an algebra provided with a non-decreasing exhaustive and separated filtration F (indexed by the integers). Recall the Cartan filtration on any object $P\varepsilon Presh(\mathfrak{S})$ is defined by $\{\tau_{\leq n}P|n\}$ where $\tau_{\leq n}$ is the cohomology truncation functor as in Chapter I. This filtration will be denoted $\mathcal{C}$. Now $\{\tau_{\leq n}F_m\mathcal{A}|n,m\}$ is a common refinement of both the filtrations: we will let the filtration $\mathcal{C}\circ F$ be defined by $(\mathcal{C}\circ F)_k\mathcal{A} = \underset{k=n+m}{+}\tau_{\leq n}F_m(\mathcal{A})$. Clearly this filtration is also exhaustive and separated.

1.1. We will now make an *assumption* that the associated graded term of the filtration $\mathcal{C}\circ F$ in bi-degree (n,m) is $Gr_{\mathcal{C},n}(Gr_{F,m}(\mathcal{A}))$ for all m and n. (We observe that the associated graded term in bi-degree (n,m) of the filtration $\mathcal{C}\circ F$ is given by $Gr_{\mathcal{C}\circ\mathfrak{F},n,m}(K) = \tau_{\leq n}F_mK/(\tau_{\leq n-1}F_mK + \tau_{\leq n}F_{m-1}K)$. Observe also that since $\tau_{\leq n}$ need not commute with taking quotients, the above assumption need not be satisfied in general.)

Observe that $Gr_F(\mathcal{A})\varepsilon Presh(\mathfrak{S})$ is also an algebra and $Gr_{\mathcal{C},n}(Gr_F(\mathcal{A}))$ is its associated graded term of degree n with respect to the Cartan filtration. i.e. $\mathcal{H}^i(Gr_{\mathcal{C},n}(Gr_F(\mathcal{A}))) = 0$ unless $i = n$. (For *example* consider the case $Presh(\mathfrak{S})$ is the category of complexes of abelian presheaves on a site $\mathfrak{S}$. Now a filtered algebra $\mathcal{A}$ in $Presh(\mathfrak{S})$ corresponds to a differential graded algebra provided with a filtration compatible with the structure of a differential graded algebra. In this case one may require the differentials of the (differential graded) algebra $\mathcal{A}$ are strict, i.e. their images and co-images are isomorphic. This condition implies that the spectral sequence (in $\mathcal{H}^*$) associated to the given filtration degenerates, which in turn implies the hypothesis 1.1 at least under the hypothesis that $\mathcal{H}^i(\mathcal{A}) = 0$ for $i << 0$ and that the filtration F is bounded below.)

In this situation, we will let $Mod_l^{filt}(\mathfrak{S},\mathcal{A})$ denote the category of all left-modules M over $\mathcal{A}$ provided with an exhaustive and separated filtration F_M compatible with the filtration $\mathcal{C}\circ F$ on $\mathcal{A}$. One defines $Mod_r^{filt}(\mathfrak{S},\mathcal{A})$ similarly. Moreover one may carry over the entire theory developed in Chapter III to this context; in particular one defines $Mod_l^{c,f.t.d}(\mathfrak{S},\mathcal{A})$ and $Mod_l^{perf}(\mathfrak{S},\mathcal{A})$ as in chapter III. One may define $Mod_{bi}^{perf}(\mathfrak{S},\mathcal{A})$ as the corresponding category of bi-modules over $\mathcal{A}$.

In this context the Bi-duality theorem of Chapter IV, section 4 applies to provide a dualizing complex for this derived category. This theorem may be restated in this context as follows.

THEOREM 1.1. *Assume the above situation. A perfect complex* $\mathbb{D}\varepsilon Mod_{bi}^{perf}(\mathfrak{S}, \mathcal{A})$ *will be a dualizing complex for the category* $Mod_l^{perf}(\mathfrak{S}, \mathcal{A})$ *provided it comes equipped with a non-decreasing filtration* F *compatible with the given filtration on* $\mathcal{A}$*, so that* $Gr_F(\mathbb{D})$ *is a dualizing complex for the category* $Mod_l^{perf}(\mathfrak{S}, \mathcal{H}^*(Gr_F(\mathcal{A})))$.

Grothendieck-Verdier style duality for derived categories associated to sites provided with filtered sheaves of rings is clearly a special case of the above framework where the Cartan filtration is trivial. The following are examples of this.

EXAMPLES 1.2. 1. Let X denote a complex non-singular algebraic variety and let $\mathcal{A} = \mathcal{D}_X =$ the sheaf of rings of differential operators on X. above theorem shows what could be candidates for a dualizing complex for the category of perfect complexes of $\mathcal{D}_X$-modules. In fact, since, every coherent $\mathcal{D}_X$-module may be given a filtration so that it is a perfect complex, this theorem shows why the usually defined dualizing complex for $\mathcal{D}_X$-modules is in fact a dualizing complex.

2. Similar considerations apply to super-varieties and shows what are possible candidates for a dualizing complex. (Recall that the structure sheaves of super-varieties are filtered so that the associated graded objects are commutative.)

2. Derived schemes

Recall that a derived scheme is given by a ringed site $(X, \mathcal{A} = \underset{i\varepsilon\mathbb{Z}}{\oplus} A_i)$ where $\mathcal{A} = \underset{i\varepsilon\mathbb{Z}}{\oplus} A_i$ is a sheaf of graded differential graded algebras so that $(X, \mathcal{A}_0)$ is a scheme (in the usual sense) and each $\mathcal{A}_i$ is a coherent $\mathcal{A}_0$ -module. (See [Kon], [CK1], [CK2] for basic details on derived schemes.) The derived versions of the quot-schemes and Hilbert schemes are constructed in [CK1] and [CK2]. A quasi-coherent (coherent) sheaf on such a derived scheme is a sheaf of graded differential graded modules $F = \underset{i}{\oplus} F_i$ so that each F_i is a quasi-coherent (coherent, respectively) sheaf on the scheme $(X, \mathcal{A}_0)$. The basic theory of Chapter IV applies now to define perfect complexes over the ringed site $(X, \mathcal{A})$. Moreover, the general theory of Grothendieck-Verdier duality as formulated in Chapter IV applies to extend the formalism of Grothendieck-Verdier duality to derived schemes.

In addition, the discussion in the last section provides a bi-duality theorem for the derived category of coherent $\mathcal{D}$-modules (defined suitably) on smooth derived schemes.

3. Derived categories of mixed (Tate) motives over a general scheme

Over a field, there has been an elegant construction of the category of mixed Tate motives by Bloch, Kriz and May. (See [Bl-3], [Bl-K] and [K-M].) This depends crucially on the construction of a differential graded algebra associated to the cycle complex for the field. There have been nontrivial difficulties in extending this construction to all smooth quasi-projective varieties over a field; these have been overcome in [J-6], by making use of the motivic complexes. (Recall the motivic complex is known to be quasi-isomorphic to the cycle complex for all smooth quasi-projective schemes. See [Voe-2].) The main idea again is to associate a DGA to the motivic complex (tensored with $\mathbb{Q}$), provide the category of finitely generated modules over this DGA with a t-structure and then take the heart of this t-category. At least for smooth linear varieties, it is shown in [J-6] that this provides a reasonable theory of relative Tate motives. Similar techniques are expected to extend to general smooth schemes: however, it seems quite likely that an appropriate sheafification of the motivic complex is required. In this context, the conjectures of Beilinson on motivic derived categories seem quite relevant.

For example, we quote (part of) what is referred to as the version 4 of Beilinson's conjecture on motivic derived categories - see [Jan] p. 280.

Let k denote a field. For every k-scheme X, there exists a triangulated $\mathbb{Q}$-linear tensor category $DM(X)$ with a t-structure so that the following hold:

(i) If $f : X \rightarrow Y$ is a map between such schemes, there exist the derived functors f_*, $f_!$, f^* and $f^!$ between these derived categories so that the usual formalism of Grothendieck-Verdier duality (i.e. the usual relations among these functors) carries over.

(ii) There exist *exact* realization functors:

$r_l : DM(X) \rightarrow D^b_m(X_{et}; \mathbb{Q}_l)$, $l \neq char(k)$ and

$r_B : DM(X) \rightarrow D^b(MH(X))$, $k = \mathbb{C}$

Here $D^b_m(X_{et}; \mathbb{Q}_l)$ is the derived category of bounded complexes of $\mathbb{Q}_l$-sheaves on the étale topology of X with mixed constructible cohomology sheaves. $MH(X)$ is the category of mixed Hodge modules on X.

(iii) There exists a t-structure on $DM(X)$ so that its heart is the $\mathbb{Q}$-linear abelian category, $M(X)$, of mixed motivic sheaves. The realization functor r_l sends $M(X)$ to the category of mixed perverse $\mathbb{Q}_l$-sheaves on X_{et}. r_B sends $M(X)$ to $MH(X)$. Moreover the above realization functors are exact and faithful on $M(X)$.

The general theory of Grothendieck-Verdier style duality developed in this paper should apply in this context to provide at least part of the conjectured formalism of Grothendieck-Verdier style duality in the setting of motivic derived categories, perhaps for a derived category of relative Tate motives.

One key issue in this setting would be the definition of a t-structure for the category of sheaves of modules over a DGA. In the setting of Tate motives over a field (or relative Tate motives for linear varieties over a field) where no sheafification is required, such a t-structure is provided easily using the theory of minimal models. This is non-trivial when sheafification is needed. Moreover the issue of defining a t-structure for sheaves of modules over DGAs is related to the following.

4. Generalized intersection cohomology theories

In fact this is a problem that had been the starting point of our interest in generalizing the Grothendieck-Verdier formalism of duality. This is stated as an open problem in [Bo], last section. Briefly stated the question is the following. *Intersection cohomology seems to be the correct variant of singular cohomology (i.e. cohomology with respect to the constant sheaf $\mathbb{Z}$) adapted to the study of singular spaces. What are the the corresponding variants of the familiar generalized cohomology theories (for example, topological K-theory) adapted to the study of singular spaces?*

A key step in the definition of such a theory would be the definition of t-structures for presheaves of module-spectra over the spectrum representing the given generalized cohomology theory. There have been partial success in this direction in [Kom] and also [J-7]. (Komezano uses cobordism theory with singularities and provides a definition of generalized intersection cohomology theories; however a detailed analysis shows that despite superficial differences, the two approaches are similar at least in principle.) In fact, using the techniques established in this work work, we hope to complete the work begun in [J-7].

5. Motivic Homotopy Theory

In this section we will show that the basic formalism adopted in Chapters I and II applies to motivic homotopy theory or more precisely the stable homotopy theory associated to the unstable $\mathbb{A}^1$-homotopy theory of [M-V]. (Here the stabilization is with respect to $T = S^1 \wedge \mathbb{G}_m$.)

Let S denote a Noetherian base scheme; we will let $(smt.schemes/S)_{Nis}$ denote the category of all schemes of finite type over S provided with the Nisnevich topology. Recall now the basic result of unstable motivic homotopy theory is the following:

THEOREM 5.1. *(See [M-V].) Let $SPresh((smt.schemes/S)_{Nis})$ denote the category of all simplicial presheaves on $(smt.schemes/S)_{Nis}$. This has the following structure of a proper simplicial model category:*

the cofibrations are monomorphisms

the weak-equivalences are the $\mathbb{A}^1$-weak-equivalence and

the fibrations are defined by right lifting property with respect to cofibrations which are also weak-equivalences.

Using smashing with T (instead of with S^1) one may define the notion of spectra in this category; these form T-spectra. Moreover one may define symmetric spectra in the category $SPresh((smt.schemes/S)_{Nis})$. We will let $SSpPresh((smt.schemes/S)_{Nis})$ denote the category of all symmetric spectra obtained this way.

THEOREM 5.2. *$SSpPresh((smt.schemes/S)_{Nis})$ is an enriched stable closed simplicial model category in the sense of Chapter II, Definition 4.11.*

PROOF. The stable simplicial model structure and the axioms on the monoidal structure as in Chapter I follow by more or less standard arguments. □

Therefore, in order, to be able to apply the results of Chapter III to this setting, it suffices to show that $SSpPresh((smt.schemes/S)_{Nis})$ has a strong t-structure as in Chapter I. We will refer to objects in

$SSpPresh((smt.schemes/S)_{Nis})$ as *motivic spectra.*

Recall that the presheaves of motivic stable homotopy groups are bi-graded by a *degree* t and *weight* s and defined as:

$$\pi_{t,s}(\Gamma(U,P)) = Hom_H(\Sigma_T(S^t \wedge \mathbb{G}_m^s)_{|U}, P_{|U}) \tag{5.0.1}$$

where Hom_H denotes Hom in an appropriate homotopy category. One lets $\pi_t(P) = \bigoplus_s \pi_{t,s}(P)$ where the latter denotes the above presheaf: by abuse of notation, we will call these the motivic stable homotopy groups. It is known that a cofiber-sequence in $SSpPresh((smt.schemes/S)_{Nis})$ provides a long-exact sequence in π_*, where $*$ denotes the degree.

The definition of the Eilenberg-Maclane functor as in Chapter I is, however not clear, since the motivic stable homotopy seems difficult to compute. To be able to define Eilenberg-Maclane spectra as in Chapter I (and therefore a strong t-structure) one needs to be able to kill off the homotopy indexed by the weight as well the degree, for example by a suitable analogue of *attaching cells.*

On the other hand, the slices introduced in [Voe-3] may be related to providing a different sort of t-structure.

APPENDIX A

Verification of the axioms for Γ-spaces and symmetric spectra

1. Γ-spaces

THEOREM 1.1. *The category of Γ-spaces endowed with the smash-product defined in [Lyd] is an enriched stable simplicial model t-category.*

The rest of this section will be devoted to a proof of this theorem. Throughout we will adopt the following *convention*. A simplicial set (a pointed simplicial set) will be denoted *space* (*pointed space*, respectively).

Let Γ^{op} denote the category with objects $n^+ = \{0, ..., n\}$. (We view n^+ as pointed by 0.) The morphisms $f : m^+ \to n^+$ are all maps so that $f(0) = 0$. A Γ-space is a functor

$\mathcal{A} : \Gamma^{op} \to$ (pointed spaces)

so that $\mathcal{A}(0) = *$. The sphere Γ-space $\mathcal{S}$ is the Γ-space defined by: $\mathcal{S}(n^+) = n^+$, for each n. A map between two Γ-spaces is a natural transformation of functors. The category of all Γ-spaces will be denoted $\mathcal{GS}$. The Hom-sets in this category will be denoted $Hom_{\mathcal{GS}}$ (or merely Hom, if there is no cause for confusion).

We first recall the following *strict simplicial model structures for Γ-spaces* from [B-F] section 3. This will define *the strict model structure* on the category of Γ-spaces.

1.1. For each fixed integer $k \geq 0$, let Γ_k^{op} denote the full sub-category of Γ consisting of objects n^+, $n \leq k$. A Γ_k-space is a functor $\Gamma_k^{op} \to$(pointed spaces). Now one defines the k-truncation

$Tr_k : (\Gamma\text{-spaces}) \to (\Gamma_k\text{-spaces})$

as the functor restricting a Γ-space $\mathcal{A}$ to the sub-category Γ_k^{op}. This has a left-adjoint denoted sk_k and a right adjoint denoted $cosk_k$. Often we will denote the composition $sk_k \circ Tr_k$ ($cosk_k \circ Tr_k$) by sk_k ($cosk_k$, respectively) as well. (See [B-F] pp.89-90 for more details.) Now a map $f : \mathcal{A} \to \mathcal{B}$ of Γ-spaces is a *cofibration* if for each n, the induced map $(sk_{n-1}\mathcal{B})(n^+) \underset{(sk_{n-1}\mathcal{A})(n^+)}{\sqcup} \mathcal{A}(n^+) \to \mathcal{B}(n^+)$ is *injective* and the symmetric group Σ_n acts freely on the simplices not in the image of the above map. A map $f : \mathcal{A} \to \mathcal{B}$ is *a fibration* if the induced map $\mathcal{A}(n^+) \to (cosk_{n-1}\mathcal{A})(n^+) \underset{(cosk_{n-1}\mathcal{B})(n^+)}{\times} \mathcal{B}(n^+)$ is a fibration of pointed spaces for each n. A Γ-space $\mathcal{A}$ is *cofibrant* (fibrant) if the obvious map $* \to \mathcal{A}$ is a cofibration ($\mathcal{A} \to *$ is a strict fibration, respectively). A map $f : \mathcal{A} \to \mathcal{B}$ of Γ-spaces is a weak-equivalence if the map $f(n^+) : \mathcal{A}(n^+) \to \mathcal{B}(n^+)$ is a weak-equivalence of pointed spaces. It is shown in [B-F] Theorem (3.5) that this defines a simplicial model structure on the category of Γ-spaces. The same proof applies to show that one may define a simplicial model category structure on the sub-categories $\Gamma_{\leq k}$-spaces = the functor category $\{\mathcal{A} : \Gamma_k^{op} \to\text{(pointed spaces)} | \mathcal{A}(0) = *\}$ in an entirely similar manner. (i.e. A map $f : \mathcal{A} \to \mathcal{B}$

in $\Gamma_{\leq k}$-spaces is a weak-equivalence (cofibration, fibration) if the corresponding conditions above are satisfied for all $n \leq k$.)

1.2. The functor $\otimes$: (*pointed simplicial sets*) $\times$ (Γ − *spaces*) $\rightarrow$ (Γ − *spaces*) is given by sending a pointed simplicial set K and a Γ-space $\mathcal{A}$ to the Γ-space $(K \wedge \mathcal{A})(n^+) = K \wedge A(n^+)$. Observe that not every monomorphism is a cofibration. However (PM4) of Chapter II, section 4 is shown to be satisfied by Lemma (3.7) of [B-F]. The above strict simplicial model structure defines the required *partial model category structure* as in Chapter II, section 4.

1.3. Next we consider the stable simplicial model structure from [B-F]. For this we recall the connection between Γ-*spaces and connective spectra*. (Here spectra mean as in [B-F] and not the more sophisticated symmetric spectra considered below.)

1.4. A *spectrum* K is given by a collection $\{K^n | n \geq 0\}$ of pointed simplicial sets provided with maps $S^1 \wedge K^n \rightarrow K^{n+1}$ for each n. (Here $S^1 = \Delta[1]/\delta(\Delta[1])$ is the simplicial one-sphere.) A map of spectra $K = \{K^n|n\} \rightarrow L = \{L^n|n\}$ is given by a compatible collection of maps $f^n : K^n \rightarrow L^n$ of pointed spaces commuting with the suspension. The homotopy groups of a spectrum K are defined by $\pi_k(K) = \underset{n\rightarrow\infty}{\operatorname{colim}} \pi_{n+k}(Sing(|K_n|)$. A map $f : K \rightarrow L$ is a stable-equivalence of spectra if it induces an isomorphism on the above homotopy groups. A spectrum K is connective (or −1-connective) if $\pi_k(K) = 0$ for all $k < 0$.

1.5. Now let $\mathcal{A}$ denote a Γ-space. One may progressively extend $\mathcal{A}$ to a functor (finite pointed sets) $\rightarrow$(pointed spaces), (pointed sets) $\rightarrow$ (pointed spaces), (pointed spaces) $\rightarrow$ (pointed spaces) in the obvious manner. (See [B-F] section 4.) Now let K denote a spectrum. One may show that there exist natural maps $S^1 \wedge \mathcal{A}(K^n) \rightarrow \mathcal{A}(S^1 \wedge K^n) \rightarrow \mathcal{A}(K^{n+1})$ for each n; these show that one may finally extend $\mathcal{A}$ to a functor (spectra) $\rightarrow$(spectra).

Let $\Sigma = \{S^0, S^1, S^2, ..., S^n, ...\}$ denote the sphere spectrum. Now given the Γ-space $\mathcal{A}$, $\mathcal{A}(\Sigma)$ is a spectrum which is clearly connective. This defines the functor:

$-(\Sigma)$: (Γ-spaces) $\rightarrow$ (connective spectra)

Given the connective spectrum K, one defines the associated Γ-space $\Phi(K)$ by

$\Phi(K)(n^+) = Map(\Sigma^n, K)$.

Here Σ^n denotes the n-fold product of the sphere spectrum and $Map(\Sigma^n, K)$ is the pointed space given in degree k as the set of pointed maps $\Sigma^n \wedge \Delta[k]_+ \rightarrow K$ of spectra. There is an adjunction:

$$Hom_{\Gamma-spaces}(\mathcal{A}, \Phi(K)) \cong Hom_{spectra}(\mathcal{A}(\Sigma), K) \tag{1.5.1}$$

Therefore one obtains natural maps $\mathcal{A} \rightarrow \Phi(\mathcal{A}(\Sigma))$ and $\Phi(K)(\Sigma) \rightarrow K$ for a Γ-space $\mathcal{A}$ and a connective spectrum K.

A map $f : \mathcal{A} \rightarrow \mathcal{B}$ of Γ-spaces is a *stable equivalence* if the induced map $f(\Sigma) : \mathcal{A}(\Sigma) \rightarrow \mathcal{B}(\Sigma)$ of connective spectra is a stable equivalence of spectra. A map $f : \mathcal{A} \rightarrow \mathcal{B}$ of Γ-spaces is a *stable cofibration* if it is a strict cofibration in the sense of 1.1 and a map $f : \mathcal{A} \rightarrow \mathcal{B}$ of Γ-spaces is a *stable fibration* if it has the right lifting property with respect to all maps that are stable cofibrations and stable-equivalences. It is shown in [B-F] Theorem (4.2) that the above structure is in fact a simplicial model category structure on the category of Γ-spaces.

Now we will adopt the above stable model structure to define the *stable simplicial model structure* as in Chapter II, section 4. Clearly the axiom (SM0) is satisfied. Clearly every strict weak-equivalence is a quasi-isomorphism: this shows axiom (SM1) is also satisfied. (See [B-F] Lemma (4.7) to see that a strict weak-equivalence is a stable-equivalence.) Moreover the above definition of stable cofibrations shows that the statement in (SM2) on cofibrations is satisfied. To complete the proof of (SM2) it suffices to check that every stable fibration is a strict fibration. This may be done by checking that every stable fibration satisfies the corresponding right lifting property for fibrations in the sense of 1.1. This is clear since every stable cofibration is a strict cofibration in the sense of (4.2) and every strict weak-equivalence is a stable weak-equivalence.

Next we consider (SM3.1) through (SM3.3). First observe that the functor $K \to Sing(|K|)$ from pointed spaces to pointed spaces has the following properties: there is a natural map $K \to Sing(|K|)$ which is a weak-equivalence of pointed spaces and moreover $Sing(|K|)$ is a fibrant simplicial set. This functor readily extends to a functor Γ-spaces → Γ-spaces and defines the functor Q. If $X = \{X_n|n\}$ is a spectrum so that each X_n is a fibrant pointed space, one may convert X to a fibrant Ω-spectrum by the (usual) functor we denote by T': $T'(X)_k = \underset{n}{colim}\Omega^n X_{n+k}$. If each X_n is not necessarily a fibrant pointed space, one may first apply $Sing \circ |\ \ |$ to X degreewise to convert it to a spectrum which consists of fibrant pointed spaces in each degree. The composition $T' \circ Sing \circ |\ \ |$ will be denoted T. We now define the functor Q^{st} as follows: let $\mathcal{A}$ denote a Γ-space. Now $Q^{st}(\mathcal{A}) = \Phi(T(\mathcal{A}(\Sigma)))$. So defined, we will now verify that the functors Q and Q^{st} satisfy the axiom (SM3). The assertions in (SM3) are easy to verify for the functor Q. We may verify the corresponding assertions for the functor Q^{st} as follows. Let $\mathcal{A}$ denote a Γ-space: to obtain a map $\mathcal{A} \to Q^{st}(\mathcal{A}) = \Phi T(\mathcal{A}(\Sigma))$, it suffices to show the existence of a map $\mathcal{A}(\Sigma) \to T(\mathcal{A}(\Sigma))$ and the latter clearly exists. Next observe that there exists a natural map $T \circ T \to T$. In view of the adjunction between the functor Φ and $\mathcal{A} \to \mathcal{A}(\Sigma)$, this suffices to define a natural map $Q^{st} \circ Q^{st} \to Q^{st}$.

The pairing $(pointed\ \ simplicial\ \ sets) \times (\Gamma - spaces) \to (\Gamma - spaces)$ is the one considered in 1.2 . This has all the properties required in (SM3.4). The hypotheses in (SM4) through (SM6)' may be verified readily at the level of spectra (where the model structures provided by [B-F] may be used). Now applying the functor $-(\Sigma)$ to pass from a Γ-space to a connective spectrum and applying the functor Φ to pass back to a stably weakly equivalent Γ-space proves these axioms are in fact satisfied.

Next consider the axioms (HCl) and (Hl). The model category-structure on the category of diagrams $\mathcal{C}^{\Delta^{op}}$ and $\mathcal{C}^{\Delta}$ when $\mathcal{C}$ is the category of pointed simplicial sets is established in [B-K]. The stable versions (i.e. when $\mathcal{C}$ is replaced by the category of spectra and Γ-spaces) may be defined as in [B-F]: we skip the details.

Now we provide the category of Γ-spaces with the smash-product defined in [Lyd]. This will be denoted ∧. We proceed to verify the axioms on the monoidal structure in Chapter I.

THEOREM 1.2. *(Lydakis) The category of* Γ*-spaces is symmetric monoidal with respect to the above smash product. Moreover the sphere* Γ*-space is a strict unit.*

The above theorem establishes the axiom (M0) for the category of Γ-spaces. The last assertion in the theorem clearly shows the sphere spectrum is a strict unit. We take $\mathfrak{F}$ to be the full subcategory of stably cofibrant objects. The axioms on the stable model structure now show that (M1) is satisfied. Moreover the sphere spectrum is known to be stably cofibrant, so that the axiom (M3) is also satisfied. The pairing required in (M3) is defined in (1.3) above. Moreover now one may readily verify the axiom (M4.0) through (M4.5). [Lyd] (3.20) shows smashing with a Γ-space preserves injective maps and it is shown in [Lyd](4.1) that if $\mathcal{A}$ and $\mathcal{B}\varepsilon$(Γ-spaces) are both stably cofibrant, the functors $\mathcal{A} \wedge -$ and

$- \wedge B$ preserve stable cofibrations and that $\mathcal{A} \wedge \mathcal{B}$ is stably cofibrant. Moreover it is shown [Lyd](4.18) that both the above functors also preserve stable weak-equivalences. These prove (M2). The assertion about $\mathcal{H}om$ is now an immediate consequence of this and the adjunction with $A \otimes -$.

Next we consider the axiom (M5) for both the functos Q and Q^{st} as defined in. First consider the functor $Sing \circ |\ \ |$ applied to pointed spaces. Let X, Y denote two pointed spaces. Now there exists a natural map $Sing|X| \wedge Sing|Y| \to Sing|X \wedge Y|$. (To see this observe that such a map is adjoint to a map $|(Sing|X|) \wedge (Sing|Y|)| \to |X \wedge Y|$. The latter exists since $|(Sing(|X|)) \wedge (Sing|Y|)| \cong (|(Sing(|X|))| \wedge |(Sing(|Y|))|)_{Kelley}$ and $|X \wedge Y| \cong (|X| \wedge |Y|)_{Kelley}$; here we have used the notation that if Z is a Hausdorff topological space Z_{Kelley} is the underlying set of Z retopologized by the finer Kelley topology. (See [G-Z] p.10, p.53.) Finally observe again that the geometric realization is left adjoint to the singular functor.) Now recall that the functor Q is a degree-wise extension of the functor $Sing \circ |\ \ |$ to Γ-spaces; therefore it has the property mentioned in (M5).

Next we show the axiom (M5) is satisfied for the functor Q^{st}. For this it is good to recall the relation between pairings of Γ-spaces and that of the associated spectra again from [Lyd]. A *spectrum with no odd terms* consists of a sequence $\{E_{2n}|n \geq 0\}$ of pointed simplicial sets and pointed maps $S^2 \wedge E_{2n} \to E_{2n+2}$, for all $n \geq 0$. One may extend all the standard notions like maps, homotopy groups, weak-equivalence etc. from spectra to spectra with no odd terms. Let E and E' denote two spectra in the usual sense; now the naive-smash product $E \wedge E'$ is the spectrum with no odd terms defined by

$$(E \wedge E')_{2n} = E_n \wedge E'_n \tag{1.5.2}$$

and where the map $S^2 \wedge (E \wedge E')_{2n} \to (E \wedge E')_{2n+2}$ is defined as the composition: $S^1 \wedge S^1 \wedge (E_n \wedge E'_n) \cong (S^1 \wedge E_n) \wedge (S^1 \wedge E'_n) \to E_{n+1} \wedge E'_{n+1} = (E \wedge E')_{2n+2}$. *Convention: the smash product of two spectra will denote this naive smash product in this section*

Any spectrum E in the usual sense defines a spectrum with no odd terms E^t by $(E^t)_{2n} = E_{2n}$. Conversely any spectrum with no odd terms $E^t = \{E^t_{2n}|n\}$ defines a spectrum E in the usual sense by $(E)_n = E^t_{2n}$. One may now readily observe that the category of spectra is equivalent to the category of spectra with no odd terms. It follows that if E, E' are two spectra in the usual sense and E^t, E'^t are the associated spectra with no odd terms, then there is an isomorphism

$$Map(E, E') \cong Map(E^t, E'^t)$$

of pointed spaces.

PROPOSITION 1.3. *Let $\mathcal{A}$, $\mathcal{B}$ denote two Γ-spaces. Now the following hold:*

(i) There exists a natural map $T\mathcal{A}(\Sigma) \wedge T\mathcal{B}(\Sigma) \to T(\mathcal{A} \wedge B)(\Sigma)^t$. (Here T is the functor considered in earlier.)

(ii) There exists natural maps $\Phi(T\mathcal{A}(\Sigma)) \wedge \Phi(T\mathcal{B}(\Sigma)) \to \Phi(T\mathcal{A}(\Sigma) \wedge T\mathcal{B}(\Sigma)) \to \Phi(T(\mathcal{A} \wedge \mathcal{B})(\Sigma)^t)$

(iii) If X , Y are spectra, then there exists a natural map $\Phi(X) \wedge \Phi(Y) \to \Phi(X \wedge Y)$ of Γ-spaces where the Γ-space on the left is the one defined using the smash product in 1.2. The smash product $X \wedge Y$ on the right is defined as in 1.5.2 .

PROOF. First, it is shown in [Lyd] section 4 that, under the above hypotheses, there exists a natural map $\mathcal{A}(\Sigma) \wedge \mathcal{B}(\Sigma) \to (\mathcal{A} \wedge \mathcal{B})(\Sigma)^t$ of spectra with no odd terms. We apply the geometric realization followed by the singular functor degree-wise to obtain the pairing:

$Sing(|A(\Sigma)|) \wedge Sing(|\mathcal{B}(\Sigma)|) \to Sing(|(\mathcal{A} \wedge \mathcal{B})(\Sigma)^t|)$.

Now we apply the functor T' to both sides. Clearly there exists a natural map $T'(Sing(|A(\Sigma)|)) \wedge T'(Sing(|\mathcal{B}(\Sigma)|)) \to T'(Sing(|A(\Sigma)|) \wedge Sing(|\mathcal{B}(\Sigma)|))$. Now the definition of the functor T as above completes the proof of (i).

We will next consider (iii). We first show that, in order to establish (iii), it suffices to show that if n, m are two non-negative integers, there exists a natural map:

$$\Phi(X)(n^+) \wedge \Phi(Y)(m^+) \to \Phi(X \wedge Y)(n^+ \bigvee m^+) \tag{1.5.3}$$

To see this recall that

$$(\Phi(X) \wedge \Phi(Y))(p^+) = \operatorname*{colim}_{n^+ \wedge m^+ \to p^+} \Phi(X)(n^+) \wedge \Phi(Y)(m^+).$$

Therefore, in order to prove (iii), it suffices to show that for each map $n^+ \wedge m^+ \to p$ in Γ^{op}, there exists an induced map

$$\Phi(X)(n^+) \wedge \Phi(Y)(m^+) \to \Phi(X \wedge Y)(p^+) \tag{1.5.4}$$

Observe there exist natural maps $n^+ \bigvee m^+ \to n^+ \wedge m^+$ and $n^+ \wedge m^+ \to p^+$. Therefore the map in 1.5.4 may be obtained by pre-composing the map $\Phi(X \wedge Y)(n^+ \bigvee m^+) = Map(\Sigma^{n^+ \vee m^+ - +}, X \wedge Y) \to Map(\Sigma^{n^+ \wedge m^+ - +}, X \wedge Y) = \Phi(X \wedge Y)(n^+ \wedge m^+) \to Map(\Sigma^{p_+}, X \wedge Y) = \Phi(X \wedge Y)(p^+)$ with the map in 1.5.3. This shows that it suffices to prove 1.5.3. Now given two maps $f : \Delta[k]_+ \wedge \Sigma^n \to X$ and $g : \Delta[k]_+ \wedge \Sigma^m \to Y$, we may define a map $f \wedge g : \Delta[k]_+ \wedge \Sigma^{n+m} \to X \wedge Y$ as the composition:

$$\Delta[k]_+ \wedge \Sigma^n \wedge \Sigma^m \stackrel{\Delta \wedge id}{\to} \Delta[k]_+ \wedge \Delta[k]_+ \wedge \Sigma^n \wedge \Sigma^m \stackrel{f \wedge Y}{\to} X \wedge Y$$

This proves (iii).

Now consider (ii). By (iii) applied to $X = T\mathcal{A}(\Sigma)$ and $Y = T\mathcal{B}(\Sigma)$, we see that there exists a natural map $\Phi(T(\mathcal{A}(\Sigma))) \wedge \Phi(T(\mathcal{B}(\Sigma))) \to \Phi(T(\mathcal{A}(\Sigma)) \wedge T(\mathcal{B}(\Sigma)))$. This provides the first map in (ii). Combining this with the pairing $T(\mathcal{A}(\Sigma)) \wedge T(\mathcal{B}(\Sigma)) \to T(\mathcal{A} \wedge \mathcal{B})(\Sigma)^t$, we obtain the second map in (ii). □

Now we may complete the proof that (M5) is satisfied by the functor Q^{st}. Recall $Q^{st}(\mathcal{A}) = \Phi(T\mathcal{A}(\Sigma))$. Therefore $Q^{st}(\mathcal{A}) \wedge Q^{st}(\mathcal{B}) = \Phi(T\mathcal{A}(\Sigma)) \wedge \Phi(T\mathcal{B}(\Sigma))$ maps naturally to $\Phi(T\mathcal{A}(\Sigma) \wedge T\mathcal{B}(\Sigma))$. By (ii) of 1.3, the latter maps naturally to $\Phi(T(\mathcal{A} \wedge \mathcal{B})(\Sigma)^t) = Q^{st}(\mathcal{A} \wedge \mathcal{B})$.

Next we verify the axioms on the strong t-structure as in Chapter I. (ST1) through (ST5). We will first consider (ST3) and (ST4). The category $\mathcal{C}_f^{\leq n \geq}$ is given by the subcategory $\{\mathcal{A} | \mathcal{A} \quad fibrant \quad and \quad \pi_k(\mathcal{A}) = 0, k \neq n\}$ of Γ-spaces. The Abelian category $\mathbf{A}$ is in fact the category of all Abelian groups. Let π denote an Abelian group. Now we consider the chain complex $\pi[n]$ which is concentrated in degree n. We may denormalize this to obtain a simplicial Abelian group $DN(\pi[n])$ which has only one homotopy group that is non-trivial, namely in degree n, and where it is π. We may deloop this simplicial Abelian group to obtain a connected spectrum: $Sp(\pi) = \{Sp(\pi)_m = B^m(DN(\pi[n])) | m \geq 0\}$. This spectrum will be denoted $K(\pi, n)$. Now we apply the functor Φ to this spectrum to obtain a Γ-space which we denote by $EM_n(\pi)$. Clearly $\pi_n(EM_n(\pi)) \cong \pi$ and $\pi_k(EM_n(\pi)) \cong 0$ if $k \neq n$. Moreover the above definition of the functors EM_n, $n \varepsilon \mathbb{Z}$ shows that if π, π' are object in $\mathbf{A}$, there exists a natural map $K(\pi, n) \wedge K(\pi', m) \to K(\pi \otimes \pi', n+m)^t$ of spectra where the smash-product on the left is defined as in 1.5.2. Now 1.3 (ii) applies to show that there exists a pairing:

$EM_n(\pi)\wedge EM_m(\pi') = \Phi(K(\pi,n))\wedge\Phi(K(\pi',m)) \to \Phi(K(\pi\otimes\pi',n+m)) = EM_{n+m}(\pi\otimes\pi')$. This proves (ST6).

Next recall that the category of all connected abelian group spectra is equivalent to the category of all chain complexes of abelian groups that are trivial in negative degrees. Moreover the category of all such abelian group spectra is equivalent to a corresponding subcategory of the category of all Γ-spaces which we call *abelian* Γ*-spaces.* These observations prove (ST3) and (ST4).

One may show axiom (ST9) holds by an argument as in Appendix II, (0.7.3). Now we consider the axioms (ST1), (ST2), (ST5), (ST7) and (ST8). We begin by recalling the functorial Postnikov truncation defined for fibrant simplicial sets. Let X denote a *fibrant* pointed simplicial set and let $n \geq 0$ denote an integer. We let P_nX be the simplicial set defined by

$(P_nX)_k = X_k$ if $k < n$ and $= X_k/\,\tilde{}$ if $k \geq n$.

Here $\tilde{}$ denotes the equivalence relation where two k-simplices of X are identified if their $n-1$-dimensional faces are all identical. Clearly there is a natural map $X \to P_nX$ of pointed simplicial sets and $X \to P_nX$ defines a functor on fibrant pointed simplicial sets and pointed maps. We let $\tilde{P}_nX =$ the fiber of the map $X \to P_nX$. Now one observes that $\pi_k(\tilde{P}_nX) \cong \pi_k(X)$ if $k \geq n$ and $\cong 0$ otherwise. Observe also that $(\tilde{P}_nX)_k = *$, $k < n$ and $= \{x_k \varepsilon X_k |$ all the $(n-1)$-dimensional faces of x_k are trivial$\}$. *As a consequence we may characterize* $|\tilde{P}_nX|$ *as the maximal pointed sub-space of* $|X|$ *having no cells except the base point in degrees* 0 *through* $n-1$. In general (i.e. in case X is not a fibrant simplicial set), one may define $\tilde{P}_nX = \tilde{P}_n(Sing|X|)$.

One may also observe that if $n < 0$, $\tilde{P}_n(Sing|X|) = Sing|X|$ and that $\underset{n}{\cap}\tilde{P}_n(Sing|X|) = *$.

Now we will extend the functor $\tilde{P}_n$ to spectra. (For this purpose, the definition we adopt needs to use the geometric realization and the singular functor; the only way to avoid this seems to be by adopting a different notion of smash product of pointed simplicial sets as in [Kan]. However this would then mean a reworking of all the foundational material on spectra and Γ-spaces that use the more familiar notion of smash products of pointed spaces. Even if one is willing to do so, the feasibility of this approach is doubtful.) Let $X = \{X^m | m \geq 0\}$ denote a degree-wise fibrant spectrum i.e. each X^m is a fibrant pointed simplicial set. Now we define

$\tilde{P}_nX = \{Sing(|\tilde{P}_{n+m}X^m|) | m \geq 0\}$.

The structure map $S^1 \wedge Sing(|\tilde{P}_{n+m}X^m|) \to Sing(|\tilde{P}_{n+m+1}X^{m+1}|)$ is defined as follows. First such a map is adjoint to a map $|S^1 \wedge Sing(|\tilde{P}_{n+m}X^m|) \cong |S^1| \wedge |Sing(|\tilde{P}_{n+m}X^m|) \to |\tilde{P}_{n+m+1}X^{m+1}|$. (See [G-Z] p.47 to see the isomorphism above.) Now $|S^1|\wedge|Sing(|\tilde{P}_{n+m}X^m|)$ maps naturally to $|S^1|\wedge|\tilde{P}_{n+m}X^m|$. Clearly the latter maps into $|S^1|\wedge|X^m| \cong |S^1\wedge X^m| \to |X^{m+1}|$ where the last map is the given map $S^1 \wedge X^m \to X^{m+1}$. We will show that the last map factors through the natural map $|\tilde{P}_{n+m=1}X^{m+1}| \to |X^{m+1}|$. To see this observe that $|S^1|\wedge|\tilde{P}_{n+m}X^m|$ is isomorphic to a space with no cells in degrees less than $n+m$ except the base point and that $|\tilde{P}_{n+m+1}X^{m+1}|$ is the maximal subspace of $|X^{m+1}|$ with no cells except the base-point in degrees less than $n+m$. The required factorization follows and provides the structure maps of the spectrum $\tilde{P}_nX$. In case X is not a degree-wise fibrant spectrum, we will first apply the functor T to convert it to a degree-wise fibrant Ω-spectrum.

If $\mathcal{A}$ is a Γ-space and n is an integer, we will define

$\tau_{\leq n}\mathcal{A} = \Phi(\tilde{P}_{-n}T(\mathcal{A}(\Sigma)))$

One may readily see that both (ST1) and (ST2) are now satisfied. (The exhaustiveness and separatedness of the filtration will follow readily from 1.5 .) We proceed to verify the axiom (ST7). For this we begin with a pairing $K \wedge L \to M$ of fibrant pointed simplicial sets and let $n,\ m \geq 0$ be two integers. Now consider $|\tilde{P}_nK| \wedge |\tilde{P}_mL|$. In general this may not have the structure of a $C.W$-complex; however, by [G-Z] p.53, $|\tilde{P}_nK| \wedge |\tilde{P}_mL| \cong (|\tilde{P}_nK| \wedge |\tilde{P}_mL|)_{Kelley}$ which is the same underlying set re-topologized using the Kelley topology. On applying the singular functor, we therefore obtain a map $Sing((|\tilde{P}_nK| \wedge |\tilde{P}_mL|)_{Kelley}) \to Sing(|\tilde{P}_nK \wedge \tilde{P}_mL|)$. Now there exists a natural map $(Sing|\tilde{P}_nK|) \wedge (Sing|\tilde{P}_mL) \to Sing((|\tilde{P}_nK| \wedge |\tilde{P}_mL|)_{Kelley})$. (Such a map is adjoint to a map : $|(Sing|\tilde{P}_nK|) \wedge (Sing|\tilde{P}_mL)| \cong (|(Sing|\tilde{P}_nK|)| \wedge |(Sing|\tilde{P}_mL|)|)_{Kelley} \to (|\tilde{P}_nK| \wedge |\tilde{P}_mL|)_{Kelley}$. This map clearly exists since the geometric realization functor is left adjoint to the singular functor.) As a result we have obtained a map:

$$(Sing|\tilde{P}_nK|) \wedge (Sing|\tilde{P}_mL|) \to Sing(|\tilde{P}_nK \wedge \tilde{P}_mL|) \cong Sing((|\tilde{P}_nK| \wedge |\tilde{P}_mL|)_{Kelley})$$

Clearly $|\tilde{P}_nK \wedge \tilde{P}_mL|$ maps naturally to $|K \wedge L|$ which maps to $|M|$ using the given pairing. We proceed to show this factors through $|\tilde{P}_{n+m+1}M|$. For this we will consider sub-spaces of the space $|\tilde{P}_nK \wedge \tilde{P}_mL|$ of the form $|F_1| \wedge |F_2|$, where F_1 is a countable pointed sub-simplicial set of $\tilde{P}_nK$ and F_2 is a countable pointed sub-simplicial set of $\tilde{P}_mL$. Now $|F_1| \wedge |F_2|$ has the structure of a $C.W$-complex; since F_1 is trivial in degrees less than n and F_2 is trivial in degrees less than m, it follows that $|F_1| \wedge |F_2|$ has no cells except the base-point in degrees less than $n+m$. Now consider $|\tilde{P}_{n+m}M|$. This is the maximal sub-space of $|M|$ having no cells except the base-point in degrees less than $n+m$. Therefore the natural map $|F_1| \wedge |F_2| \to |M|$ factors through $|\tilde{P}_{n+m}M|$.

Now consider the natural map $|\tilde{P}_nK| \wedge |\tilde{P}_mL| \xrightarrow{\phi} |M|$. The above argument shows that for every countable pointed sub-simplicial set F_1 of $\tilde{P}_nK$ and F_2 of $\tilde{P}_mL$, $\phi(|F_1| \wedge |F_2|)$ has no cells except the base-point in degrees less than $n+m$ and hence is contained in $|\tilde{P}_{n+m}M|$. Therefore the same conclusion holds for $\bigcup_{F_1,F_2} \phi(|F_1| \wedge |F_2|)$. It follows that the map $|\tilde{P}_nK| \wedge |\tilde{P}_mL| \to |M|$ factors through the natural map $|\tilde{P}_{n+m}M| \to |M|$.

Now consider a pairing $K \wedge L \to M$ of degree-wise fibrant spectra. (i.e. we may view $K \wedge L$ as a spectrum with no odd terms and we have a map from this to the spectrum with no odd-terms associated to M.) The above arguments show that if $n,\ m$ are two integers, one obtains a pairing:

$\tilde{P}_nK \wedge \tilde{P}_mL \to \tilde{P}_{n+m}$.

These readily show that if $\mathcal{A} \wedge \mathcal{B} \to \mathcal{C}$ is map of Γ-spaces, and $n,\ m$ are two integers, one obtains an induced pairing of Γ-spaces:

$\tau_{\leq n}\mathcal{A} \otimes \tau_{\leq m}\mathcal{B} \to \tau_{\leq n+m}\mathcal{C}$ and $F_n(Q^{st}(\mathcal{A})) \otimes F_m(Q^{st}(\mathcal{B})) \to F_{n+m}(Q^{st}(\mathcal{C}))$.

On taking the associated graded terms of the Cartan filtrations, one obtains (ST7). It remains to verify the axioms (ST5) and (ST8): these are established in the following proposition.

PROPOSITION 1.4. *(i) Let X denote a spectrum. Let n denote an integer so that $\pi_i(X) \cong 0$ for all $i \neq n$. Now there exists natural maps of presheaves $X \to \mathbb{Z}X[-\infty,\ n] \to K(\pi_n(X), n)$ of abelian spectra which are weak-equivalences. (Here $[-\infty,\ n]$ is the* *functorial Postnikov-truncation that kills the homotopy above degree n defined as the homotopy cofiber* $\tilde{P}_{n-1}(X) \to Sing|X|$.*)*

(ii) If $\mathcal{A}$ *is a gamma-space so that* $\pi_i(\mathcal{A}) = 0$ *if* $i \neq n$*, there exists a natural map* $\mathcal{A} \to EM_n(\pi_n(\mathcal{A}), n)$ *which is a stable weak-equivalence.*

(iii) If $\mathcal{A} \wedge \mathcal{B} \to \mathcal{C}$ *is a pairing of* Γ*-spaces, the induced pairings* $Gr_{\mathcal{C}}(Q^{st}(\mathcal{A})) \wedge Gr_{\mathcal{C}}(\mathcal{B}) \to Gr_{\mathcal{C}}(\mathcal{C})$ *and* $\pi_*(\mathcal{A}) \otimes \pi_*(B) \to \pi_*(\mathcal{C})$ *are compatible under the map in (ii).*

PROOF. The existence of the first map in (i) is clear; that it is a weak-equivalence follows from an application of the Hurewicz theorem. Since the last two are presheaves of abelian group spectra, these are both of the form $A = \{A^n|n\}$, where each A^n is a simplicial abelian group and one is given maps $A^n \to Map(S^1, A^{n+1})$ of simplicial abelian groups, for all n. The above map is adjoint to a map $A^n \otimes S^1 \to A^{n+1}$ of simplicial abelian groups, where $(A^n \otimes S^1)_k = \underset{(S^1)_k}{\oplus} (A^n)_k$. Observe that on taking the normalizations, one obtains the map $N(\mathbb{Z}(S^1)) \otimes N(A^n) \to N(A^{n+1})$ of chain-complexes. Observe that $N(\mathbb{Z}(S^1))$ is the chain-complex with $\mathbb{Z}$ in degree 1 and 0 elsewhere. One may view tensoring with this complex as a suspension functor for chain complexes.

Using the normalization functor, one may now view both $\mathbb{Z}X[-\infty, n]$ and $K(\pi_n(X), n)$ as systems of complexes $\{A^n|n\}$ of abelian groups commuting with the above suspension. Let D denote such a chain complex. Now the Cartan filtration on D may be identified with $(D.[m, \infty])_j = Ker(d : D_j \to D_{j-1})$ if $j = m$, $= D_j$ if $j > m$ and $= 0$ otherwise. Moreover $\pi_i(D.) = ker(d : D_i \to D_{i-1})/Im(d : D_{i+1} \to D_i)$ which is a quotient of $(D.[i, \infty]/D.[i+1, \infty]) \cong D.[i, i]$. It follows that the existence of the last map in the lemma is clear when the simplicial abelian groups are replaced by their associated chain complexes. We may therefore apply the denormalization functor (see Appendix II, (0.1)) finally to obtain the required map. That it is a weak-equivalence is clear. This proves (i). To obtain (ii) we take X in (i) to be $T(\mathcal{A}(\Sigma))$. Finally apply the functor Φ to the map $T(\mathcal{A}(\Sigma)) \to K(\pi_n(X), n)$ to obtain a weak-equivalence: $\mathcal{A}(\Sigma) \to \Phi(T(\mathcal{A}(\Sigma))) \to EM_n(\pi_n(X))$.

Now we consider (iii). Using the functor T as before, we may first assume that X, Y and Z are fibrant Ω-spectra and that there is a pairing $X \wedge Y \to Z^t$ as in 1.5.2. This induces a pairing $Gr_{\mathcal{C}}(X) \wedge Gr_{\mathcal{C}}(Y) \to Gr_{\mathcal{C}}(Z^t)$ as in 1.5.2 and also a pairings $\mathbb{Z}(Gr_{\mathcal{C}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}(Gr_{\mathcal{C}}(Y)) \to \mathbb{Z}(Gr_{\mathcal{C}}(Z))$. As in the proof of (i), the latter pairing may be interpreted as a pairing of chain-complexes commuting with the suspension $\otimes\mathbb{Z}(S^1)$. Therefore one may readily verify that the above pairing is compatible with the natural map to $GEM(\pi_*(X)) \otimes GEM(\pi_*(Y)) \to GEM(\pi_*(Z))$. □

These complete the verification of the axioms for the case of Γ*-spaces.*

2. The axioms for symmetric spectra

We will prove the following theorem .

THEOREM 2.1. *The category of symmetric spectra with the smash product of symmetric spectra defines an enriched stable simplicial model t-category.*

PROOF. Since many of the arguments are similar to that of Γ-spaces we will verify the axioms rather briefly. We define the strict (partial) model structure as follows. A map $f : X = \{X^n|n\} \to Y = \{Y^n|n\}$ of symmetric spectra is a strict cofibration (strict weak-equivalence) if for each n, the map $f^n : X^n \to Y^n$ is a co-fibration (weak-equivalence, respectively). The fibrations defined by the right lifting property with respect to cofibrations that are also weak-equivalences. This defines a strict simplicial model structure. Now axioms (PM1), (PM2) and (PM3) follow, while (PM4) is clear. In fact every monomorphism is a strict cofibration.

The stable cofibration, stable fibrations and stable weak-equivalences are defined as in [H-S-S] section (3.4). This defines the structure of a simplicial model category on symmetric spectra. It is shown in [H-S-S] (See Propositions (3.3.8) and (3.4.3)) that every strict weak-equivalence is a stable weak-equivalence and every stable cofibration is a strict cofibration. To show every stable fibration is a strict fibration, we will simply observe that the free-functor Fr_n left-adjoint to the evaluation functor Ev_n, send cofibrations (weak-equivalences) of pointed spaces to stable cofibrations (stable weak-equivalences, respectively) of symmetric spectra. We have essentially verified the axioms (SM1) through (SM3.3). Observe from [H-S-S] Corollary (3.4.1.3) that a degree-wise fibrant symmetric spectrum is stably fibrant if and only if it is an Ω-spectrum. Now we may let $Q = Sing \circ |\ \ |$ extended to symmetric spectra; $(Q^{st}X)^n = \operatorname*{colim}_m \Omega^{n+m}(QX)^m$.

The usual smash product functor between pointed spaces extends to define the operation $\otimes$ in (SM3.4). As in the case of Γ-spaces, all of the axiom (SM3.4) are direct consequences of the simplicial model structures provided by the strict and stable simplicial model structures. Since this does not appear in [H-S-S] we will sketch an argument to show that the stable structure on symmetric spectra is in fact a simplicial model category structure. Recall we have defined stable cofibrations to be the ones with left lifting property for all degree-wise fibrations that are also strict weak-equivalences. As in [B-F] p. 84 one may now see readily that a map $i : K \to L$ of symmetric spectra is a stable cofibration if and only if the maps $K^{n+1} \underset{S^1\wedge K^n}{\sqcup} S^1 \wedge L^n \to L^{n+1}$ is a level cofibration. Now one may readily verify the axiom denoted (SM7)(b) in [Qu]. Let $L \to K$ denote a cofibration of finite pointed simplicial sets and let $A \to B$ denote a stable cofibration of symmetric spectra. Now we need to show that the induced map

$$K \wedge A \underset{L\wedge A}{\sqcup} L \wedge B \to K \wedge B$$

is a *stable cofibration* which is also a stable weak-equivalence if the map $A \to B$ is a stable weak-equivalence. One may check the first assertion readily using the characterization of stable cofibrations mentioned above. To show that the above map is a stable weak-equivalence, one may consider the commutative diagram:

$$\begin{tikzcd}
K \wedge A \arrow[r] \arrow[d, "id"] & K \wedge A \underset{L\wedge A}{\sqcup} \wedge B \arrow[r] \arrow[d, "\simeq"] & (B/A) \wedge L \arrow[d, "\simeq"] \\
K \wedge A \arrow[r] & K \wedge B \arrow[r] & (B/A)K
\end{tikzcd}$$

The two rows are distinguished triangles in the sense of section 1. If $A \to B$ is a stable cofibration which is also a stable weak-equivalence, B/A is stably weakly-equivalent to $*$ and hence so are $B/A \wedge L$ and $B/A \wedge K$. It follows therefore that the middle map is also a stable weak-equivalence. Now the axiom (SM6) and (SM7) follow readily as in the case of Γ-spaces. (See [H-S-S] (3.3.11).)

The tensor product $\otimes$:(symmetric spectra) $\times$(symmetric spectra) $\to$(symmetric spectra) is defined by the symmetric smash product over the symmetric sphere spectrum defined in [H-S-S](2.2.3). Now (M1) and (M3) are clear. Observe that the unit in (M3) is now the symmetric sphere spectrum. The functors Q and Q^{st} are straightforward adaptations of the corresponding functors for spectra. i.e. $Q = Sing \circ |\ \ |$ extended to symmetric spectra and $Q^{st} = T' \circ Q$ where T' is the functor considered in (4.8). To see (M2) we proceed as follows. Let Fr: (pointed spaces) →(symmetric spectra) be the functor considered in [H-S-S](3.4.1). Now $Fr(X)$ is a stably cofibrant symmetric spectrum for any pointed space X. Now axiom (M2) holds if A or B is of the form $Fr(X)$: $Fr(X) \wedge - \simeq X \otimes -$ where $\otimes$ on the right is the tensor product defined in [H-S-S](2.1.3). Clearly the latter preserves stable weak-equivalences and stable cofibrations. To complete the proof of (M2) it suffices to show that the class of symmetric spectra that satisfy (M2) is closed retractions and under transfinite compositions of maps (as in [Sch] appendix A) that are stable cofibrations. This follows readily if one observes that filtered colimits are in fact homotopy colimits. (See [B-K] p. 332.)

Next we consider the axioms on the t-structure. The functors $\tau_{\leq n}$ are defined as follows.

$\tau_{\leq n} X = \tilde{P}_{-n} T(X)$, X = a symmetric spectrum

To see this applies to symmetric spectra, observe that $K \to \tilde{P}_n K$ is a functor from pointed fibrant simplicial sets to pointed fibrant simplicial sets, for each $n \geq 0$. Therefore, if a (symmetric) group Σ_k acts on K $\tilde{P}_n K$ has an induced action by Σ_k. Now it is clear that (ST1) and (ST2) are satisfied. (The exhaustiveness and separatedness of the filtration follows as in 1.5 .) To see that (ST7) is also satisfied, it suffices to make the following observations. Let $X = \{X^n | n \geq 0\}$, $Y = \{Y^n | n \geq 0\}$ and $Z = \{Z^p | p\}$ denote symmetric spectra which are degree-wise fibrant and let k, l denote two fixed integers. Assume there exists a pairing $X \otimes Y \to Z$ where $\otimes$ now denotes the tensor-product of symmetric spectra as in [H-S-S] (2.1.3). Recall $(X \otimes Y)_p = \underset{n+m=p}{\wedge} \Sigma_p^+ \underset{\Sigma_n \times \Sigma_m}{\wedge} (X_n \wedge Y_m)$ As observed above, $\tilde{P}_{k+n} X_n \wedge \tilde{P}_{l+m} X_m$ has an induced action by the group $\Sigma_n \times \Sigma_m$ and so does $\tilde{P}_{k+l+n+m}(Z_{n+m})$ so that the induced map (as in 1.5) $\tilde{P}_{k+n} X_n \wedge \tilde{P}_{l+m} X_m \to \tilde{P}_{k+l+n+m}(Z_{n+m})$ is $\Sigma_n \times \Sigma_m$-equivariant. This shows that axiom (ST7) is satisfied if we use the tensor product of symmetric spectra as in [H-S-S] (2.1.3). Recall the smash product of the symmetric spectra X and Y is defined as the co-equalizer of the two maps $X \otimes S \otimes Y \underset{\nu \otimes id}{\overset{id \otimes \mu}{\rightrightarrows}} X \otimes Y$. Since taking the associated graded terms of a filtration commute with respect to taking co-equalizers, we obtain (ST7).

In (ST4) we take $\mathbf{A}$ to be the whole category of all abelian groups. Now we define, in outline, the functors EM_n. First we consider a different suspension for simplicial abelian groups. Let $K(\mathbb{Z}, 1)$ denote the simplicial abelian group obtained by denormalizing the chain complex $\mathbb{Z}[1]$ concentrated in degree 1. Now observe that there exists a natural map from the simplicial sphere to $K(\mathbb{Z}, 1)$, since $\pi_1(S^1) = \mathbb{Z}$. Moreover if A denotes a simplicial abelian group, there exists natural maps $S^1 \wedge A \to K(\mathbb{Z}, 1) \wedge A \to K(\mathbb{Z}, 1) \otimes A$ where $K(\mathbb{Z}, 1) \otimes A$ denotes the degree-wise tensor product of the two simplicial abelian groups $K(\mathbb{Z}, 1)$ and A. One may readily verify that $K(\mathbb{Z}, 1) \otimes A \simeq BA$: we view this as the suspension functor for simplicial abelian groups. Starting with a simplicial abelian group A one now obtains the suspension spectrum $\{K(\mathbb{Z}, 1)^{\otimes n} \otimes A | n\}$. We may view this as a symmetric spectrum by letting the symmetric group Σ_n act on $K(\mathbb{Z}, 1)^{\otimes n} \otimes A$ by letting it act on $K(\mathbb{Z}, 1)^{\otimes n}$ by permuting the n-factors in the tensor product. So defined we obtain a functor

Sp:(simplicial abelian groups) →(symmetric spectra)

One may readily see that this functor is *faithful.* The functor EM_n is the restriction of the above functor to the sub-category of simplicial abelian groups of the form $DN(\pi[n])$, where π is an abelian group, $\pi[n]$ the corresponding complex concentrated in degree n, and DN is the denormalization functor sending a chain-complex to a simplicial abelian group. This proves (ST4). The functor $Sp \circ DN$ now defines a faithful functor from the category of all chain complexes that are trivial in negative degrees to that of symmetric spectra. One may readily extend this functor to a faithful functor from the category of all unbounded chain-complexes of abelian groups to the category of symmetric spectra - however the details are skipped. This proves (ST3). The remaining properties, (ST5), (ST6) and (ST8) are established by arguments very similar to those in the case of Γ-spaces and are therefore skipped. Appendix I, (0.7.3) readily adapts to the case of symmetric spectra to prove axiom b(ST9). □

2.1. The spectra of K-theory. We conclude this section by making some observations that the spectra of algebraic and topological K-theory are in fact symmetric spectra.

THEOREM 2.2. *(Geisser-Hasselholt: see [G-H]) Let S denote a category with cofibrations and weak-equivalences in the sense of Waldhausen (i.e. [Wald]). Then the associated K-theory spectrum $K(S)$ is a symmetric spectrum. If, in addition, S is also a symmetric monoidal category with a tensor product that preserves cofibrations and weak-equivalences in both arguments, the associated K-theory spectrum $K(S)$ is an algebra in the category of symmetric spectra (i.e. a ring object so that the unit map from the sphere spectrum is a map of ring objects).*

COROLLARY 2.3. *It follows that if X is an algebraic variety or a scheme, the spectrum of the K-theory of vector bundles on X is an algebra in the category of symmetric spectra. If X is a suitable topological space, the spectrum of topological complex K-theory on X is also an algebra in the category of symmetric spectra.*

2.2. Completions of symmetric spectra. Often, especially in considering presheaves of spectra on the étale site of schemes, it will become necessary to assume that their presheaves of homotopy groups are all l-primary torsion, for a prime l different from the residue characteristics. However the common operation of smashing with a Moore-spectrum often does not preserve the category of ring spectra. Therefore, it will be necessary to perform completions in the sense of [B-K] or localizations. The following result shows this is possible.

THEOREM 2.4. *(i) Completions (and localizations) at a set of primes in the sense of [B-K] extend to symmetric spectra and preserve the sub-category of ring spectra.*

(ii) Moreover, if R is a symmetric ring spectrum and R_l denotes its completion at the prime l, the functor of completion at l sends the category of module spectra over R to the category of module-spectra over R_l.

(iii) If R is an E^∞ ring object in the category of symmetric spectra, its l-completion is also an E^∞-object in the category of symmetric spectra.

PROOF. (i) The first assertion follows from [Hirsch] section 3. The main observation is that one may construct a new model category structure on the same underlying category of symmetric spectra, where weak-equivalences are replaced by weak-equivalences on the l-completions. The cofibrations will be the same as in the original category and the fibrations will be defined by lifting property with respect to cofibrations that are also weak-equivalences on l-completions. Since the underlying category is the same as the original one, namely the category of symmetric spectra, it follows that the completion functor sends symmetric spectra to symmetric spectra.

Now we consider the second assertion in (i). For this we need to recall some results in [SS]. Accordingly a ring object in the category of symmetric spectra is an algebra over the monad (or triple) defined by:

$T(K) = S \sqcup K \sqcup K^{\wedge 2} ... \sqcup K^{\wedge (n)}..$

where $\wedge$ denotes the smash product of symmetric spectra and S is the sphere spectrum. Now the observation that l-completion has the property $(A \wedge B)_l = A_l \wedge B_l$ shows that $T(K)_l = S_l \sqcup K_l \sqcup K_l^{\wedge 2} ... \sqcup K_l^{\wedge (n)} ...$ (The above property of the completion may be checked using the definition of the smash product of symmetric spectra and will ultimately reduce to showing the l-completion commutes with the smash product of two pointed simplicial sets.) Therefore, if R is a ring object, in the category of symmetric spectra, so is its l-completion. This proves the second statement in (i). (ii) may be checked easily in a similar manner.

No we consider (iii). For this we proceed as above replacing the monad by the monad $T(K) = \mathbf{1} \sqcup K \sqcup E\Sigma_2 \underset{\Sigma_2}{\times} K^{\wedge 2} \sqcup ... \sqcup \underset{\Sigma_n}{\times} K^{\wedge n} ...$

taking into consideration the action of the symmetric group Σ_n as well. An E^∞-ring object may be identified with an algebra over this monad. □

3. Presheaves with values in Γ-spaces, symmetric spectra

Observe that all our constructions in the last two sections were functorial. Therefore, they extend to presheaves on any site satisfying the basic hypotheses as in Chapter II. i.e. we obtain the following theorem.

THEOREM 3.1. *Let $\mathfrak{S}$ denote a site as in Chapter I. Let $Presh(\mathfrak{S})$ denote the category of presheaves of Γ-spaces or symmetric spectra on the site $\mathfrak{S}$. Then the category $Presh(\mathfrak{S})$ satisfies all the axioms of Chapter I.*

REMARK 3.2. The category of spectra in the $\mathbb{A}^1$-local category of simplicial presheaves on the big Zariski or Nisnevich site of schemes of finite type over a Noetherian base scheme (using a suitable suspension functor) satisfies many of the axioms of Chapter I. See Chapter VI for a brief discussion of this.

APPENDIX B

Chain complexes and simplicial objects

Since the functor Sp appearing in Definition 4.6 of Chapter I is defined in terms of a homotopy inverse limit, one has to first replace this (upto weak-equivalence) by a suitable homotopy direct limit so that it will pull out of the $Hom_{Sp(\pi_*(E))}$-functor above, so that the axiom (ST9) will be satisfied. In an abelian category, this is clearly feasible provided the above homotopy inverse limit is a homotopy inverse limit of a finite diagram; nevertheless, for our purposes it is necessary to obtain the precise relationship between these homotopy inverse limits and direct limits and relate them to the *total-complex-construction.* Much of the work in the first few sections are expended in this direction. While the results we obtain are probably well-known and part of the folklore, many of the details do not exist in the literature. (See [T-1] (4.2.32) for a brief discussion.) One may skip the details and only read the main results, which are Proposition 0.3, Lemma 0.4, 0.7 and 0.6.3. (See 0.8 for a technique that is used often in this section.)

0.1. Let $\mathbf{A}$ denote an abelian category closed under all small limits and colimits and where filtered colimits are exact; a chain complex $K_\bullet$ (co-chain complex $K^\bullet$) in $\mathbf{A}$ will denote a sequence $K_i \varepsilon \mathbf{A}$ ($K^i \varepsilon \mathbf{A}$) provided with maps $d : K_i \rightarrow K_{i-1}$ ($d : K^i \rightarrow K^{i+1}$) so that $d^2 = 0$. Let $C_+(\mathbf{A})$ ($C^+(\mathbf{A})$) denote the category of chain complexes (co-chain complexes, respectively) in $\mathbf{A}$ that are trivial in negative degrees. One defines denormalizing functors: $DN_\bullet : C_+(\mathbf{A}) \rightarrow (Simplicial\ objects\ in\ \mathbf{A})$ and $DN^\bullet : C^+(\mathbf{A}) \rightarrow (Cosimplicial\ objects\ in\ \mathbf{A})$ as in [Ill] pp. 8-9. $DN_\bullet$ will be inverse to the normalizing functor:
$N : (Simplicial\ objects\ in\ \mathbf{A}) \rightarrow C_+(\mathbf{A})$ defined by $(NK_\bullet)_n = \underset{i \neq 0}{\cap} ker(d_i : K_n \rightarrow K_{n-1})$ with $\delta : (NK)_n \rightarrow (NK)_{n-1}$ being induced by the map d_0. $DN^\bullet$ will be inverse to the functor $N : (Cosimplicial\ objects\ in\ \mathbf{A}) \rightarrow C^+(\mathbf{A})$ defined by $(NK^\bullet)^n = \underset{i \neq 0}{+} coker(d^i : K^n \rightarrow K^{n+1})$ with $\delta : (NK)^n \rightarrow (NK)^{n+1}$ induced by d^0. A map $f : K'_\bullet \rightarrow K_\bullet$ of simplicial objects in $\mathbf{A}$ will be called a *weak-equivalence* if it induces an isomorphism on the associated homology objects. To simplify our discussion, we may assume that $\mathbf{A}$ is in fact the category of all abelian groups.

0.2. The definition of the normalization and denormalizing functors work also in more general settings. If $\mathbf{C}$ is a category with a zero-object with finite limits and colimits one may define a chain complex in $\mathbf{C}$ to be a sequence of objects $\{K_i|i\}$ in $\mathbf{C}$ provided with maps $K_i \overset{d}{\rightarrow} K_{i-1}$, $i \geq 1$ so that $d^2 = 0$. Co-chain complexes may be defined similarly. For chain complexes (co-chain complexes) that are trivial in negative degrees one may define denormalizing functors that produce simplicial (cosimplicial, respectively) objects by the same formulae. There are also normalizing functors defined similarly. For us, the important case will be when $\mathbf{C}$ is the category of spectra. In this case the results in [Rect] show that the normalizing and denormalizing functors are weak-inverses.

0.3. In addition we will often view a simplicial object $K_\bullet$ (cosimplicial object $K^\bullet$) in $\mathbf{A}$ as a chain complex (co-chain complex, respectively) with the differential given by

$\delta = \Sigma_{i=0}^{i=n} d_i : K_n \to K_{n-1}$ ($\delta = \Sigma_{i=0}^{i=n} d^i : K^{n-1} \to K^n$, respectively). (Now $N(K_\bullet)$ ($N(K^\bullet)$) is a sub-complex of the complex $K_\bullet$ ($K^\bullet$, respectively) with the above differential.)

Let $K_{\bullet\bullet}$ denote a double chain-complex in **A** that is trivial in negative degrees. Let $TOT(K_{\bullet\bullet})$ denote the total complex defined by

$$TOT(K_{\bullet\bullet})_n = \underset{i+j=n}{\oplus} K_{i,j}, \quad \delta(k_{i,j}) = (\delta_1(k_{i,j}) + (-1)^i \delta_2(k_{i,j})) \tag{0.3.1}$$

where δ_l denotes the differential in the l-th index, $l = 1, 2$.

Let $DN_\bullet \circ DN_\bullet(K_{\bullet\bullet})$ denote the double simplicial object in **A** obtained by applying the denormalizing functors in both directions to $K_{\bullet\bullet}$. Let $\Delta(DN_\bullet \circ DN_\bullet(K_{\bullet\bullet}))$ denote its diagonal; we view this as a chain complex as above. Now the theorem of Eilenberg-Zilber-Cartier (see [Ill] p.7) shows there exists a natural map (the Alexander-Whitney map) $\Delta(DN_\bullet \circ DN_\bullet(K_{\bullet\bullet})) \to TOT(K_{\bullet\bullet})$ that is a weak-equivalence.

0.4. Relationship between homotopy colimits for double simplicial objects in an abelian category and the total complex construction. Next assume the situation of Chapter II section 1. We will apply the above result to a double complex $\bar{K}_{\bullet\bullet}$ of abelian sheaves on a site $\mathfrak{S}$ as in Chapter II, section 1 that is trivial in negative degrees. (An abelian sheaf will denote a sheaf with values in any abelian category satisfying the hypotheses as above; we will assume once again, for simplicity, that the abelian category is in fact the category of all abelian groups.) The above arguments show that if $DN_\bullet(TOT(\bar{K}_{\bullet\bullet}))$ is the resulting simplicial object, one obtains a natural weak-equivalence (of simplicial objects):

$$\underset{\Delta}{\mathrm{hocolim}}(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})) \overset{\simeq}{\to} \Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})) \overset{\simeq}{\to} DN_\bullet(TOT(\bar{K}_{\bullet\bullet}))$$

(To see that the last map is a map of simplicial objects, observe that there is a natural map $N(\Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet}))) \to \Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet}))$ of complexes (where the latter is provided with the differential as above) The map in the above paragraph maps the complex $\Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet}))$ to the complex $TOT(\bar{K}_{\bullet\bullet})$. On applying the denormalizing functor to this, one obtains the map $\Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})) \cong DN \circ N(\Delta(DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet}))) \to DN(TOT(\bar{K}_{\bullet\bullet}))$ of simplicial objects.)

0.5. Relationship between the Tot constructions of [B-K] and [Br] for cosimplicial simplicial objects in an abelian category with the total complex. (See Proposition 0.3 and Lemma 0.4 for the final result.) Next let $\bar{K}_\bullet^\bullet$ denote a double complex in **A** that is trivial in negative degrees and where the differentials in the first (second) index are of degree 1 (-1, respectively). We will say $\bar{K}^\bullet.$ is *bounded* if $\bar{K}_j^i = 0$ for all but finitely many indices i and j. We will let $TOT(\bar{K}_\bullet^\bullet)$ be the total complex with differentials of degree -1 and defined by

$$(TOT(\bar{K}_\bullet^\bullet))_n = \underset{p}{\Pi} \bar{K}_{p+n}^p \tag{0.5.1}$$

and with differentials defined by $d(k_{p+n}^p) = (d_1(k_{p+n}^p) + (-1)^p d_2(k_{p+n+1}^{p+1}))$. This is a chain complex and is *trivial in negative degrees* if $\bar{K}_j^i = 0$ for all $i > j$. Let $DN^\bullet \circ DN_\bullet(\bar{K}_\bullet^\bullet)$ denote the cosimplicial simplicial object in **A** obtained by applying the denormalizing functors to $\bar{K}_\bullet^\bullet$. We may view this as a double complex with differentials:

$\delta_1 : (DN^\bullet \circ DN_\bullet(\bar{K}_\bullet^\bullet))_q^p \to (DN^\bullet \circ DN_\bullet(\bar{K}_\bullet^\bullet))_q^{p+1}$ given by $\delta_1 = \Sigma_{i=0}^{i=p+1}(-1)^i d^i$

($\delta_2 : (DN^\bullet \circ DN_\bullet(\bar{K}_\bullet^\bullet))_q^p \to (DN^\bullet \circ DN_\bullet(\bar{K}_\bullet^\bullet))_{q-1}^p$ given by $\delta_2 = \Sigma_{i=0}^{i=q}(-1)^i d_i$, respectively)

Let $TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ denote the chain complex defined by

$$TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_n = \prod_p (DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))^p_{p+n}$$

with the differential defined by $\delta(k^p_{p+n}) = \delta_1(k^p_{p+n}) + (-1)^p \delta_2(k^{p+1}_{p+n+1})$. The definition of the denormalizing functors provides a natural map

$\Phi : TOT(\bar{K}^\bullet_\bullet) \to TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ of chain complexes

One may verify that there are spectral sequences

$$E_1^{s,t}(1) = H^t(\bar{K}^s) \Rightarrow H^{s+t}(TOT(\bar{K}^\bullet_\bullet)) \quad \text{and} \tag{0.5.2}$$

$$E_1^{s,t}(2) = H^t((DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))^s) \Rightarrow H^{s+t}(TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))) \tag{0.5.3}$$

which converge strongly if $\bar{K}^\bullet_\bullet$ is bounded as assumed. The map Φ, being natural, induces a map of these spectral sequences which is clearly an isomorphism at the E_2-terms since $E_2^{s,t}(1) = E_2^{s,t}(2) = H^s($ the co-chain complex $n. \to H_t(\bar{K}^n.))$. It follows that Φ induces a weak-equivalence if $\bar{K}^\bullet_\bullet$ is bounded in the first or second index.

Let $Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ denote the chain complex defined by

$$Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_n = \{(k^p_{p+n}) \varepsilon \prod_p (DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))^p_{p+n} |$$
$$d^i(k^{p-1}_{p+n-1}) = d_{p-i}(k^p_{p+n}), s^i(k^{p+1}_{p+n+1}) = s_{p-i}(k^p_{p+n}),\ 0 \le i \le p\}$$

and where $\delta : Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_n \to Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_{n-1}$ is given by $\delta((k^p_{p+n})) = \Sigma_{i=0}^{i=n-1}(-1)^i d_{i+p+1}(k^p_{p+n})$. One readily verifies that the map sending a tuple $(k^p_{p+n}) \varepsilon Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ to the same tuple $(k^p_{p+n}) \varepsilon TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ defines a map of chain complexes. We will denote this map by Ψ. We may view $Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet.))$ also as a presheaf of pointed simplicial objects where the face maps $d_i : Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_n \to Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))_{n-1}$ is given by $d_i((k^p_{p+n})) = (d_{i+p+1}(k^p_{p+n}))$ for all $i \le n-1$ and $d_n = *$ (for non-degenerate simplices). The degeneracies are defined similarly. One may also view $Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$ as a presheaf of simplicial spectra um-object in the sense of Kan by letting the higher d_i and s_i be the trivial maps. (Recall that a spectrum $S.$ in the sense of Kan (see [Kan]) is given by a sequence $\{S_{(q)}|q\}$ of pointed sets along-with structure maps $d_i : S_{(q)} \to S_{(q-1)}$, $s_i : S_{(q-1)} \to S_{(q)}$ defined for all i and satisfying the usual relations. It is also assumed that for each $s \varepsilon S_{(q)}$ all but finitely many $d_i(s)$ are different from $*$. Clearly one may view any pointed simplicial set as a spectrum in the sense of Kan; this will correspond to the suspension spectrum of the original simplicial set. Observe that Kan's definitions apply to any pointed category; such an object in a pointed category will be referred to as a spectrum object in the sense of [Kan].)

One may define a filtration of Tot_1 by Tot_1^m which is defined in a manner similar to Tot_1, except that one considers only those (k^p_{p+n}) with $p \le m$. Since $DN^\bullet \circ DN_\bullet(K^\bullet.)$ is a cosimplicial object of presheaves of simplicial abelian *groups*, the stalks are *fibrant cosimplicial objects* by [B-K] p. 276 and hence the map $Tot_1^m(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \to Tot_1^{m-1}(DN^\bullet \circ DN_\bullet(K^\bullet_\bullet))$ is a fibration at each stalk (see [Br] p. 457). The presheaf of homotopy groups of the fiber of the map may now be identified with degree-k terms of the normalization of the cosimplicial abelian presheaf $p \to \pi_*(DN^\bullet \circ DN_\bullet(\bar{K}^p.))$. It follows that, one obtains a spectral sequence:

$$E_2^{s,t} = H^s(\text{the co-chain complex } n \to H_t(\bar{K}^n.))$$
$$\Rightarrow \pi_{-s+t}(Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \cong H^{s-t}(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$$

The map Ψ induces a map of the above spectral sequence to the second spectral sequence in 4.0.5 and this is an isomorphism at the E^2-terms. If $\bar{K}^\bullet$. is *bounded*, both spectral sequences converge strongly and therefore the map Ψ induces a weak-equivalence $Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \to TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$

Next recall that if A and B are two pointed simplicial sets, one may define their *join* $A * B$ (see [K-W] p. 242) to be the simplicial set given by: $(A * B)_{n+1} = \bigvee_{i+j=n} A_i \wedge B_j$, $d_i(a \wedge b) = d_{p-i}(a) \wedge b$ if $a \varepsilon A_p$, $0 \le i \le p$, $d_i(a \wedge b) = a \wedge d_{i-p-1}(y)$, if $a\varepsilon A_p$, $i > p$ and $s_i(a \wedge b) = s_{i-p}(a) \wedge b$ if $a\varepsilon A_p$, $0 \le i \le p$, $s_i(a \wedge b) = a \wedge s_{i-p-1}(y)$, if $a \varepsilon A_p$, $i > p$. One obtains a homeomorphism of $|A*B|$ with $\Sigma(|A \wedge B|)$, where $A \wedge B = (A \times B)/(* \times B \cup A \times *)$ and suspension is simply smash product with S^1. One may now define $\Omega(A * B)$. denote the simplicial set given by $(\Omega(A * B))_n = \{x\varepsilon(A * B)_{n+1} | d_n(x) = *\}$ and where the face maps $d_i : (\Omega(A * B))_n(\Omega(A * B))_{n-1}$ and the degeneracies $s_i : (\Omega(A * B))_{n-1} \to (\Omega(A * B))_n$, $0 \le i \le n-1$, are the restrictions of the corresponding maps of $A * B$. It follows that if one views $A * B$ and $A \wedge B$ as the associated simplicial spectra in the sense of Kan, one obtains a natural weak-equivalence

$: A \wedge B$ with $\Omega(A * B)$.

(To see this more clearly one needs to use a different suspension S for a simplicial set T which performs an upward shift. Now $|ST| \cong \Sigma(|T|)$ - we skip the remaining details.)

Let $\bar{L}^\bullet_\bullet$ denote a cosimplicial simplicial object of abelian sheaves on $\mathfrak{S}$. We will view this as a cosimplicial object of presheaves of abelian spectra in the sense of Kan on the site $\mathfrak{S}$. For each integer m, let $\Delta[m]$ denote the constant presheaf with stalks isomorphic to the simplicial set $\Delta[m]$; we will view this also as a presheaf of spectra in the sense of Kan in the obvious manner. Now we let

$$Tot_2(\bar{L}^\bullet_\bullet)_n = Hom(\Omega(\Delta[\bullet]_+ * \Delta[n]_+), \mathcal{G}\bar{L}^\bullet_\bullet)$$

where $\bar{L}^\bullet_\bullet$ is viewed as a cosimplicial object of presheaves of spectra in the sense of Kan and the Hom is in the category of cosimplicial objects. Let $Tot_2^m(\bar{L}^\bullet_\bullet)_n = Hom(\Omega(sk_k(\Delta[\bullet])_+ * \Delta[n]_+), \mathcal{G}\bar{L}^\bullet_\bullet)$. Since $\bar{L}^\bullet$. is a cosimplicial object of simplicial abelian groups, the stalks are fibrant as cosimplicial spaces (see [B-K] p. 276) and therefore the obvious maps $Tot_2^m(\bar{L}^\bullet_\bullet) \to Tot_2^{m-1}(\bar{L}^\bullet_\bullet)$ are fibrations. $Tot_2(\bar{L}^\bullet_\bullet)$ is the inverse limit of this tower of fibrations. To see the relationship of this with the stable Tot of Bousefield-Kan, one may proceed as follows. Recall $Tot(\bar{L}^\bullet_\bullet)_n = Hom(\Delta[.] + \wedge\Delta[n], \mathcal{G}\bar{L}^\bullet_\bullet)$ and $Tot^m(\bar{L}^\bullet_\bullet)_n = Hom(sk_m(\Delta[\bullet]_+) \wedge (\Delta[n]_+), \mathcal{G}\bar{L}^\bullet_\bullet)$. Now $Tot(\bar{L}^\bullet_\bullet)$ is the homotopy inverse limit of the tower $Tot^m(\bar{L}^\bullet_\bullet) \to Tot^{m-1}(\bar{L}^\bullet_\bullet)$. One uses the natural weak-equivalences $\Omega(sk_m(\Delta[p])_+ * \Delta[n]_+) \simeq sk_m(\Delta[p])_+ \wedge \Delta[n]_+$, for all m, to obtain a map of the corresponding homotopy inverse limit spectral sequences. As in [B-K] pp. 281-283, one may identify the E_1-terms of both the above spectral sequences with the normalization of the cosimplicial abelian presheaf $p \to \pi_*(\bar{L}^p_\bullet)$. If $\bar{L}^\bullet_\bullet = DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)$ where $\bar{K}^\bullet_\bullet$ denotes a *bounded* double complex of abelian sheaves on $\mathfrak{S}$, both spectral sequences converge strongly; since we clearly obtain an isomorphism at the E_1-terms, it follows that one now obtains a weak-equivalence

$$Tot_2(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \simeq Tot(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)$$

Let $\bar{K}^\bullet_\bullet$ denote a bounded double complex of abelian sheaves on $\mathfrak{S}$. Now one obtains a natural map

$$Tot_2(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \to Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$$

of presheaves of spectra by sending a map: $\Omega(\Delta[p]_+ * \Delta[n]_+) \to DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)^p$ to the $p+n$ simplex which is the image of the $p+n$ simplex $i_p \wedge i_n \varepsilon (\Delta[p]_+)_p \wedge (\Delta[n]_+)_n$ where i_p (i_n) generates $\Delta[p]$ ($\Delta[n]$, respectively). We have defined Tot_1 in such a manner so that the image of the $p+n$-simplex $i_p \wedge i_n$ satisfies the conditions in 0.5 defining Tot_1. Recall from 4.0.5 the Bousefield-Kan type spectral sequence for Tot_1 whose $E-2$-terms are given by $E_2^{s,t} = H^s($ the co-chain complex $n \to H_t(\bar{K}^n.))$. *If the double complex* $\bar{K}^\bullet_\bullet$ *is bounded as in 0.5, it is clear that the above spectral sequence will converge strongly.* The construction of the usual Bousefield-Kan spectral sequence readily applies to provide a spectral sequence that converges to the homotopy groups of Tot_2; the E_2-terms of this spectral sequence will be also given by the same description as above. The map in 0.5 induces a map of these spectral sequences thereby showing that it is a weak-equivalence *provided* $\bar{K}^\bullet_\bullet$ *is bounded.*

Now we summarize our results in the following proposition.

PROPOSITION 0.3. *Let $\bar{K}^\bullet_\bullet$ denote a double complex of abelian sheaves on $\mathfrak{S}$ that is trivial in negative degrees and where the differentials in the first (second) index are of degree 1 (-1, respectively). Assume further that $\bar{K}^i_j = 0$ if $i > j$ and that $\bar{K}^\bullet_\bullet$ is bounded i.e. $\bar{K}^i_j = 0$ for all but finitely many indices i and j.*

Now one obtains the following weak-equivalences of presheaves of simplicial abelian groups (natural in $\bar{K}^\bullet_\bullet$):

$$DN_\bullet(TOT(\bar{K}^\bullet_\bullet)) \simeq DN_\bullet(TOT_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))) \simeq Tot_1(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet))$$

$$\simeq Tot_2(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet_\bullet)) \simeq Tot(DN^\bullet \circ DN_\bullet(\bar{K}^\bullet.))$$

where the last Tot is the stable Bousefield-Kan Tot-functor and the others are the ones defined above.

PROOF. The arguments above clearly prove the assertion. □

0.5.4. Let $\bar{K}_{\bullet\bullet}$ denote a double complex of abelian sheaves on $\mathfrak{S}$ so that the following conditions are satisfied. There exists a large positive integer m so that $\bar{K}_{i,j} = 0$ if $j < m - i$ or $i < 0$ or $j < 0$. Let $\{\tilde{K}^i_j | i,j\}$ denote the double complex so that $\tilde{K}^i_j = \bar{K}_{m-i,j}$. The differentials in the first index are now of degree $+1$ while those in the second index are of degree -1. Let $TOT(\bar{K}_{\bullet\bullet})$ ($TOT(\tilde{K}^\bullet_\bullet)$)) denote the chain complex defined as in 0.3.1 (0.5.1, respectively). (The assumptions on $\bar{K}_{\bullet\bullet}$ ensure that $TOT(\bar{K}_{\bullet\bullet})_n = 0$ if $n < m$ and $TOT(\tilde{K}^\bullet_\bullet)_n = 0$ if $n < 0$. Now $DN^\bullet \circ DN_\bullet(\tilde{K}_{\bullet\bullet})$ is a cosimplicial simplicial object and $DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})$) is a double simplicial object of abelian sheaves on the given site.

LEMMA 0.4. *Assume the above situation. Now one obtains a natural weak-equivalence:*

$$\Sigma^m Tot(DN^\bullet \circ DN_\bullet(\tilde{K}^\bullet_\bullet)) \simeq \underset{\Delta}{hocolim} DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})$$

PROOF. Let $DN_\bullet(TOT(\bar{K}_{\bullet\bullet}))$ and $DN_\bullet(TOT(\tilde{K}^\bullet_\bullet))$ denote the obvious simplicial objects. Now the left-hand side is weakly-equivalent to $\Sigma^m DN_\bullet(TOT(\tilde{K}^\bullet_\bullet))$ (see 4.0.5) while the right-hand side is weakly equivalent to $DN_\bullet TOT(\bar{K}_{\bullet\bullet})$. (See 0.5.1.) It is clear that the complex $TOT(\bar{K}_{\bullet\bullet})$ is the complex $TOT(\tilde{K}^\bullet_\bullet)$ shifted up m-times. Now 0.7 shows $\Sigma^m DN_\bullet(TOT(\tilde{K}^\bullet_\bullet)) \simeq DN_\bullet(TOT(\bar{K}_{\bullet\bullet}))$. □

0.6. The functor Sp for presheaves of spectra. In the rest of this section, we will show that one may define a functor Sp on bounded below complexes of sheaves of abelian groups taking values in the category of presheaves of spectra and satisfying the hypotheses as in Chapter I. Accordingly *Presh* will denote the category of presheaves of spectra (i.e. simplicial spectra as in [B-F]) on a site $\mathfrak{S}$ satisfying the hypotheses of Chapter I, section 1. We may identify the cohomological functor $\mathcal{H}^*$ as in Chapter I with the functor $P \mapsto \pi_{-*}(P)$, where the latter is the presheaf of stable homotopy groups. Let $\bar{M}^\bullet$ denote a co-chain complex of abelian sheaves on $\mathfrak{S}$ and let $m >> 0$ be an integer so that $\bar{M}^i = 0$ if $i < 0$ or if $i > m$. Let l denote an integer $> m$. For each integer $n \geq 0$, let $K(\bar{M}^n, l)$ denote the presheaf of Eilenberg-Maclane spaces so that $\pi_i(K(\bar{M}^n, l))\tilde{} \cong \bar{M}^n$ if $i = l$ and $\cong 0$ otherwise. Now $\tilde{K}^\bullet_\bullet = \{N_\bullet(K(\bar{M}^n,\ l))|n\}$ is a double complex with differentials in the first (second) index of degree $+1$ (-1, respectively). Let $\bar{K}_{\bullet\bullet}$ denote the double chain-complex defined by $\bar{K}_{i,j} = \tilde{K}_j^{m-i}$. Now $\tilde{K}^\bullet_\bullet$ and $\bar{K}_{\bullet\bullet}$ are complexes that satisfy the hypotheses of 0.3 (since $N(K(\bar{M}^n,\ l))_i = 0$ if $i < l$ and for all n). Moreover observe that $DN_\bullet(\tilde{K}^\bullet_\bullet) = DN_\bullet \circ N(K(\bar{M}^\bullet, l)) \cong K(\bar{M}^\bullet, l)$. Therefore one obtains the weak-equivalence:

$$\Sigma^m Tot(DN^\bullet K(\bar{M}^\bullet, l)) \simeq \underset{\Delta}{\mathrm{hocolim}} DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet})$$

Next observe the following. Let $\bar{M}$ denote an abelian sheaf on the site $\mathfrak{S}$ and let i denote an integer. Let l denote any fixed integer. One may define the presheaf of Eilenberg-Maclane spectra $\widehat{Sp}(K(M,i))$ be given by the sequence of presheaves of Eilenberg-Maclane spaces defined by

$$(\widehat{Sp}(K(\bar{M}, i)))_j := *, \quad j < l - i \tag{0.6.1}$$

$$:= K(M, j+i), \quad j \geq l - i \tag{0.6.2}$$

One may observe readily that $\widehat{Sp}(K(M,i)) = EM_i(M)$, where EM_i is the functor defined in Appendix B for presheaves of symmetric spectra. Therefore, the present discussion applies equally well to presheaves of symmetric spectra.

Let E denote a ring object in the category of symmetric spectra; we will also denote by E the obvious constant presheaf associated to E. Let $\bar{M}^\bullet = \prod_i \bar{M}^\bullet(i) \varepsilon D^b(Mod_r(\mathfrak{S}; \pi_*(E)))$. *Assume that* $\bar{M}^n = 0$ *if* $n > m$ *or if* $n < 0$ *and that* $l > m$. We let $\widehat{Sp}(\bar{M}^\bullet)) = \prod_i \widehat{Sp}(K(\bar{M}^\bullet(i), i))$. Now observe that for each fixed n, each presheaf of spaces forming the presheaf of spectra $\widehat{Sp}(K(\bar{M}^n(i), i))_j$ is given by $K(\bar{M}^n(i), j+i)$, if $j \geq l - i$ and $= *$ otherwise. Therefore the hypotheses of 0.6 are satisfied with l replaced by $j + i$ and one obtains:

$$\Sigma^m Tot DN^\bullet(K(\bar{M}^\bullet(i), j+i)) \simeq \underset{\Delta}{\mathrm{hocolim}} DN_\bullet \circ DN_\bullet(\bar{K}_{\bullet\bullet}(j))$$

where $\bar{K}_{\bullet\bullet}(j)$ is the double chain complex defined by $(\bar{K}_{\bullet\bullet}(j))_{s,t} = N_\bullet(K(\bar{M}^\bullet(i), j+i))_t^{m-s} = N_\bullet(K(\bar{M}^{m-s}(i), j+i))_t$. Observe that inner denormalizing functor $DN_\bullet$ is inverse to the functor $N_\bullet$ that produces $N_\bullet(K(\bar{M}^{m-s}(i), j+i))$ from the simplicial abelian sheaf $K(\bar{M}^{m-s}(i), j+i)$. (We will use the second subscript of $\bar{K}_{\bullet\bullet}(j)$ to denote this direction.) Therefore, if $K(\bar{M}^\bullet(i), j+i)[m_h]$ is the chain-complex of presheaves of Eilenberg-Maclane spaces defined by

$$(K(\bar{M}^\bullet(i), j+i)[m_h])_s = K(\bar{M}^{m-s}(i), j+i)$$

and with the obvious differential induced by that of $\bar{M}^\bullet(i)$ and $DN_\bullet$ denotes denormalizing along the second direction, one obtains:

$$DN_\bullet(\bar{K}_{\bullet\bullet}(j)) = K(\bar{M}^\bullet(i), j+i)[m_h]$$

Thus, for all j, we obtain the weak-equivalences:

$$\Sigma^m TotDN^\bullet(K(\bar{M}^\bullet(i), j+i)) \simeq \underset{\Delta}{\text{hocolim}} DN_\bullet(K(\bar{M}^\bullet(i), j+i)[m_h])$$

Recall that $\widehat{Sp}(K(\bar{M}^\bullet(i), i)) = \{K(\bar{M}^\bullet(i), j+i) | j \geq l-i\}$ and $\widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h])$ $= \{K(\bar{M}^\bullet(i), j+i)[m_h] | j \geq l-i\}$. Since the second equality of 0.6 holds for all $j \geq l-i$, we obtain:

$$\Sigma^m TotDN^\bullet(\widehat{Sp}K(\bar{M}^\bullet(i), i)) \simeq \underset{\Delta}{\text{hocolim}} DN_\bullet(\widehat{Sp}K(\bar{M}^\bullet(i), i)[m_h])$$

for all i. (The chain-complex $K(\bar{M}(i), i)[m_h]$ is defined as above.) Now

$$\Sigma^m TotDN^\bullet \widehat{Sp}(\bar{M}^\bullet) \simeq \prod_i \Sigma^m TotDN^\bullet(\widehat{Sp}(K(\bar{M}^\bullet(i), i)))$$

$$\simeq \prod_i \underset{\Delta}{\text{hocolim}} DN_\bullet \widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h])$$

Now one observes that for each fixed integer k there are only finitely many terms in the above product with nontrivial homotopy groups in degree k. (To see this, first observe that by the hypotheses, $\bar{M}^\bullet$ has bounded cohomology; therefore if one considers the spectral sequence:

$$E^2_{s,t} = H_s(\pi_t(DN_\bullet \widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h]))\tilde{}\ )$$

$$\Rightarrow \pi_{s+t}(\underset{\Delta}{\text{hocolim}} DN_\bullet \widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h]))\tilde{}$$

there exists a uniform bound m (independent of i) so that $E^2_{s,t} = 0$ if $s > m$, $s < 0$ or if $t \neq i$. It follows that $\pi_k(\underset{\Delta}{\text{hocolim}} DN_\bullet \widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h]))\tilde{} = 0$ unless $i \leq k \leq i+m$.) Therefore the product in the last term above may be replaced by a $\bigvee_i$; now such a $\bigvee_i$ commutes with homotopy colimits and with the denormalizing functor for simplicial objects (which also involve only sums). Therefore the last term above may be replaced by $\underset{\Delta}{\text{hocolim}} DN_\bullet \prod_i \widehat{Sp}(K(\bar{M}^\bullet(i), i)[m_h]) \simeq \underset{\Delta}{\text{hocolim}} DN_\bullet \widehat{Sp}(\bar{M}^\bullet[m_h])$. (Here $\bar{M}^\bullet[m_h]$ is the chain complex in $Mod_r(\mathfrak{S}, \pi_*(E))$ given by $(\bar{M}^\bullet[m_h])_s = \bar{M}^{m-s}$ and $\widehat{Sp}$ is applied degree-wise to this complex to produce a chain complex in $Mod(\mathfrak{S}, \widehat{Sp}(\pi_*(E)))$.) Recalling the definition of the functor Sp from Chapter I, we now obtain a weak-equivalence:

$$\Sigma^m Sp(\bar{M}^\bullet) = \Sigma^m TotDN^\bullet(\widehat{Sp}K(\bar{M}^\bullet)) \simeq \underset{\Delta}{\text{hocolim}} DN_\bullet \widehat{Sp}(\bar{M}[m_h]) \tag{0.6.3}$$

for any $\bar{M}^\bullet \varepsilon D^b(Mod_r(\mathfrak{S}; \pi_*(E)))$ so that $M^n = 0$ if $n < 0$ or if $n > m$.

Shifts of complexes and suspension. We conclude the paper with a discussion on shifts of complexes and how they relate to suspensions (loopings) of the associated simplicial and cosimplicial objects. *For this, we will assume the context of Chapter III, 1.2.*

If S and T are both pointed simplicial sets (or simplicial presheaves), one defines $S \otimes T$ to be the pointed simplicial object defined by $(S \otimes T)_n = \bigvee_{S_n - *} T_n$ with the base points of all T_n identified with the common base point and with the obvious structure maps induced from those of S and T. If S is a pointed simplicial set and T is a simplicial object in an abelian category $\mathbf{A}$, $S \otimes T$ will denote the simplicial object in $\mathbf{A}$ defined by $(S \otimes T)_n = \underset{S_n - *}{\oplus} T_n$ with the base points of all T_n identified with the common base point and with the obvious structure maps induced from those of S and T. If S and T are both simplicial objects in $\mathbf{A}$, $S \otimes T$ will denote the simplicial object in $\mathbf{A}$ that is the diagonal of the bisimplicial object $\{S_n \otimes T_m | n, m \geq 0\}$ of $\mathbf{A}$, where $\otimes$ has the usual meaning. Given a pointed simplicial set S and a pointed simplicial presheaf P, $S \wedge P$ denotes the simplicial presheaf defined by $\Gamma(U, S \wedge P) = S \wedge \Gamma(U, P)$, U in the given site.

0.7. Shifts for chain complexes. If S^1 denotes the simplicial sphere as above, one first observes the isomorphism $\mathbb{Z}(S^1) \cong DN_\bullet(\mathbb{Z}[1]_\bullet)$, where $\mathbb{Z}(S^1)$ is the free abelian group functor applied to the pointed simplicial set S^1, $\mathbb{Z}[1]_\bullet$ denotes the chain-complex that is trivial in all degrees except 1 and where it is $\mathbb{Z}$ and $DN_\bullet$ is the denormalizing functor applied to this chain complex. Next let $K_\bullet$ denote a chain complex of abelian sheaves on the site $\mathfrak{S}$ that is trivial in negative degrees. Now one obtains the isomorphisms:

$$DN_\bullet(K_\bullet[1]) \cong DN_\bullet(\mathbb{Z}[1]) \otimes DN_\bullet(K_\bullet) \cong \mathbb{Z}(S^1) \otimes DN_\bullet(K_\bullet) \cong S^1 \otimes DN_\bullet(K_\bullet)$$

Moreover there is a natural map $S^1 \wedge DN_\bullet(K_\bullet) \to S^1 \otimes DN_\bullet(K_\bullet)$ of simplicial objects. *This is a weak-equivalence.*

Let $\mathcal{A}$ denote an algebra in $Presh$ and let Let $\bar{M}. = \prod_i \bar{M}.(i) \varepsilon Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$; now $\widehat{Sp}(\bar{M}.) = \prod_i Sp(\bar{M}.(i), i)$ is a chain-complex in $Mod(\mathfrak{S}, Sp(\mathcal{H}^*(E)))$. In this case the map above induces a map $S^1 \wedge DN_\bullet(Sp(\bar{M}.)) \to S^1 \otimes DN_\bullet(Sp(\bar{M}.))$ of simplicial objects in $Mod(\mathfrak{S}, Sp(\mathcal{H}^*(\mathcal{A})))$. One may readily show this induces a weak-equivalence on taking the homotopy colimits of the corresponding simplicial objects in $Mod(\mathfrak{S}, Sp(\mathcal{H}^*(\mathcal{A})))$ Combining this with the earlier isomorphisms, we obtain a weak-equivalence :

$$\underset{\Delta}{\mathrm{hocolim}} DN_\bullet(Sp(\bar{M}_\bullet[1])) \simeq S^1 \wedge \underset{\Delta}{\mathrm{hocolim}} DN_\bullet(Sp(\bar{M}_\bullet))$$

Shifts for co-chain complexes. Let $\Delta[n]_+$ denote the obvious constant presheaf on the site $\mathfrak{S}$ as before. If S is a pointed simplicial set, recall that $S \otimes \Delta[n]_+$ is the pointed simplicial set given by:

$$(S \otimes \Delta[n]_+)_p = \bigvee_{S_p - *} (\Delta[n]_+)_p$$

and with the obvious structure maps. Let P denote a presheaf of pointed simplicial sets on the site $\mathfrak{S}$. Now we will let $\mathcal{M}ap(S, P)$ denote the presheaf of pointed simplicial sets denoted P^S in Chapter I, (M4.1). If S is a pointed set viewed as a constant pointed simplicial set, one may observe the natural isomorphisms:

$$\mathcal{M}ap(S, P) = \prod_S (P).$$

Next let S denote a pointed simplicial set. Let $P^\bullet$ denote a cosimplicial object of presheaves of pointed simplicial sets. Let $(P^\bullet)^S$ denote the cosimplicial presheaf given in cosimplicial degree n by $(P^n)^{S_n}$ where the last term has the meaning as above when S_n is viewed as a constant simplicial set. The structure maps are the obvious induced maps. This is the diagonal of a double cosimplicial object given in degrees m and n by $\mathcal{M}ap(S_m, P^n)$ when each S_m is viewed as a constant simplicial object. Thus $\underset{\Delta}{\mathrm{holim}}\, \Delta(\mathcal{M}ap(S, P^\bullet)) = \underset{\Delta}{\mathrm{holim}}(P^{\bullet S})$ where $\mathcal{M}ap(S, P^\bullet)$ denotes the double cosimplicial object considered above. Let the first (second) cosimplicial indices for this double cosimplicial object be in the direction of S (the cosimplicial direction of $P^\bullet$, respectively). If ${}^i\operatorname{holim}_\Delta$ denotes the $\operatorname{holim}_\Delta$ in the first (second) direction if $i = 1$ ($i = 2$, respectively). Now one obtains the following chain of natural maps that are weak-equivalences:

$$\mathcal{M}ap(S, \operatorname{holim}_\Delta(P^\bullet)) \overset{\simeq}{\to}{}^2 \operatorname{holim}_\Delta \mathcal{M}ap(\Delta(S), P^\bullet) \overset{\simeq}{\to}{}^2 \operatorname{holim}_\Delta \circ^1 \operatorname{holim}_\Delta \mathcal{M}ap(S, P^\bullet)$$

Moreover the latter maps naturally to $\operatorname{holim}_\Delta((P^\bullet)^S)$ by a weak-equivalence.

Examples 0.5. (i) Let $S = \Delta[1]_+$. Now we obtain the natural weak-equivalence: $Tot((P^\bullet)^S) \overset{\simeq}{\leftarrow} \mathcal{M}ap(\Delta[1]_+, Tot(P^\bullet))$

(ii) Let $S = (\Delta[1]/\overset{\circ}{\Delta}[1])$ = the 1-dimensional simplicial sphere S^1. Now we obtain the natural weak-equivalence: $Tot((P^\bullet)^{S^1}) \overset{\simeq}{\leftarrow} \mathcal{M}ap(S^1, Tot(P^\bullet))$.

Let $\mathcal{A}$ denote an algebra in $Presh$ and let $K_\bullet$ denote a chain complex *either* in $Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$ or in $\underset{n}{\oplus} Mod_r(\mathfrak{S}, \mathcal{A})_f^{\leq n \geq}$ (the latter as in Chapter I, (ST4)) that is trivial in negative degrees as above. Let $DN_\bullet(K_\bullet)$ denote the associated simplicial object. We will view this as a cosimplicial simplicial object constant in the cosimplicial direction. Let $K_\bullet^\bullet[-1_v]$ denote the double complex $(K[-1_v])^1_j = K_j$ and $(K[1_v])^i_j = 0$ for all j and all $i \neq 1$. One may now readily observe the isomorphism of cosimplicial objects : $DN^\bullet \circ DN_\bullet(K[-1_v]) \cong (DN^\bullet \circ DN_\bullet(K_\bullet))^{S^1}$ where the term $DN^\bullet \circ DN(K_\bullet))$ is the simplicial object $DN_\bullet(K_\bullet)$ viewed as a constant cosimplicial simplicial object in the obvious manner.

Now we consider the more general case where $\tilde{K}_\bullet^\bullet$ is a double chain complex in $Mod_r(\mathfrak{S}, \mathcal{H}^*(\mathcal{A}))$ that is trivial in negative degrees and where the differentials in the first (second) index are of degree $+1$ (-1, respectively). We may view the complex $\tilde{K}_\bullet^\bullet$ as sitting in the first quadrant with the cosimplicial (simplicial) direction along the x (y-axis, respectively). For each fixed n, let $\tilde{K}^n[-1_z]$ denote the chain complex K^n shifted up one-step in the direction of the positive z-axis. (As n varies, we now obtain a triple complex, trivial everywhere except in the plane $z = 1$.) Now observe the isomorphism (from the previous paragraph), for each fixed n: $DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^n)^{S^1} \cong DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^n[-1_z])$. (Here the cosimplicial-denormalization is along the z-axis.) Now we denormalize in the x-direction to get a double cosimplicial simplicial object: $DN^\bullet \circ DN^\bullet \circ DN_\bullet(\tilde{K}^\bullet)^{S^1}$.

Let $\tilde{K}[-1_v]$ denote the double complex given by $(\tilde{K}[-1_v])^i_j = \tilde{K}^{i-1}_j$ if $i \geq 1$ and $(\tilde{K}[-1_v])^0_j = 0$ for all j. Let $DN^\bullet \circ DN_\bullet(\tilde{K}[-1_v])$ denote the corresponding cosimplicial simplicial object. Now one may show readily that there is natural map $\Delta DN^\bullet \circ DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^\bullet)^{S^1} \to DN^\bullet \circ DN_\bullet(\tilde{K}[-1_v])$ of cosimplicial simplicial objects that induces a weak-equivalence on applying $\operatorname{holim}_\Delta$s. (To see this simply observe that the total complex in the x, z-directions of the triple complex $\{\tilde{K}^n[-1_z]_\bullet | n\}$ maps into the double complex $\tilde{K}^\bullet_{bullet}[-1_v]$ and that this is a weak-equivalence on taking the total complexes.) Therefore one obtains the following natural maps that are weak-equivalences:

$$\operatorname{holim}_\Delta(DN^\bullet \circ DN_\bullet(\tilde{K}[-1_v])) \overset{\simeq}{\leftarrow} \operatorname{holim}_\Delta(\Delta DN^\bullet \circ DN^\bullet DN_\bullet(\tilde{K}^\bullet)^{S^1})$$

$$\overset{\simeq}{\leftarrow} \mathcal{M}ap(S^1, \operatorname{holim}_\Delta(DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^\bullet))).$$

the last $\simeq$ follows from the second example above.

Next observe from Chapter I, Remark 3.2 that $\mathcal{M}ap(S^1, \underset{\Delta}{\operatorname{holim}}(DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^\bullet))) \simeq \Omega \underset{\Delta}{\operatorname{holim}}(DN^\bullet \circ DN_\bullet(\tilde{K}_\bullet^\bullet))$ where Ω is used in the sense of Chapter I, Definition 2.3.

Convention. In view of the above, shifting a chain-complex $K_\bullet$ to the right k times will be denoted $K_\bullet[k]$; for a co-chain complex $K^\bullet$, the corresponding shift will be denoted $K^\bullet[-k]$.

Finally we return to the setting of presheaves of spectra as in 0.6.3. Let $\bar{M}^\bullet = \prod_i \bar{M}^\bullet(i)$ denote a bounded co-chain complex in $Mod_r(\mathfrak{S}, \pi_*(E))$ that is trivial in negative degrees. Let $l > 0$ be such that $\bar{M}^k = 0$ if $k \geq l$. Now, as in the discussion preceding 0.6, $\widehat{Sp}(K(\bar{M}^n(i), i))$ is the presheaf of spectra given by $\widehat{Sp}(K(\bar{M}^n(i), i))_j = K(\bar{M}^n(i),\, j+i)$, if $j \geq l-i$ and $= *$ otherwise. For each j, $\tilde{K}_\bullet^\bullet(j, \bar{M}^\bullet) = N_\bullet K(\bar{M}^\bullet(i), j+i)$ is now a double complex satisfying the hypotheses on $\tilde{K}_\bullet^\bullet$ as above. Moreover if $\bar{M}^\bullet[-1]$ is the co-chain complex given by $(\bar{M}^\bullet[-1])^i = \bar{M}^{i-1}$, one may observe the isomorphism (using the notation from above):

$$\tilde{K}_\bullet^\bullet(j, \bar{M}^\bullet[-1]) = \tilde{K}_\bullet^\bullet(j, \bar{M}^\bullet)[-1_v], \text{ for each } j.$$

Therefore, 0.7 provides the weak-equivalence:

$$Tot(DN^\bullet \widehat{Sp}(K(\bar{M}^\bullet[-1](i), i))_j) \simeq Map(S^1, Tot(DN^\bullet(K(\widehat{Sp}(\bar{M}^\bullet(i), i))_j))), \text{ for all } j.$$

It follows that

$$Sp(\bar{M}^\bullet[-1])) = \prod_i Tot(DN^\bullet K(\widehat{Sp}(\bar{M}^\bullet[-1](i), i)))$$

$$\simeq \prod_i \Omega Tot(DN^\bullet K(\widehat{Sp}(\bar{M}^\bullet(i), i))) = \Omega Sp(\bar{M}^\bullet).$$

0.8. Co-chain complexes to chain complexes and vice-versa by shifts. A device that we have frequently used is the following technique for converting a bounded co-chain complex that is trivial in negative degrees to a chain complex that is also trivial in negative degrees and bounded. Let $K^\bullet$ denote a bounded *co-chain* complex in an category $\mathbf{C}$ with a zero object and with finite limits and colimits. Assume that $m > 0$ is an integer so that $K^i = 0$ for all $i < 0$ and all $i > m$. Now we let $K[m]$ denote the *chain-complex* defined by $(K[m])_i = K^{m-i}$ and with the differentials induced from those of K. One may apply the same technique (in reverse) to produce a co-chain complex from a chain-complex that is trivial in negative degrees and that is bounded.

Bibliography

[Ad] J.F. Adams, : *Stable homotopy and generalized cohomology* , Chicago Lectures in mathematics, Univ. of Chicago Press (1971)

[B-G] K. Brown and S. Gersten: *Algebraic K-theory and generalized sheaf cohomology*, Lect. in Math., **341**, (1973), 419-458

[BFM] P. Baum, W. Fulton and R. Macpherson: *Riemann Roch for singular varieties*, Publ. IHES, **45**, (1975), 100-145

[BGSV] A. Beilinson, A. Goncharov, V. Schechtman and A. Varchenko: *Aomoto Dilogarithms, Mixed Hodge Structures and Motivic cohomology of pairs of triangles in the plane*, Groth. Feschrift, vol I, 136- 172, Birkhauser, (1990)

[BMS] A. Beilinson, R. Macpherson and V. Schechtman: *Notes on motivic cohomology*, Duke Math. J, **54**, 679-710, (1987)

[Bl-1] S. Bloch: *Algebraic Cycles and Higher K-theory*, Adv. Math., **61**, 267-304, (186)

[Bl-2] S. Bloch: *Algebraic cycles and the Beilinson conjectures*, Contemp. Math., **58**, (1986), 65-79

[Bl-3] S. Bloch: *Algebraic cycles and the Lie algebra of mixed Tate motives*, J.A.M.S, **4**, (1991), 771-791

[Bl-K] S. Bloch and I. Kriz: *Mixed Tate motives*, Annals of Math., (1996)

[Bo] A. Borel, et al: *Intersection cohomology*, Progress in Math, Birkhauser-Boston, (1985).

[B-B-D] A. Beilinson A., J. Bernstein and P. Deligne : *Faisceaux perverse*, Asterisque no. 100, (1981), Societé Mathematiques de France

[Board] J.M. Boardman: *Conditionally convergent spectral sequences*, in Contemp. Math, **239**, AMS, (1998), 49-84

[B-F] A. K. Bousfield and E. Friedlander : *Homotopy Theory of* Γ*-spaces, spectra and bisimplicial Spaces*, Springer Lect. Notes, vol 658, pp. 801-30, (1977), Springer

[B-K] A. K. Bousfield and D. M. Kan : *Homotopy limits, completions and localizations*, Springer Lect. Notes. No. 304, (1972), Springer

[Bryl-1] J. L. Brylinski: *Equivariant intersection cohomology*, in: Kazhdan-Lusztig theory and related topics, Contemp. Math. vol. 139, AMS, Providence (1992)

[Bryl-2] J. L. Brylinski: *Transformations canoniques...*, in: Asterisque, vol. 140, 3-134, (1986)

[B-B-Mac] W. Borho, J. L. Brylinski, R. Macpherson: *Nilpotent Orbits, Primitive ideals and Characteristic classes*, Progress in Math, vol 78, Birkhauser-Boston, (1989)

[Br] K. S. Brown: *Abstract homotopy and generalized sheaf cohomology*, Trans. AMS., vol, (1973)

[Bryl-K] J. L. Brylinski and M. Kashiwara: *The Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math., vol 64, (1981)

[CK1] I.C-Fontanine and M. M. Kapranov: *Derived Quot schemes*, preprint, (1998)

[CK2] I. C-Fontanine and M. M. Kapranov: *Derived Hilbert Schemes*, preprint, (1999)

[Cu] E. Curtis: *Simplicial homotopy theory*, Advances in Math, vol 6, (1971)

[E-K-M-M] T. Elemendorff, I. Kritz, M. Mandell and P. May: *Rings, modules and algebras in stable homotopy theory* , preprint (1996)

[Freyd] P. Freyd: *Abelian categories*, Harper and Row, (1970)

[Fr] E. Friedlander : *Etale homotopy of simplicial schemes*, Ann. Math Study, **104**, (1983)

[FSV] E. Friedlander, A. Suslin and V. Voevodsky: *Cycles, transfers and motivic homology*, Ann. Math. Study **143**, (2000)

[F-1] W. Fulton: *Rational Equivalence on singular varieties*, Publ. Math. IHES, **45**, pp 147-167, (1975)

[F-2] W. Wulton: *Introduction to toric varieties*, Ann. Math. Study, **131**, (1993), Princeton

[G-H] T. Geisser and L. Hasselholt: *Topological Cyclic Homology of Schemes*, Preprint, (2000).

[G-Z] P. Gabriel and M. Zisman: *Calculus of fractions and homotopy theory*, Ergbnisse der Math, vol , (19), Springer.

[Hirsch] P. Hirschorn: *Localization of Model categories*, preprint (see http://www.math-mit.edu/ psh)

[H-S-S] M. Hovey, B. Shipley and J. Smith: *Symmetric spectra*, May, preprint, (1997)

[Hu] A. Huber, A. Huber: *Mixed Motives and their realization in derived categories*, Lect. Notes in Math, vol 1604, Springer, (1995)

[Ill] L. Illusie: *Complex cotangent et deformations*, Springer Lect. Notes, vol 239, (1971).

[Iv] B. Iverson: *Sheaf Cohomology*, Springer, (1986).

[Jan] U. Janssen: *Motivic sheaves and filtrations of Chow groups*, Proc. Symposia in Pure Math., vol 55 (1994), part I, pp. 239-301

[J-1] R. Joshua: *Becker-Gottlieb transfer in etale homotopy* , Amer. J. Math, vol 109, pp. 453-498, (1987)

[J-2] R. Joshua: *Spanier-Whitehead duality in Etale homotopy*, Trans. AMS, vol 296, pp. 151-166, (1986)

[J-3] R. Joshua: *Generalized Verdier duality for presheaves of spectra-I*, Journ of Pure and applied algebra, vol 70, pp.273-289, (1991)

[J-4] R. Joshua: *Vanishing of odd dimensional intersection cohomology*, Math. Z., vol 207, pp. 239-253, (1987).

[J-5] R. Joshua: *Modules over convolution algebras from equivariant derived categories-I*, Journ. Algebra, (1997)

[J-6] R. Joshua: *The motivic DGA*, preprint , (2001).

[J-7] R. Joshua: *Generalized Intersection Cohomology*, preprint , (1994).

[Kan] D.M.. Kan: *Semi-simplicial spectra*, Ill. J. Math., vol 7, (1963), 463-478

[K-S-1] M. Kashiwara and P. Schapira: *Microlocal study of sheaves*, Asterisque, vol 128, (1991)

[K-S-2] M. Kashiwara and P. Schapira: *Sheaves on Manifolds*, Springer Verlag, (1989)

[K-M] I. Kritz and P. May: *Operads, algebras, modules and motives*, Asterisque, vol 233, (1995)

[Kom] G. Komezano: *Generalized Intersection cohomology theories*, preprint, (1995)

[Kon] M. Kontsevich: *Enumeration of rational curves via torus actions*, in: Moduli Spaces of Curves, 335-368, Birkhauser, (1995)

[Lau] G. Laumon: *Transformation de Fourier ...*, Publ. Math. IHES, vol 65, (1987), 131-210

[Lusz] G. Lusztig: *Cuspidal local systems and graded Hecke algebras-I*, Publ. Math., vol 67, pp. 145-202, (1987)

[Lyd] M. Lydakis, *Smash products and Γ-spaces*, preprint, (1996)

[Mac] S. Maclane: *Categories for the working mathematician*, GTM, vol 5, Springer-Verlag, (1971)

[M1] F. Morel: *Théorie homotopique de schemas* , Asterisque, vol, Societé mathematiques de France, **256**, (1999)

[M2] F. Morel: *Voevodsky's proof of the Milnor conjecture*, Bul. AMS, **35**, (1998), 123-143

[M-V] F. Morel and V. Voevodsky: *A^1-homotopy theory*, preprint, (1998)

[Neem] A. Neeman: *A Grothendick duality theorem via Bousfield's techniques and Brown representability*, JAMS, **9**, (1996), 205-236

[N-B] A. Neeman and M. Bockstedt: *Homotopy limits in triangulated categories*, Compositio Math, **86**, (1993), 209-234

[Qu] D. Quillen: *Homotopical algebra*, Springer Lect. Notes, vol 43, (1967)

[Rect] D. Rector: *Steenrod Operations in the Eilenberg-Moore spectral sequence*, Commentari. Math., vol 45, (1970), 540-552

[Rob] A. Robinson: Derived tensor products in stable homotopy theory,

[Seg] G. Segal:*The representation ring of a compact Lie group* , Publ. Math. vol 34, pp.113-128, (1968)

[SS] S. Schwede and B. Shipley: *Algebras and modules in model categories*, Proc. Lond. Math. Soc., **80**, 2000, 491-511

[SGA4] SGA4:Seminaire de geometrie algebrique, Springer Lect. notes vols. 269, 270 and 305, Springer, (1971)

[$SGA4_{1/2}$] SGA4.5:Seminaire de geometrie algebrique, *Cohomologie etale* Springer Lect. notes in Math, vol 569, Springer, (1977).

[SGA5] SGA5:Seminaire de geometrie algebrique, *Cohomologie ℓ-adic et Fonctions L*, Springer Lect. Notes, No. 589, Springer Verlag, (1977)

[SGA6] SGA6:Seminaire de geometrie algebrique, *Théorie des intersections et Théorem de Riemann-Roch*, Springer Lect. Notes, # 225, Springer Verlag, (1971)

[Simp-1] C. Simpson, *Flexible sheaves*, preprint, (1997)

[Simp-2] C. Simpson, *A Giraud-type characterization of the simplicial categories associated to closed model categories as ∞-pretopoi*, preprint, (1997)

[Simp-3] C. Simpson, *The topological realization of a simplicial presheaf*, preprint, (1997)

[Spalt] N. Spaltenstein: *Resolutions of unbounded complexes*, Compositio Math., **65**, (1988), 121-154

[Sp] E. Spanier, *Algebraic Topology*, McGrawHill, (1966)

[T-1] R. Thomason: *Algebraic K-theory and étale cohomology*, Ann. Sci. Ecol. Norm. Sup., **18**, (1985), 437-552

[T-2] R. Thomason: *Comparison of equivariant algebraic and topological K-theory*, Duke Math. J. vol 53, pp.795-825, (1986)

[T-3] R. Thomason:*First quadrant spectral sequences in algebraic K-theory via homotopy colimits*, Comm. Algebra, pp. , (1983)

[Toe-1] B. Toen: Schematization of homotopy types, preprint , (2000)

[Toe-2] B. Toen: Toward a Galoisian interpretation of homotopy theory, preprint, (2000)

[Verdier] J. L. Verdier: *Classe d'homologie d'un cycle*, Asterisque, No. 32-33, Expose VI, pp. 101-152.

[Voe-1] V. Voevodsky: *The Milnor conjecture*, preprint, (1996)

[Voe-2] V. Voevodsky: *Triangulated categories of motives over a field*, Ann. Math. Study, **143**, (2000), 188-238.

[Voe-3] V. Voevodsky: *Open problems in the motivic stable homotopy theory, I*, Preprint, (2000).

[Vogt-1] R. Vogt:*Homotopy limits and colimits*, Math. Z., vol 134, pp.11-52, (1973)

[Vogt-2] R. Vogt: *Convenient categories of topological spaces for homotopy theory*, Archiv der Math, vol 22, pp.545-555, (1971)

[Wh] G. Whitehead: *Generalised cohomology theories*, Trans. AMS., vol 102, pp.227-283, (1962)

Index