Higher Grassmann Codes III: Quantum Variants

Mahir Bilen Can and Roy Joshua

***Abstract*—We show that it is possible to break up Higher Grassmann codes we constructed in earlier work to a sequence of affine Reed-Muller codes so that various operations can be reduced to performing these operations for the component affine Reed-Muller codes. Then we consider quantum codes produced from a pair of Higher Grassmann codes and discuss also their implementation aspects in detail.**

***Index Terms*—Error correcting codes, quantum communication.**

I. Introduction

THE present paper is a follow-up to our earlier work where we constructed what are called Higher Grassmann codes and estimated their parameters. The Grassmannians are usually imbedded into projective spaces making use of the Plücker imbedding. The (usual) Grassmann codes considered in the literature as in [19] make use of such imbeddings. Higher Grassmann codes are algebraic geometric codes defined by starting with Grassmann varieties defined over finite fields, and using higher imbeddings of such Grassmannians into projective spaces: these are obtained by composing a diagonal Plücker imbedding with a corresponding Segre imbedding. By varying such imbeddings we obtain a family of Higher Grassmann codes.

The first part of this paper is devoted to a detailed analysis of such Higher Grassmann codes, with the main details discussed in section III, and section II serving as an introduction to our techniques. We show that it is possible to break up the Higher Grassmann code into a sequence of affine Reed-Muller codes in an iterative manner, so that various operations can be reduced to performing these operations for the component affine Reed-Muller codes.

In section IV, we provide a rather detailed comparison of the parameters of the Higher Grassmann codes with those of the projective Reed-Muller codes. In the remaining part of the paper, namely sections V through 7, we focus on quantum codes constructed from pairs of such Higher Grassmann codes, for example, by making use of the CSS-construction. Section V discusses the basic construction of such quantum codes. We also show that such quantum codes inherit several nice properties from the Higher Grassmann codes used in their construction. Section VI discusses how to obtain the stabilizer matrices for the quantum codes from the parity check matrices of the Higher Grassmann codes. Finally section VII discusses certain applications to fault tolerant quantum computation making use of an affine variant of the Higher Grassmann codes.

Received 16 November 2023; revised 17 March 2025; accepted 25 July 2025. Date of publication 6 August 2025; date of current version 17 September 2025. The work of Mahir Bilen Can was supported by the Louisiana Board of Regents. The work of Roy Joshua was supported by the Simons Foundation. *(Corresponding author: Roy Joshua.)*

Mahir Bilen Can is with the Department of Mathematics, Tulane University, New Orleans, LA 70118 USA (e-mail: mahirbilencan@gmail.com).

Roy Joshua is with the Department of Mathematics, The Ohio State University, Columbus, OH 43210 USA (e-mail: joshua@math.ohio-state.edu).

Communicated by S. R. Ghorpade, Associate Editor for Coding and Decoding.

Digital Object Identifier 10.1109/TIT.2025.3596479

II. Geometry of Grassmannians

This section is essentially a quick review of the Higher Grassmann codes constructed in our prior work: see [4] and [5]. Throughout the following discussion, we will fix a finite field $\mathbb{F}_q$ and consider objects defined over it. Let V denote an m-dimensional vector space over $\mathbb{F}_q$ and let $\{e_1, \ldots, e_m\}$ denote a chosen basis of V. Note that the rows of any $l \times m$ matrix $A = A(M)$, with the rank of A equal to l generates an l dimensional subspace of V. Two such matrices A_1, A_2 span the same vector subspace if and only if there exists $g \in \mathbf{GL}_l$ such that $A_1 = gA_2$. This provides the following description of the Grassmannian of l-planes in V. Let $\mathbf{Mat}_{l,m}$ denote the space of $l \times m$ matrices over $\mathbb{F}_q$ and let $\mathbf{Mat}^0_{l,m}$ denote the Zariski open subset consisting of rank l matrices. Then $\mathbf{GL}_l$ acts by left matrix multiplication on $\mathbf{Mat}^0_{l,m}$, and the quotient is precisely the Grassmann variety $Gr(l, V)$.

Finally, let us point out the fact that $Gr(l, V)$ is a homogeneous space for $\mathbf{SL}_m$ as well as for $\mathbf{GL}_m$:

$$\begin{aligned} Gr(l, V) &\cong \mathbf{SL}_m/\mathrm{Stab}_{\mathbf{SL}_m}(\langle e_1, \ldots, e_l\rangle) \\ &\cong \mathbf{GL}_m/\mathrm{Stab}_{\mathbf{GL}_m}(\langle e_1, \ldots, e_l\rangle). \end{aligned}$$

Here, $\langle e_1, \ldots, e_l\rangle$ is the l-dimensional subspace spanned by $e_1, \ldots, e_l$ in V, and $\mathrm{Stab}_{\mathbf{SL}_m}(\langle e_1, \ldots, e_l\rangle)$ (resp. $\mathrm{Stab}_{\mathbf{GL}_m}(\langle e_1, \ldots, e_l\rangle)$ stands for the stabilizer subgroup of $\langle e_1, \ldots, e_l\rangle$ in $\mathbf{SL}_m$ (resp. in $\mathbf{GL}_m$).

A. Plücker Coordinates and the Plücker Imbedding

Let

$$\begin{aligned} I(l, m) = \{\alpha = (\alpha_1, \cdots, \alpha_l) | 1 \\ \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l \leq m\}. \end{aligned} \tag{II-A.1}$$

One defines a partial order $\leq$ on $I(l, m)$ by defining $\alpha \leq \beta$, if and only if for each i, $1 \leq i \leq l$, $\alpha_i \leq \beta_i$. In this case we will define $\alpha \leq \beta$. Observe that an $l \times m$-matrix A has rank l if and only if there exist l columns indexed by an element $\alpha \in I(l, m)$ which are linearly independent, or equivalently the corresponding minor of A has nonzero determinant. Thus $I(l, m)$ denotes the set of *Plücker coordinates* for the Grassmannian $Gr(l, V)$, where the Plücker coordinate corresponding to $\alpha \in I(l, m)$ is the determinant of the corresponding minor.

Observe that one now obtains a closed immersion $Gr(l, V) \to \mathbb{P}(\wedge^l V)$ by sending the l-dimensional subspace spanned by vectors $\{v_{\alpha_1}, \cdots, v_{\alpha_l}\}$ to the line $v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_l}$. This can also be viewed as the mapping sending an $l \times m$ matrix of

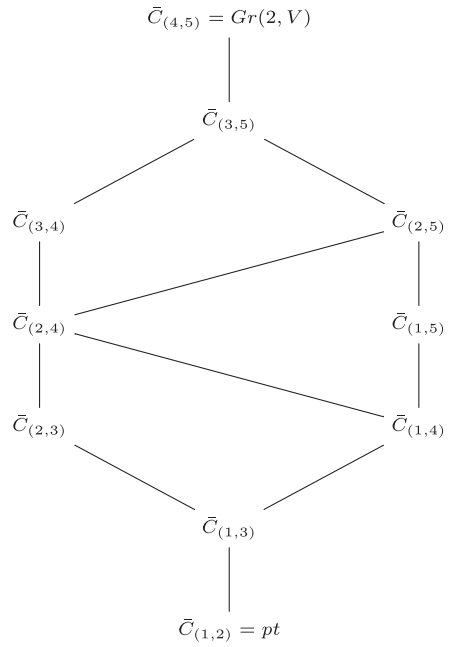

Fig. 1. The Grassmann variety $Gr(2,V)$ and its Schubert varieties, where $\dim_k V = 5$.

rank l to the sequence $(det(A_\alpha)|\alpha \in I(l,m))$, where $\alpha \in I(l,m)$ denotes the columns of A which are linearly independent. This is the *Plücker imbedding*.

B. The Schubert Cell Decomposition of $Gr(l,V)$

We will fix a basis $\{e_1, \cdots, e_m\}$ for the vector space V, and consider the full flag $(V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m = V)$ in V, where $V_i = Span(e_1, \cdots, e_i)$. Then for each $\alpha \in I(l,m)$, we let the Schubert cell indexed by α be

$$C_\alpha = \{W \in Gr(l,V) | dim(W \cap V_j) = i, \text{ if } \alpha_i \le j < \alpha_{i+1}, 1 \le j \le m, 0 \le i \le l\} \quad \text{(II-B.1)}$$

As is well-known each Schubert cell C_α is an affine space of dimension $\Sigma_{i=1}^m \alpha_i - l(l+1)/2$. The closure of the Schubert cell C_α will be denoted $\bar{C}_\alpha$ and is the corresponding *Schubert variety*. One may now show that given two Schubert cells C_α and C_β, $C_\alpha \subseteq \bar{C}_\beta$ if and only if $\alpha \le \beta$. This order is called the Bruhat-Chevalley order. The Hasse diagram of the Bruhat-Chevalley order is given in Figure 1 for the special case when $dim_k V = 5$ and $l = 2$.

In the above example, the top cell $C_{(4,5)}$ is of dimension 6, the next cell $C_{(3,5)}$ is of dimension 5, with the dimension of the cells in each successive lower level, being one less than the cells in the previous level. Thus the cell $C_{(3,5)}$ has dimension 5, the cells $C_{(3,4)}$ and $C_{(2,5)}$ of dimension 4, the cells $C_{(2,4)}$ and $C_{(1,5)}$ of dimension 3, the cells $C_{(2,3)}$ and $C_{(1,4)}$ of dimension 2, the cell $C_{(1,3)}$ of dimension 1 and the cell $C_{(1,2)}$ of dimension 0.

In general, it is well-known that

$$Gr(l,V) = \cup_\alpha C_\alpha. \quad \text{(II-B.2)}$$

It is also not difficult to check from (II-B.1) that the Schubert cell $C_{(m-l+1,m-l+2,\ldots,m)}$ is open and dense in $Gr(l,V)$, and that, there is a unique codimension one Schubert subvariety $\bar{C}'$ in $Gr(l,V)$. The indexing set of the Schubert cells in $\bar{C}'$ is given by $(m-l, m-l+2, m-l+3, \ldots, m)$.

C. Projective Embeddings of $Gr(l,V)$

We quickly review the following key definitions from [4, 2.1].

Let $\mathbf{T}_m$ denote the maximal diagonal torus of $\mathbf{SL}_m$. The root system of the pair $(\mathbf{SL}_m, \mathbf{T}_m)$ will be denoted by R. Explicitly, it is given by the set of vectors $R = \{\varepsilon_i - \varepsilon_j : 1 \le i, j \le m, i \ne j\}$, where $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is the standard basis for the m-dimensional Euclidean $\mathbb{Q}$-vector space. The system of positive roots determined by $\mathbf{B}_m$, denoted by R^+, is given by $R^+ = \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le m\}$. The subset of simple roots in R^+ will be denoted by S; it is given by $S = \{\alpha_i : \alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \le i \le m-1\}$. The duals of the basis vectors α_i $(1 \le i \le m-1)$ are denoted by $\alpha_i^\vee$, and the *fundamental weights* ϖ_i $(1 \le i \le m-1)$ are defined by equations

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha_j \in S$. Note that the i-th fundamental weight ϖ_i is the highest weight vector of the i-th fundamental representation $\wedge^i k^m$ of $\mathbf{SL}_m$. Here, k is the underlying field on which the representation is defined, and $\wedge^i$ denotes the i-th exterior power of the vector space k^m. The submonoid generated by ϖ_i $(1 \le i \le m-1)$ in $X(\mathbf{T}_m)$, denoted by $X(\mathbf{T}_m)_+$, is the *monoid of dominant weights*. Then we have,

$$X(\mathbf{T}_m)_+ = \{\lambda \in X(\mathbf{T}_m) : \langle \lambda, \alpha_i^\vee \rangle \ge 0 \text{ for every } \alpha_i \in S\}.$$

It is well-known that for every finite dimensional irreducible representation W of $\mathbf{SL}_m$, there is a unique dominant weight $\lambda \in X(\mathbf{T}_m)_+$ called the *highest weight of* W, see [14, Chapter II.2]. In other words, simple $\mathbf{SL}_m$-modules are parametrized by the elements of $X(\mathbf{T}_m)_+$.

Let V denote the m-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q^m$ with the standard basis $\{e_1, \cdots, e_m\}$. The l-th fundamental representation of $\mathbf{SL}_m$ is given by the l-th exterior power of V. It is well-known that the Picard group of $Gr(l,V)$ is generated by the ample line bundle $\mathcal{L}(\varpi_l)$, corresponding to the weight ϖ_l.

The dual of the space of global sections, that is $H^0(Gr(l,V), \mathcal{L}(\varpi_l))^*$, is isomorphic to $\wedge^l V$. Therefore, the Plücker coordinates on $Gr(l,V)$ are given by the restrictions of the coordinate functions on the affine space $\mathbb{A}^{\binom{m}{l}} \cong \wedge^l V$. In general, our higher imbeddings are given as follows, where $0 \le \nu \le (q-1)(l(m-l))$:

$$\iota : Gr(l,V) \longrightarrow \mathbb{P}\left(\prod^\nu (\wedge^l V)\right) \longrightarrow \mathbb{P}((\bigwedge^l V)^{\otimes \nu}). \quad \text{(II-C.1)}$$

Let $\mathcal{O}_{\mathbb{P}((\bigwedge^l V)^{\otimes \nu})}(1)$ denote the first Serre twist of the structure sheaf of $\mathbb{P}((\bigwedge^l V)^{\otimes \nu})$. The pullback of this line bundle under the Segre embedding is equal to the ν-fold tensor product $\mathcal{O}_{\mathbb{P}(\bigwedge^l V)}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}(\bigwedge^l V)}(1)$. The restriction of this product to the diagonal, which is isomorphic to $\mathbb{P}(\bigwedge^l V)$, is given by the multiplication of the sections of the factors; it lands in $\mathcal{O}_{\mathbb{P}(\bigwedge^l V)}(\nu)$. Therefore, we notice that a degree one hyperplane in $\mathbb{P}((\bigwedge^l V)^{\otimes \nu})$ determines a degree ν hypersurface in $\mathbb{P}(\bigwedge^l V)$.

We will discuss this in more detail by considering the following example, where we take $\nu = 2$. Now we first embed

$Gr(l,V)$ into $\mathbb{P}^s \times \mathbb{P}^s$, where $s = \binom{m}{l} - 1$; this embedding is given by the composition of the diagonal embedding of $Gr(l,V)$ into $Gr(l,V) \times Gr(l,V)$ followed by the doubled Plücker embedding. Then we use the Segre embedding to embed the doubled projective space into a bigger projective space. We denote the morphism defined by these compositions by ι. In summary, we have the following diagram:

$$\iota : Gr(l,V) \xrightarrow{\text{diag}} Gr(l,V) \times Gr(l,V) \xrightarrow{\mathbf{p}\times\mathbf{p}} \mathbb{P}^s \times \mathbb{P}^s \xrightarrow{\text{Segre}} \mathbb{P}^{s^2+2s}. \tag{II-C.2}$$

We will describe explicitly the image of (II-C.2).

Let M be a point from $Gr(l,V)$, and let $(m_1, m_2, \ldots, m_{s+1})$ denote its image under the Plücker embedding. Then we have

$$\iota : \; M \xrightarrow{\text{diag}} (M,M) \xrightarrow{(\mathbf{p},\mathbf{p})} ((m_1,\ldots,m_{s+1}),(m_1,\ldots,m_{s+1})) \xrightarrow{\text{Segre}} (m_i m_j)_{\substack{i=1,\ldots,s+1 \\ j=1,\ldots,s+1}}. \tag{II-C.3}$$

Equivalently, $(m_i m_j)_{\substack{i=1,\ldots,s+1 \\ j=1,\ldots,s+1}}$ is the point that is represented by the tensor product $M \otimes M$ in $\mathbb{P}(\bigwedge^l V \otimes \bigwedge^l V)$.

D. Tableaux, Standard Monomials and the Homogeneous Coordinate Ring of $Gr(l,V)$

(i) A *tableaux* in $I(l,m)$ of length n (or a tableaux on $Gr(l,V)$) is a sequence $(\alpha(1), \cdots, \alpha(n))$, with each $\alpha(i) \in I(l,m)$. Such a tableaux is *standard* if $\alpha(1) \geq \alpha(2) \geq \cdots \geq \alpha(n)$.

(ii) A *standard monomial* on $Gr(l,V)$ of length n is a formal expression $p_{\alpha(1)} p_{\alpha(2)} \cdots p_{\alpha(n)}$, where each $p_{\alpha(i)}$ are Plücker coordinates and $(\alpha(1), \cdots, \alpha(n))$ is a standard tableaux. (See [25, Definition 1.3.1].)

(iii) The homogeneous coordinate ring of the Grassmannian $Gr(l,V)$ is the graded ring

$$\oplus_{\nu \geq 0} \Gamma(Gr(l,V), \mathcal{O}_{Gr(l,V)}(\nu)),$$

where $\Gamma(Gr(l,V), \mathcal{O}_{Gr(l,V)}(\nu))$ is given by the span of all the standard monomials of length ν. (See [25, Proposition 1.3.6].)

We conclude this section by recalling two main results from [5].

Theorem 1: (See [5, Theorem 6.2].) Let $C_{Gr(l,V)}(\nu)$ denote the q-ary degree ν Higher Grassmann code on $Gr(l,V)$. Let r and s be the non-negative integers defined by $\nu - 1 = r(q-1) + s$, where $0 \leq s < q-1$. If $\nu < (q-1)l(m-l)$, then the minimum distance d of $C_{Gr(l,V)}(\nu)$ is bounded as follows:

$$(q-s)q^{l(m-l)-r-1} \leq d \leq q^{l(m-l)} - sq^{l(m-l)-r-1}.$$

Theorem 2: (See [5, Theorem 7.4].) Let $\nu \in [(q-1)l(m-l)]$. Let $C_{Gr(l,V)}(\nu)$ denote the q-ary degree ν Grassmann code on $Gr(l,V)$. If μ is defined by the equation $\mu := (q-1)l(m-l) - \nu$, then the dual of $C_{Gr(l,V)}(\nu)$ is given by one of the following two cases:

$$C_{Gr(l,V)}(\nu) = \begin{cases} C_{Gr(l,V)}(\mu) & \text{if } \nu \neq 0 \mod (q-1), \\ \overline{C_{Gr(l,V)}(\mu)} & \text{otherwise,} \end{cases} \tag{II-D.1}$$

where $\overline{C_{Gr(l,V)}(\mu)}$ denotes the extension of the code $C_{Gr(l,V)}(\mu)$ by adding the code-word that is 1 everywhere.

Remark 1: Observe that there was a misstatement in the upper bound in Theorem 1. The misstatement in [5, Theorem 7.4] can be traced back to [5, Lemma 7.2(1)]. For $\nu \in 0, 1, ..., (q-1)l(m-l)$, let $\kappa(\nu)$ denote the number of standard monomials $p_{\alpha(j_1)}^{i_1} \cdots p_{\alpha(j_r)}^{i_r}$ such that $\Sigma_{j=1}^r i_j = \nu$ and $i_j \in \{0, \cdots, q-1\}$ for all $j \in \{1, \cdots, r\}$. In this notation, [5, Lemma 7.2 (1)] states the following: The numbers $\kappa(\nu)$ satisfy the equalities $\kappa(\nu) = \kappa((q-1)l(m-l) - \nu)$. The corrected statement of [5, Lemma 7.2 (1)] asserts the following: the numbers $\kappa(\nu)$ satisfy the equalities $\kappa(\nu) = \kappa((q-1)(l(m-l)+1) - \nu)$. These appear in [6]. See also [11] for related material.

III. The Decomposition of the Higher Grassmann Codes

Let α_{max} (α_{min}) denote the maximal element $(m-l+1, m-l+2, \cdots, m)$ (the minimal element $(1, 2, \cdots, l)$ of $I(l,m)$. As we observed above $p_{\alpha_{max}}$ is the Plücker coordinate corresponding to the open cell and $p_{\alpha_{min}}$ is the Plücker coordinate corresponding to the smallest cell, which is zero dimensional.

Let $p_{\alpha(1)} \cdots p_{\alpha(n)}$ denote a standard monomial of degree ν, viewed as a section in

$$\Gamma(Gr(l,V), \mathcal{O}_{Gr(l,V)}(\nu)).$$

Let C_{α_0} denote the Schubert cell corresponding to the fixed Plücker coordinate α_0. Then, the restriction of the standard monomial $p_{\alpha(1)} \cdots p_{\alpha(n)}$ to C_{α_0} is obtained by performing the following operations on $p_{\alpha(1)} \cdots p_{\alpha(n)}$ (see [25, 1.3.1 Definition (iii)], as well as [25, 1.2.10 Proposition, 1.3.12 Proposition]):

(i) one puts the Plücker coordinate $p_{\alpha_0} = 1$,

(ii) any $\alpha \neq \alpha_0$ on any path starting from α_{max} and ending at α_0 in the corresponding Hasse diagram, correspond to Schubert cells whose closures contain C_{α_0}. Therefore, one puts the Plücker coordinate $p_\alpha = 0$ for such α, and

(iii) one puts all the Plücker coordinates p_β which are *not* on any path starting at p_{α_0} and ending at $p_{\alpha_{min}}$ in the corresponding Hasse diagram to be 0.

Let $\mathrm{HG}(\nu)$ denote the Higher Grassmann code on $Gr(l,V)$ of degree ν.[1] Given a Schubert cell C_{α_0} (a Schubert variety $\bar{C}_\alpha$) let $\mathrm{HG}(\nu)_{|C_{\alpha_0}}$ ($\mathrm{HG}(\nu)_{|\bar{C}_\alpha}$) denote the restriction of the code $\mathrm{HG}(\nu)$ to the rational points lying in C_{α_0} ($\bar{C}_\alpha$, respectively). Let $\bar{C}_{\alpha_i}$, $i = 1, \cdots, n$ denote the Schubert varieties appearing in the boundary of the Schubert cell C_{α_0}.

The code $\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}}$ is obtained by evaluating all linear combinations of the standard monomials at the rational points on the Schubert cell C_{α_0} and on the lower dimensional Schubert cells forming the boundary. Therefore, restricting a linear combination of standard monomials to the rational points in the boundary of the Schubert cell $C(\alpha_0)$ (denoted $\delta\bar{C}_{\alpha_0}$) defines a restriction map:

$$res_{\alpha_0} : \mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}} \to \mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}. \tag{III-.1}$$

It may be important to observe that the restriction map res_{α_0} is an instance of *puncturing* the code $\mathrm{HG}(\nu)_{|\bar{C}(\alpha_0)}$ by evaluating only on the rational points of $Gr(l,V)$ contained in $\cup_{i=1}^n \bar{C}(\alpha_i)$. (Observe from [12, Theorem 1.5.1] that puncturing need not preserve the minimum distance.)

[1] In [5], this code was denoted $C_{Gr(l,V)}(\nu)$. However this notation would become a bit too inconvenient for the purposes of this paper, as we have to focus on restrictions of this code to the various Schubert cells.

Let $\mathrm{RM}_{C_{\alpha_0}}(\nu-1)$ denote the affine Reed-Muller code of degree $\nu-1$ on the Schubert cell C_{α_0}: recall this is generated by monomials of total degree $\leq \nu-1$ in the coordinates of the affine cell C_{α_0}. Observe that, $p_{\alpha_0}=1$ everywhere on the cell C_{α_0}. Therefore, any polynomial of degree at most $\nu-1$ in the Plücker coordinates other than p_{α_0} restricted to the cell C_{α_0} may be viewed as a homogeneous polynomial of degree ν in the Plücker coordinates including p_{α_0}. This defines the injective map of the code $\mathrm{RM}_{C_{\alpha_0}}(\nu-1)$ into $\mathrm{HG}(\nu)_{\bar{C}(\alpha_0)}$, sending a polynomial of degree at most $\nu-1$ in the Plücker coordinates other than p_{α_0} to the corresponding homogeneous polynomial of degree ν involving the Plücker coordinate p_{α_0}.

$$i_{\alpha_0}:\mathrm{RM}_{C_{\alpha_0}}(\nu-1)\to \mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}}. \tag{III-.2}$$

Now we have the following Lemma.

Lemma 1: Let $p_{\alpha_0}^{i_0}p_{\alpha_1}^{i_1}\cdots p_{\alpha_j}^{i_j}$ denote a standard monomial which vanishes at every point on the boundary of the Schubert cell C_{α_0}. Then the exponent $i_0\geq 1$.

Proof: Observe that the boundary of the open Schubert cell is a union of lower dimensional Schubert cells, where the variables $p_{\alpha_1},\cdots,p_{\alpha_j}$ take on all possible values, with p_{α_0} always zero. Now suppose $i_0=0$ and let $i_{j_1},\cdots,i_{j_k}$ denote the exponents among $\{i_1,\cdots,i_j\}$ which are all non-zero. Then there is a point on the boundary of the open Schubert cell whose $p_{\alpha_{j_1}},\cdots,p_{\alpha_{j_k}}$-coordinates are all nonzero. Therefore, the given monomial will not vanish at that point on the boundary of the open Schubert cell, contradicting our assumption. □

Theorem 3: (i) The diagram

$$0\to \mathrm{RM}_{C_{\alpha_0}}(\nu-1)\xrightarrow{i_{\alpha_0}}\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}}\xrightarrow{res_{\alpha_0}}\mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}\to 0 \tag{III-.3}$$

is an exact sequence of codes in the sense that the map res_{α_0} is a surjective map of codes whose kernel is the image of the map i_{α_0}.

(ii) One obtains an isomorphism: $\mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}\cong \Pi_{i=1}^{n}\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_i}}$ where the product denotes an iterated fibered product of the restriction of the code $\mathrm{HG}(\nu)$ to the Schubert varieties forming the boundary of the Schubert variety $\bar{C}_{\alpha_0}$.

Proof: Observe that we only need to consider the restriction of standard monomials $p_{\alpha_0}p_{\alpha(1)}\cdots p_{\alpha(n)}$, with $p_{\alpha_0}\geq p_{\alpha(1)}\geq\cdots\geq p_{\alpha(n)}$. Now restricting such a standard monomial to the boundary of the cell C_{α_0}, corresponds to putting the Plücker coordinate $p_{\alpha_0}=0$ in all such standard monomials. Any section of $\mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}$ is given by a linear combination of the standard monomials of the above form and of degree ν that does not involve the Plücker coordinate p_{α_0}. Clearly any such linear combination of the standard monomials is in the image of the restriction map

$$\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}}\xrightarrow{res_{\alpha_0}}\mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}.$$

These observations prove that the restriction res_{α_0} is surjective: in fact this follows from Standard monomial theory. (See [25, 1.3]).

Next we proceed to show that the kernel of the above restriction map res_{α_0} identifies with the image of $\mathrm{RM}_{C_{\alpha_0}}(\nu-1)$ under the map i_{α_0}. Lemma 1 shows that every standard monomial in the kernel of the restriction map res_{α_0} must have p_{α_0} as a factor. On the other hand, $p_{\alpha_0}=1$ everywhere on the cell C_{α_0}. The net effect is that any linear combination of standard monomials in the kernel of the restriction map res_{α_0} in (III-.1) is a homogeneous polynomial of degree ν with p_{α_0} appearing in every monomial of this homogeneous polynomial: such a homogeneous polynomial identifies with a polynomial of degree $\leq \nu-1$ in the Plücker coordinates other than p_{α_0} when p_{α_0} is inverted, i.e., on the cell C_{α_0} where p_{α_0} is 1. Therefore, this shows that the kernel of the restriction map res_{α_0} identifies with the image of the map $i_{\alpha_0}:\mathrm{RM}_{C_{\alpha_0}}(\nu-1)\to\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_0}}$ and completes the proof of the first statement in the theorem.

To prove the second statement, one needs to show that the obvious map

$$\mathrm{HG}(\nu)_{|\delta\bar{C}_{\alpha_0}}\to\Pi_{i=1}^{n}\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_i}}$$

is an isomorphism. For this first observe that the boundary $\delta\bar{C}_{\alpha_0}$ is an iterated union of the Schubert cells forming the boundary $\delta\bar{C}_{\alpha_0}$. Since the code $\mathrm{HG}(\nu)$ is obtained by evaluating linear combinations of the standard monomials of degree ν at the rational points on this iterated union of Schubert cells, the resulting code becomes an iterated fibered product of the codes $\mathrm{HG}(\nu)_{|\bar{C}_{\alpha_i}}$, where the Schubert varieties $\bar{C}_{\alpha_i}$ are the closures of the Schubert cells forming the boundary of the Schubert variety $\bar{C}_{\alpha_0}$. □

Remarks 1:

(i) It may also be important to observe that the inclusion i_{α_0} corresponds to a *shortening* of the code $\mathrm{HG}(\nu)_{|\bar{C}(\alpha_0)}$.

(ii) Observe that, since any short exact sequence of vector spaces over a field splits, the above sequence splits as a short exact sequence of vector spaces over the given field $\mathbb{F}_q$. However, it does *not* split as a short exact sequence of codes.

(iii) One can see this clearly in the example worked out below for the Grassmannian $Gr(2,V)$, where $dim_{\mathbb{F}_q}(V)=5$: see Example 1.

(iv) The analogue of the above theorem is discussed in [16, 5.1], in the special case when the Grassmannian is a projective space.

We will explain the above results by means of the example of the Grassmannian in Figure 1.

Example 1: Consider the Grassmannian of 2-planes in 5-dimensional space as in Figure 1. Let $p_{(i,j)}$, $1\leq i<j\leq 5$ denote the corresponding Plücker coordinates. Now observe that the Plücker coordinate $p_{(4,5)}$ corresponds to the open cell, while the Plücker coordinate $p_{(1,2)}$ corresponds to the smallest Schubert cell, which is in fact 0 dimensional.

Then putting $p_{(4,5)}\neq 0$ defines the open Schubert cell, putting $p_{(4,5)}=0$, $p_{(3,5)}\neq 0$ defines the next highest dimensional Schubert cell, putting $p_{(4,5)}=p_{(3,5)}=p_{(2,5)}=0$ and $p_{(3,4)}\neq 0$ defines the Schubert cell corresponding to $p_{(3,4)}$, etc.. More generally to consider the Schubert cell corresponding to the variable $x_{(i,j)}$, one needs to put $x_{(i,j)}=1$ and all other variables $x_{(i',j')}$, other than $x_{(i,j)}$ and those variables that are on a path leading downwards from the node corresponding to $x_{(i,j)}$ to be zero.

We then obtain the following short exact sequences of codes:

(i) $0\to \mathrm{RM}_{C_{(4,5)}}(\nu-1)\xrightarrow{i_{(4,5)}}\mathrm{HG}(\nu)\xrightarrow{res}\mathrm{HG}(\nu)_{|\bar{C}_{(3,5)}}\to 0$,

(ii) $0\to \mathrm{RM}_{C_{(3,5)}}(\nu-1)\xrightarrow{i_{(3,5)}}\mathrm{HG}(\nu)_{|\bar{C}_{(3,5)}}\xrightarrow{res}\mathrm{HG}(\nu)_{|\bar{C}_{(3,4)}}\times_{\mathrm{HG}(\nu)_{|\bar{C}_{(2,4)}}}\mathrm{HG}(\nu)_{|\bar{C}_{(2,5)}}\to 0$,

(iii) $0\to \mathrm{RM}_{C_{(3,4)}}(\nu-1)\xrightarrow{i_{(3,4)}}\mathrm{HG}(\nu)_{|\bar{C}_{(3,4)}}\xrightarrow{res}\mathrm{HG}(\nu)_{|\bar{C}_{(2,4)}}\to 0$,

(iv) $0 \to \mathrm{RM}_{C_{(2,5)}}(\nu-1) \xrightarrow{i_{(2,5)}} \mathrm{HG}(\nu)_{|\bar{C}_{(2,5)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(2,4)}} \times_{\mathrm{HG}(\nu)_{|\bar{C}_{(1,4)}}} \mathrm{HG}(\nu)_{|\bar{C}_{(1,5)}} \to 0,$

(v) $0 \to \mathrm{RM}_{C_{(2,4)}}(\nu-1) \xrightarrow{i_{(2,4)}} \mathrm{HG}(\nu)_{|\bar{C}_{(2,4)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(2,3)}} \times_{\mathrm{HG}(\nu)_{|\bar{C}_{(1,3)}}} \mathrm{HG}(\nu)_{|\bar{C}_{(1,4)}} \to 0,$

(vi) $0 \to \mathrm{RM}_{C_{(1,5)}}(\nu-1) \xrightarrow{i_{(1,5)}} \mathrm{HG}(\nu)_{|\bar{C}_{(1,5)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(1,4)}} \to 0,$

(vii) $0 \to \mathrm{RM}_{C_{(2,3)}}(\nu-1) \xrightarrow{i_{(2,3)}} \mathrm{HG}(\nu)_{|\bar{C}_{(2,3)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(1,3)}} \to 0,$

(viii) $0 \to \mathrm{RM}_{C_{(1,4)}}(\nu-1) \xrightarrow{i_{(1,4)}} \mathrm{HG}(\nu)_{|\bar{C}_{(1,4)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(1,3)}} \to 0,$ and

(ix) $0 \to \mathrm{RM}_{C_{(1,3)}}(\nu-1) \xrightarrow{i_{(1,3)}} \mathrm{HG}(\nu)_{|\bar{C}_{(1,3)}} \xrightarrow{res} \mathrm{HG}(\nu)_{|\bar{C}_{(1,2)}} \to 0.$

A. The Generator and Parity Check Matrices for the Higher Grassmann Codes

We begin with the following observation. Let $\mathcal{P}$ denote the set of cells of the Grassmannian $Gr(l, V)$, partially ordered by the Bruhat order. Then [27, Proposition 3.5.2] shows the existence of the order preserving map from $\mathcal{P} \to \{1, \ldots, |\mathcal{P}|\}$ so that we can enumerate the cells C_α in the Grassmannian $Gr(l, V)$ by numbers $\alpha \in \{1, \ldots, |\mathcal{P}|\}$ in such a way that if a cell C_α is contained in the closure of another cell C_β, then $\alpha < \beta$ holds in $\{1, \ldots, |\mathcal{P}|\}$. Such a map on a poset is called a *linear extension* of the poset under consideration. Note that $|\mathcal{P}|$ is the number of Young diagrams, including the empty diagram, that fit into an $l \times (m-l)$ grid.

We now fix a linear extension $\alpha_0, \ldots, \alpha_{|\mathcal{P}|}$ of the Bruhat-Chevalley order on the cells of $Gr(l, V)$. Note that the Bruhat-Chevalley order on the Schubert cells is a self-dual poset meaning that if we reverse the Bruhat-Chevalley order, then we get an isomorphic poset. It follows that the linear extension could be constructed for the opposite poset as well. With this fact in mind, we proceed with the assumption that $C_{\alpha_0} = C_{\alpha_{max}}$ denotes the open cell, C_{α_1} denotes the next highest dimensional cell, etc., in the enumeration of the Schubert cells described in the last paragraph.

Using this extension, together with the decomposition of Higher Grassmann codes described in the previous section, we arrive at the following description of the generator matrices for these codes. In line with the convention in [12, Chapter 1, 1.2], we assume that the rows of the generator matrix form a basis for the code. Moreover the columns from left to right correspond to the rational points on the Grassmannian, beginning with the rational points in the open cell in the left-most columns, then the rational points in the next higher dimensional cell, etc.

$$G(\nu) = \begin{pmatrix} G_{C_{\alpha_0}}(\nu-1) & 0 & 0 & 0 \cdots 0 \\ * & G_{C_{\alpha_1}}(\nu-1) & 0 & 0 \cdots 0 \\ * & * & G_{C_{\alpha_2}}(\nu-1) & 0 \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \cdots \vdots \\ * & * & * & * \cdots * \end{pmatrix}. \quad \text{(III-A.1)}$$

Here $C_{\alpha_0} = C_{\alpha_{max}}$ denotes the open cell, C_{α_1} denotes the next highest dimensional cell, etc., in the enumeration of the Schubert cells described in the last paragraph. The entries * in (III-A.1) indicate the fact that starting with each code word forming a row of the matrix $G_{C(\alpha_i)}$, for $i \geq 1$, one needs to lift them to code words in the Higher Grassmann code. Thus the generator matrix for the Higher Grassmann code can be assembled by starting with the generator matrices for the affine Reed-Muller codes associated to each of the Schubert cells, and then lifting each row to a code word in the Higher Grassmann code. This is possible iteratively, making use of the Hasse diagram of the boundary structure of each Schubert variety, and in view of the fact that the restriction map *res* in the short exact sequence in (III-.3) is surjective. It is important to realize that the above generator matrix is formed as follows:

(i) Start with the generator matrices for the affine Reed-Muller codes for each of the Schubert-cells.

(ii) Then *starting with the lowest dimensional cell or cells*, extend code words forming the rows of the generator matrix for each of the corresponding affine Reed-Muller codes to a code word in the Higher-Grassmann code. The fact that the restriction maps denoted res_{α_0} in Theorem 3 are surjective shows this is possible in an iterative manner, starting with the smallest dimensional cell(s).

In fact the above decomposition into Schubert-cells also work in the case of projective spaces. The only difference between what happens for a general Grassmannian and a projective space is that the boundary structure for a Schubert cell for the projective space is simpler: the boundary of a Schubert cell of dimension n in projective space is a Schubert cell of dimension $n-1$ and Schubert cells of dimension strictly smaller than $n-1$. In the case of a general Grassmannian, the boundary of a Schubert cell of dimension n may be the (disjoint) union of several Schubert cells of dimension $n-1$ and several Schubert cells of dimension strictly lower than $n-1$.

Recall from [5, Theorem 7.4] that the parity check matrix for the Higher Grassmann code of degree ν is the generator matrix for the dual code, which is also a Higher Grassmann code of degree μ, where $\mu = (q-1)(l(m-l)) - \nu$ in case $\nu \neq 0 \mod (q-1)$ and is the extension of the above code by adding the word which is 1 everywhere, otherwise. One may observe that the corresponding generator matrix for the dual code is then obtained as in the last paragraph when $\nu \neq 0 \mod (q-1)$ or obtained by adding the word that is 1 everywhere to such a matrix. Therefore, the parity check matrix for the given Higher Grassmann code may be obtained as the generator matrix for the corresponding dual code.

It follows that the parity check matrix for the above higher Grassmann code also has the same block form as above, i.e., may be represented as follows:

$$H(\nu) = \begin{pmatrix} H_{C_{\alpha_0}}(\nu-1) & 0 & 0 & 0 \cdots 0 \\ * & H_{C_{\alpha_1}}(\nu-1) & 0 & 0 \cdots 0 \\ * & * & H_{C_{\alpha_2}}(\nu-1) & 0 \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \cdots \vdots \\ * & * & * & * \cdots * \end{pmatrix}, \quad \text{(III-A.2)}$$

with possibly the last row consisting of all 1s. Now one may inductively see, that as we go down the rows of the above matrix starting at the top, that the matrices denoted $H_{C_{\alpha_i}}(\nu-1)$, $i = 0, 1, \cdots$ are the parity check matrices for the affine Reed-Muller codes $\mathrm{RM}_{C_{\alpha_i}}(\nu-1)$.

We will now apply the above discussion to obtain the parity check matrix for the example in Example 1. (III-A.3), shown at the bottom of the page, and with an additional last row of all 1 in case $\nu = 0 \mod (q-1)$.

IV. Comparison of the Parameters of the Higher Grassmann Codes With the Parameters of the Projective Reed-Muller Code

The computation of the dimension of the Higher Grassmann codes is discussed in [5, Theorem 5.3]: moreover, it is shown there that this computation is a direct extension of the computation of the dimension of the Projective Reed-Muller codes discussed also in [5, Proposition 3.9]. A comparison of the proofs of the above two results should readily show that the dimensions of the Higher Grassmann codes, corresponding to the Grassmannian of l-planes in $m+1$-space is always higher than the dimension of projective Reed-Muller code associated to $\mathbb{P}^{m+1}$. Here are some specific details on this. The discussion in the proof of [5, Theorem 5.3] shows that the dimension calculation proceeds by taking the count of the number of supports of the relevant standard monomials of length r (as specified by the function $h(r, m+1, l)$), multiplying that by the function $g(t, r, q-1)$ which gives the number of positive exponents that can be placed on r-variables so that the resulting monomial has degree t, and then taking the resulting sum as t varies as in [5, Theorem 5.3]. One may see that the formula for the dimension of the projective Reed-Muller code is given by a very similar sum in [5, Proposition 3.9], with the notable difference being that the function $h(r, m+1, l)$ is replaced by the function $h_{PRM}(m+1, r) = \binom{m+1}{r}$. One may now observe readily that the function $h(r, m+1, l) \geq h_{PRM}(m+1, r)$, the simple explanation being that the Grassmannian of l-planes in $m+1$-space has dimension $\binom{m+1}{l} \geq m+1$, and also because of the fact that the Bruhat order on this Grassmannian is not a total order. These mean that there are more ways of choosing r-variables to form a monomial than in the case of the projective space $\mathbb{P}^{m+1}$. This will show the dimension of the Higher Grassmann code of degree ν associated to the Grassmannian of l-planes in $m+1$-space (over $\mathbb{F}_q$) is always larger (and in fact considerably larger, as a more detailed analysis will show) than the dimension of the Projective Reed-Muller code on $\mathbb{P}^{m+1}$ of degree ν, and over $\mathbb{F}_q$.

Regarding the other parameters, *the length n* and *the minimum distance d*, we have written certain functions in SageMath to compute them in specific examples. This provides the following table.

In the following table, the lines HGC (PRM) denote the Higher Grassmann codes (the Projective Reed-Muller codes, respectively).

Conclusions. One may therefore draw the following conclusions from the above discussion and the above table.

(i) A key observation is that the Higher Grassmann codes generally have better absolute parameters than the corresponding projective Reed-Muller codes.

(ii) This is already quite clear from the above table in terms of the minimum distance and also the length, with the improvement in minimum distance being very highly pronounced when considering Grassmannians of l-dimensional subspaces of m-dimensional spaces, with l and m moderately large, and l roughly of the order of $m/2$, and over moderately large fields. One may also see this conceptually as follows. In view of the formula for the lower bound for the minimum distances as given in Theorem 1 which is $(q-s)q^{l(m-l)-r-1}$ vs. $(q-s)q^{m-r-1}$ for the projective Reed-Muller codes, the exponent of q there is already significantly larger than for the projective Reed-Muller codes for the choice of l as above, and therefore the minimum distance becomes even larger as the last example in the above table shows.

(iii) In view of the fact that the minimum distances of the quantum codes depend strongly on the minimum distances of the classical codes that are input into the CSS construction, these advantages carry over to the resulting quantum codes as well, when they are constructed using the CSS construction.

V. Quantum Codes From Higher Grassmann Codes

In this section we show by using our higher Grassmann codes that there are quantum error correcting codes of desirable parameters and information rates. An overly simplified diagrammatic representation of our main idea is depicted in Figure 2.

We proceed to review the standard notation of the quantum error correcting codes. All details of the relevant notation, constructions, and more can be found in [9, Chapter 9].

Let q denote, as before, a prime power. Throughout the following discussion, we will let $\mathbb{Z}_+$ denote the set of positive integers. For each $n \in \mathbb{Z}_+$, we denote by $(\mathbb{C}^q)^{\otimes n}$ the qn dimensional complex vector space defined by the n-fold tensor product of $\mathbb{C}^q$ by itself. A *q-ary quantum code of length n* is a complex vector subspace C of $(\mathbb{C}^q)^{\otimes n}$. A *quantum error* for C is an element of the space of all operators defined on $\mathbb{C}^q$, that is, End($\mathbb{C}^q$).

$$H(\nu) = \begin{pmatrix} H_{C_{(4,5)}}(\nu-1) & 0 & 0 & 0 & \cdots & 0 \\ * & H_{C_{(3,5)}}(\nu-1) & 0 & 0 & \cdots & 0 \\ * & * & H_{C_{(3,4)}}(\nu-1) & 0 & \cdots & 0 \\ * & * & H_{C_{(2,5)}}(\nu-1) & 0 & \cdots & 0 \\ * & * & * & H_{C_{(2,4)}}(\nu-1) & \cdots & 0 \\ * & * & * & H_{C_{(1,5)}}(\nu-1) & \cdots & 0 \\ * & * & * & * & H_{C_{(1,4)}}(\nu-1)\cdots & 0 \\ * & * & * & * & H_{C_{(2,3)}}(\nu-1)\cdots & 0 \\ * & * & * & * & * & H_{C_{(1,3)}}(\nu-1) \end{pmatrix} \quad \text{(III-A.3)}$$

TABLE I
COMPARISON TABLE

	ν	l	m	q	length	min distance (lower bound)
HGC	10	2	5	7	140050	9604
PRM	10	1	5	7	2801	1372
HGC	15	2	5	7	140050	1715
PRM	15	1	5	7	2801	245
HGC	20	2	5	7	140050	294
PRM	20	1	5	7	2801	42
HGC	120	2	5	32	1109461025	6144
PRM	120	1	5	32	1082401	192
HGC	120	3	7	25	$6.22\text{x}10^{16}$	12207031250
PRM	120	1	7	25	$2.54\text{x}10^{8}$	1250
HGC	150	5	10	25	$9.27\text{x}10^{34}$	291038304567337036132812500
PRM	150	1	10	25	$3.97\text{x}10^{12}$	312500

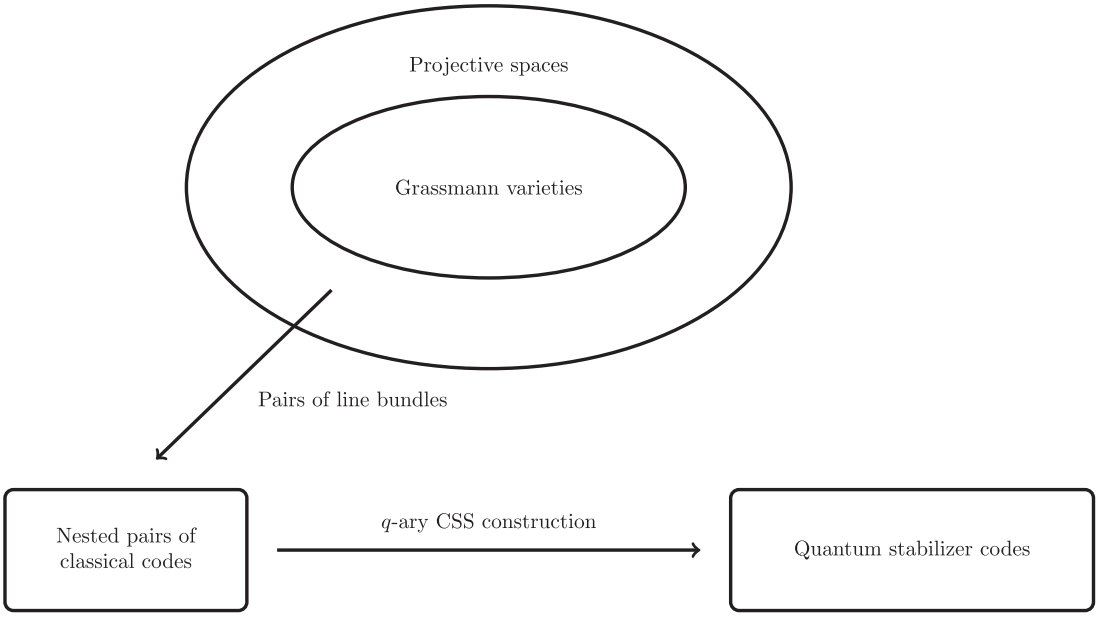

Fig. 2. Schematic of quantum Grassmann codes as sub-codes of the quantum Projective Reed-Muller codes.

Let $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the trace function defined by $\text{tr}(\alpha) = \sum_{i=0}^{m} \alpha^{p^i}$, where m is defined by $q = p^m$. We fix a primitive p-th root of unity, $\omega := exp(2\pi i/p) \in \text{C}$. Let a and b be elements from $\mathbb{F}_q$. We denote by $X(a) : \mathbb{C}^q \to \mathbb{C}^q$ and $Z(b) : \mathbb{C}^q \to \mathbb{C}^q$ the following unitary operators on $\mathbb{C}^q$:

$$X(a)(|x\rangle) := |x + a\rangle \qquad \text{and} \qquad Z(b)(|x\rangle) := \omega^{\text{tr}(bx)}|x\rangle,$$

where $|x\rangle$ is a vector in $\mathbb{C}^q$. Let $\mathcal{E}$ denote the set defined by be

$$\mathcal{E} := \{X(a)Z(b) \mid a, b \in \mathbb{F}_q\}.$$

We call $\mathcal{E}$ a *nice error basis*. Indeed, $\mathcal{E}$ is a basis for $\text{End}(\mathbb{C}^q)$.

For $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{F}_q^n$, we set

$$X(\mathbf{a}) := X(a_1) \otimes \cdots \otimes X(a_n) \qquad \text{and}$$
$$Z(\mathbf{a}) := Z(a_1) \otimes \cdots \otimes Z(a_n).$$

Similarly to the set $\mathcal{E}$, we define a *nice error basis for* $\text{End}((\mathbb{C}^q)^{\otimes n}))$ by setting

$$\mathcal{E}_n := \{X(\mathbf{a})Z(\mathbf{b}) \mid \{\mathbf{a}, \mathbf{b}\} \subset \mathbb{F}_q^n\}. \tag{V-.1}$$

In (V-.1), the product $X(\mathbf{a})Z(\mathbf{b})$ is understood as $(X(a_1)Z(b_1)) \otimes \cdots \otimes (X(a_n)Z(b_n))$. The *error group* associated with the nice error basis for $\mathcal{E}_n$, denoted G_n, is defined by

$$G_n := \{\omega^c X(\mathbf{a})Z(\mathbf{b}) \mid \{\mathbf{a}, \mathbf{b}\} \subset \mathbb{F}_q^n,\ c \in \mathbb{F}_q\}.$$

Since G_n is a subgroup of $\mathrm{GL}_n((\mathbb{C}^q)^{\otimes n})$, we may consider $(\mathbb{C}^q)^{\otimes n}$ as a defining representation of $(\mathbb{C}^q)^{\otimes n}$.

Let $\mathbf{a} := (a_1, \ldots, a_n)$ and $\mathbf{b} := (b_1, \ldots, b_n)$ be two vectors from $\mathbb{F}_q^n$. We form a new vector in $\mathbb{F}_q^{2n}$ by concatenation, $(\mathbf{a}|\mathbf{b}) \in \mathbb{F}_q^{2n}$. The *symplectic weight* of $(\mathbf{a}|\mathbf{b})$ is defined by

$$\mathrm{swt}((\mathbf{a}|\mathbf{b})) := |\{k \in \{1, \ldots, n\} \mid (a_k, b_k) \neq (0, 0)\}|.$$

If $E \in G_n$ is an element of the form $E := \omega^c X(\mathbf{a}) Z(\mathbf{b})$, where $c \in \mathbb{F}_q$, then the *weight* of E is defined by

$$\mathrm{w}(E) := \mathrm{swt}((\mathbf{a}|\mathbf{b})). \tag{V-.2}$$

Let $\mathrm{U}(q^n)$ denote the group of complex unitary matrices of size q^n. A very useful fact about quantum error-correcting codes is that a quantum code Q is able to detect an error $E \in \mathrm{U}(q^n)$ if and only if the condition $\langle c_1|E|c_2\rangle = \lambda_E \langle c_1|c_2\rangle$ holds for every $\{c_1, c_2\} \subset \mathrm{Q}$, where λ_E is a constant depending on E. Finally, we come to the definition of a notion of minimum distance for a quantum code. A quantum code Q is said to have *minimum distance* d if it can detect all errors in G_n of weight less than d but cannot detect some error of weight d. In this terminology, a quantum code Q is said to be an $[\![n, k, d]\!]_q$ code if Q is a q^k-dimensional subspace of $(\mathbb{C}^q)^{\otimes n}$ that has minimum distance d. If the quantum code Q is already defined, then we call $[\![n, k, d]\!]_q$ the *parameters* of Q.

Let S be a subgroup of G_n. Then the *quantum stabilizer code associated with* S is a nonzero subspace $\mathrm{Q} \subseteq (\mathbb{C}^q)^{\otimes n}$ such that

$$\mathrm{Q} := \bigcap_{\mathrm{E} \in \mathrm{S}} \{\mathrm{v} \in (\mathbb{C}^{\mathrm{q}})^{\otimes \mathrm{n}} \mid \mathrm{Ev} = \mathrm{v}\}.$$

In other words, Q is the weight space of weight 1 for the representation of S on $(\mathbb{C}^q)^{\otimes n}$. Let $t \in \mathbb{Z}_+$. A quantum code Q associated with S is said to be *pure to* t if S does not contain non-scalar matrices of weight less than t.

We now recall a well-known theorem in quantum information theory, called the *q-ary CSS construction*. The acronym comes from the authors Calderbank and Shor [7], and Steane [28]. We state its version that is proven in [15].

Lemma 2: (CSS Code Construction) Let $\{n, k_1, k_2, d_1, d_2\} \subseteq \mathbb{Z}_+$. Let C_1 and C_2 be two linear codes over $\mathbb{F}_q$ with parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$, respectively. If the inclusions $C_2 \subseteq C_1 \subseteq \mathbb{F}_q^n$ hold, then there exists a quantum stabilizer code C with parameters

$$[\![n, k_1 - k_2, d = min\{wt(c) | c \in (C_1 \setminus C_2 \cup C_2^{\perp} \setminus C_1^{\perp})\}]\!]_q$$

that is pure to $\min\{d_1, d_2\}$.

Next, we will outline two different constructions of quantum error correcting codes based on the higher Grassmann codes.

A. The First Construction

Let ν_1 and ν_2 be two positive integers such that $\nu_1 = \nu_2 + s(q-1)$, where s is a non-negative integer. We will consider the higher Grassmann codes $\mathrm{HG}(\nu_i)$, $i \in \{1, 2\}$. We know from [4] these codes are obtained by evaluating the sections of the line bundles $\mathcal{O}_{Gr}(\nu_1)$ and $\mathcal{O}_{Gr}(\nu_2)$ on the elements of $\mathbf{Gr}(\mathbb{F}_q)$. Since these sections are given by the homogeneous polynomials of degrees ν_i ($i \in \{1, 2\}$), there is an injective linear map

$$\theta : H^0(Gr, \mathcal{O}_{Gr}(\nu_2)) \hookrightarrow H^0(Gr, \mathcal{O}_{Gr}(\nu_1)). \tag{V-A.1}$$

Indeed, the assignment that sends a standard monomial $p_{\alpha(i_1)}^{a_1} \cdots p_{\alpha(i_j)}^{a_j}$ of degree $\sum_{i=1}^{j} a_i = \nu_2$ to the standard monomial $p_{\alpha(i_1)}^{a_1 + s(q-1)} \cdots p_{\alpha(i_j)}^{a_j}$ extends linearly to an injective map as in (V-A.1). Since the standard monomials $p_{\alpha(i_1)}^{a_1} \cdots p_{\alpha(i_j)}^{a_j}$ and $p_{\alpha(i_1)}^{a_1 + s(q-1)} \cdots p_{\alpha(i_j)}^{a_j}$ have equal values, the linear map θ preserves the distance between the codewords. Thus, in light of these observations, we proceed with the assumption that $\mathrm{HG}(\nu_2) \subseteq \mathrm{HG}(\nu_1)$.

Theorem 4: If $\mathrm{HG}(\nu_1)$ and $\mathrm{HG}(\nu_2)$ are the higher Grassmann codes as defined above, then there exists a quantum stabilizer code $C_{Gr(\mathbb{F}_q)}$ with parameters

$$\left[\!\!\left[\begin{bmatrix} m \\ l \end{bmatrix}_q, k_1 - k_2, d \right]\!\!\right]_q,$$

where k_i is the dimension of $\mathrm{HG}(\nu_i)$ for $i \in \{1, 2\}$, and d is bounded from below by the minimum of the minimum distance of the code $\mathrm{HG}(\nu_1)$ and the minimum distance of the code $\mathrm{HG}(\nu_2)^{\perp}$.

Proof: The proof is a straightforward application of the q-ary CSS construction applied to the nested pair $(\mathrm{HG}(\nu_1), \mathrm{HG}(\nu_2))$. □

Advantages of such quantum codes. First, the dimensions of the higher Grassmann codes $\mathrm{HG}(\nu_1)$ and $\mathrm{HG}(\nu_2)$ are much larger (in fact, several times larger) than the corresponding Grassmann codes as the calculation of the dimension of the higher Grassmann codes in [4] shows. In view of the calculation of the dimensions of the resulting quantum codes as above, this advantage also shows up in the dimensions of the resulting quantum codes. A comparison of the minimum distance of the higher Grassmann codes with the minimum distances of the corresponding Projective Reed-Muller code shows that the minimum distances for the higher Grassmann codes are typically much larger; this translates in to much larger minimum distances for the resulting quantum codes as well. Moreover larger dimensions and larger minimum distances translate into codes that perform much better.

Another advantage is that, as there is no restriction on the degrees ν_1 and ν_2, we obtain large families of quantum error correcting codes this way.

B. The Second Construction

In the second construction we restrict our attention to the line bundles $\mathcal{O}_{\mathbf{Gr}}(\nu)$, where ν satisfies the following conditions:

$$1 \leq \nu \leq \left\lfloor \frac{l(m-l)(q-1)}{2} \right\rfloor \quad \text{and} \quad 2\nu \equiv 0 \mod (q-1).$$

Recall here that $l(m-l)$ denotes the dimension of the Grassmannian variety. Let us define $\nu^{\perp}$ by setting $\nu^{\perp} := l(m-l)(q-1) - \nu$. Equivalently, $\nu^{\perp}$ is given by $\nu + s(q-1)$, where $s = l(m-l) - (2\nu/(q-1))$. Then we observe the following:

1) this construction fits into the framework of the first construction with $\nu_2 := \nu$ and $\nu_1 := \nu^{\perp}$, so that $\mathrm{HG}(\nu) \subseteq \mathrm{HG}(\nu^{\perp})$.
2) By [5, Theorem 7.4], $\mathrm{HG}(\nu^{\perp}) = \mathrm{HG}(\nu)^{\perp}$.

Therefore, Theorem 4 produces quantum codes whose length and dimension are as in the first construction above,

and the minimum distance is given by the minimum of the dual code $\mathrm{HG}(\nu)^{\perp}$.

Advantages of the resulting quantum codes. These codes share several of the features of the quantum codes in the first construction, hence, they have their advantages.

VI. Implementation Details

In this section we will discuss the construction of the stabilizer codes starting with a pair of Higher Grassmann codes, C_1 and C_2, with C_2 a sub-code of C_1. We will first consider the binary case. Then as shown in [20, pp. 42-43], the resulting quantum code Q is the subspace of $\mathbb{C}^{2^n}$ defined as the image of a projector operator P defined in terms of the operators $E = \omega^c X(\mathbf{a})Z(\mathbf{b})$, where $\mathbf{a} \in C_1^{\perp} \subseteq \mathbb{F}_2^n$, $\mathbf{b} \in C_2 \subseteq \mathbb{F}_2^n$ and $c \in \mathbb{F}_2$, with ω a primitive 2-th root of unity. Denoting the parity check matrices for the codes C_1 and C_2 by $H(C_1)$ and $H(C_2)$, the resulting quantum code produced by invoking the CSS construction is a stabilizer code where the stabilizer matrix can be obtained from the matrix $S = \begin{pmatrix} H(C_2^{\perp}) & 0 \\ 0 & H(C_1) \end{pmatrix}$, by applying the $Z(\ \ )$-operator to the vectors forming the rows of the matrix $H(C_2^{\perp})$ and by applying the $X(\ \ )$-operator to the vectors forming the rows of the matrix $H(C_1)$. We have already discussed in section III-A, how to produce the parity check and generator matrices for the Higher Grassmann codes, quite efficiently by decomposing the Higher Grassmann code into a sequence of affine Reed-Muller codes. Therefore, what remains is to essentially invoke the CSS construction with the above matrix S to produce the stabilizer matrix. (See [20, pp. 42-43] for additional details.)

When the base field is no longer $\mathbb{F}_2$, the situation is a bit more involved. Assume therefore that the base field is $\mathbb{F}_q$, for $q = p^n$, for some prime number p. Then the required construction of the stabilizer matrices is again discussed in [21] and [15, Theorem 13 and Lemma 20]. Given two classical codes C_1 and C_2, each contained in $\mathbb{F}_q^n$, one lets $C = C_1^{\perp} \times C_2 \subseteq \mathbb{F}_q^{n^2}$. Then $C^{\perp} = C_1 \times C_2^{\perp}$ and $C \subseteq C^{\perp}$. The resulting quantum code Q is the subspace of $\mathbb{C}^{q^n}$ defined as the image of a projector operator P defined in terms of the operators $E = \omega^c X(\mathbf{a})Z(\mathbf{b})$, where $\mathbf{a} \in C_1^{\perp} \subseteq \mathbb{F}_q^n$, $\mathbf{b} \in C_2 \subseteq \mathbb{F}_q^n$ and $c \in \mathbb{F}_p$, with ω a primitive p-th root of unity.

The stabilizer matrix of the corresponding quantum code is formed by taking some of the operators E defined above. What one may observe is that whenever there is a linear dependence relation

$$\lambda_1 \mathbf{a}_1 + \cdots + \lambda_k \mathbf{a}_{\mathbf{k}} = 0 \in \mathbb{F}_q^n, \lambda_i \in \mathbb{F}_q,$$

then the composition $X(\lambda_1 \mathbf{a}_1) \circ \cdots \circ X(\lambda_k \mathbf{a}_k) = id$. Moreover, the converse of the above implication holds and any two operators $X(\lambda_1 \mathbf{a}_1)$ and $X(\lambda_2 \mathbf{a}_2)$ commute as can be seen from the definition of the operators $X(\lambda_1 \mathbf{a}_1)$ and $X(\lambda_2 \mathbf{a}_2)$.

Similarly, whenever there is a linear dependence relation

$$\lambda_1 \mathbf{b}_1 + \cdots + \lambda_k \mathbf{b}_{\mathbf{k}} = 0 \in \mathbb{F}_q^n, \lambda_i \in \mathbb{F}_q$$

then the composition $X(\lambda_1 \mathbf{b}_1) \circ \cdots \circ X(\lambda_k \mathbf{b}_k) = id$. Moreover, the converse of the above implication holds, and again any two operators $X(\lambda_1 \mathbf{b}_1)$ and $X(\lambda_2 \mathbf{b}_2)$ commute as can be seen from the definition of the operators $X(\lambda_1 \mathbf{b}_1)$ and $X(\lambda_2 \mathbf{b}_2)$.

The upshot of these observations is that, in order to define the projection operator P associated to the quantum code Q, one may restrict the operators E to where the $X(\mathbf{a}_i)$ are obtained from the vectors $\mathbf{a}_i$ belonging to a basis for the classical code $C_1^{\perp}$ and where the $Z(\mathbf{b}_j)$ are obtained from the vectors $\mathbf{b}_j$ belonging to a basis for the classical code C_2. Moreover, one cannot omit any of the vectors $\mathbf{a}_i$ belonging to a basis for the classical code $C_1^{\perp}$ and one cannot omit any of the vectors $\mathbf{b}_j$ belonging to a basis for the classical code C_2. Therefore, we can conclude in this case that the stabilizer matrix is obtained by applying the $Z(\ \ )$-operator to the rows of the matrix $H(C_2^{\perp})$ and the $X(\ \ )$-operator to the rows of $H(C_1)$, where $H(C_1)$ ($H(C_2^{\perp})$) denotes the parity check matrix of the code C_1 ($C_2^{\perp}$). Thus we are once again able to obtain the stabilizer matrix for the quantum code Q from the matrix: $S = \begin{pmatrix} H(C_2^{\perp}) & 0 \\ 0 & H(C_1) \end{pmatrix}$.

VII. Higher Affine Grassmann Codes and Application to Fault Tolerant Quantum Computation

Fault tolerant quantum computation is a critical step in the development of practical quantum computers. Unfortunately, not every quantum error correcting code can be used for fault tolerant computation. Rengaswamy et. al. (see [22] and [23]) define CSS-T codes, which are CSS codes that admit the transversal application of the T-gate, which is a key step in achieving fault tolerant computation. They then present a family of quantum Reed-Muller fault tolerant codes.

In this section we will show that affine variants of the Higher Grassmann codes define CSS T-codes which are CSS codes that admit the transversal application of the T-gate. First we clarify what we mean by an affine variant of the Higher Grassmann code.[2]

Definition 1: Let V denote a vector space of dimension m over $\mathbb{F}_q$ and let $l \leq m$ denote a positive integer. Then $Gr(l, V)$ denotes the Grassmannian of l-planes in V. Let $\mathcal{O}$ denote the open affine cell in $Gr(l, V)$, which is an affine space of dimension $d = l(m - l)$. Let $\nu < d(q - 1)$. Then we define the Higher affine Grassmann code of order ν to be the code obtained by evaluating all the polynomials of degree $\leq \nu$ in d-variables at the $\mathbb{F}_q$-rational points in $\mathcal{O}$. We will denote this code by $C_{Gr_{aff}(l,V)}(\nu)$.

We may observe that such affine variants of the Higher Grassmann codes appear in two somewhat different contexts in the literature.

(i) Viewing the open cell in a Grassmannian as simply an affine space of suitable dimension, this open cell can also be then viewed as the open cell in a sufficiently large dimensional projective space. Then the generalized affine Reed-Muller codes studied in [8] are exactly the codes constructed by evaluating polynomials at all the rational points of such an open cell.

(ii) In [2], they studied *affine Grassmann codes* which are defined by evaluating polynomials up to certain degree at all the rational points of the same affine space, but now viewed as the open cell in a Grassmannian, which is not necessarily a projective space.
Therefore, such codes may also be viewed as restriction of the Higher Grassmann codes to the affine open cell

[2]In future work, we hope to explore other related constructions such as the hull-variation problem and its applications to quantum codes (see [24]), but in the context of Higher Grassmann codes or closely related codes.

in the Grassmannian, i.e., one takes the same global sections, but restricts such homogeneous polynomials to the affine open cell and then evaluates them only at the rational points belonging to the affine open cell. (One may observe that the restriction of a homogeneous polynomial of degree ν to the open cell makes it a non-homogeneous polynomial of degree at most ν.)

As vector spaces the above codes in (i) and (ii) identify. But by viewing them in the context of [8], one may invoke the duality results proved in [8, Theorem 2.2.1] to conclude the following.

Proposition 1: Let $0 \leq \mu < d(q-1)$ so that $\mu+\nu = d(q-1)-1$. Then $C_{Gr_{aff}(l,V)}(\nu)^{\perp} = C_{Gr_{aff}(l,V)}(\mu)$.

Next recall that a linear code C over $\mathbb{F}_q$ is *self-orthogonal* if $C \subseteq C^{\perp}$. A self-orthogonal code C is *self-dual* if $C = C^{\perp}$.

Now the following result is well-known: see [17, p. 26].

Proposition 2: A self-orthogonal code C over $\mathbb{F}_q$ with length n and dimension k is self-dual if and only if n is even and $k = n/2$.

From now on-wards we will assume that $q = 2^m$, for some positive integer m. Then if one chooses d, which is the dimension of the given Grassmannian to be *odd*, $d(2^m - 1)$ is odd, so that $d(2^m - 1) - 1$ is even. Therefore, if one chooses $\nu = \frac{d(2^m-1)-1}{2}$, then the corresponding Higher affine Grassmann code $C_{Gr_{aff}(l,V)}(\nu)$ is *self-dual*. (Observe that if we choose q to be the power of an odd prime, then the above property fails: this is the reason we restrict to the case q is a power of 2 in the following discussion.)

As the Gottesman-Knill Theorem (see [10]) indicates that in order to achieve universal fault tolerant computation, we must be able to implement a non-Clifford gate on the encoded qubits by manipulating the physical qubits. Not every stabilizer code can achieve this, so we will restrict to the class of stabilizer codes called CSS T-codes. Let C denote a binary code of length n defined over $\mathbb{F}_{2^m}$, for some $m > 0$. We say a code word $c \in C$ is supported on a vector $s \in \mathbb{F}_{2^m}^n$, if the bit-wise multiplication $c \times s = c$.

Definition 2:

(i) For a linear code C defined over $\mathbb{F}_{2^m}$ and of length n and binary string $s \in \mathbb{F}_{2^m}^n$, we define *the restriction of C to (the support of) s*, denoted $C|_s$, to be the code created by removing from C all code-words which are not supported on s, and then shortening the code by removing the indices corresponding to zeros in s from the remaining code-words.

(ii) A CSS T-code is a CSS code defined by codes $C_2 \subseteq C_1$ such that the length of C_2 is even and for every $x \in C_2$, $C_1^{\perp}|_x$contains a self-dual code.

Theorem 5: Let $C_1 = C_{Gr_{aff}(l,V)}(\nu)$, with $\nu = \frac{d(2^m-1)-1}{2}$. If $C_2 \subseteq C_1$ denotes the code that has only one word, namely the word that is 1 at all the rational points in the open affine cell of the corresponding Grassmannian, then $C_2 \subseteq C_1$ is a CSS T-code.

Proof: Let $x \in C_2$ be the only nonzero code-word. Since x is the code-word consisting of only ones, every vector in $\mathbb{F}_{2^m}^n$ is supported in x, so we see $C_1^{\perp}|_x = {C_1}^{\perp}$. Since these codes are equal, their duals are equal: $C_1 = (((C_1)^{\perp})|_x)^{\perp}$. Recall that C_1 is self-dual, so we have $C_1^{\perp}|_x = C_1^{\perp} = C_1 = (((C_1)^{\perp})|_x)^{\perp}$. Hence, $(C_1)^{\perp}|_x$ contains a self-dual sub-code, namely, itself.□

REFERENCES

[1] E. Abbe, A. Shpilka, and M. Ye, "Reed–Müller codes: Theory and algorithms," 2020, *arXiv:2002.03317*.

[2] P. Beelen, S. R. Ghorpade, and T. Hoholdt, "Affine Grassmann codes," *IEEE Trans. Inf. Theory*, vol. 56, no. 7, pp. 3166–3176, Jul. 2010, doi: 10.1109/TIT.2010.2048470.

[3] P. Beelen and P. Singh, "Point-line incidence on Grassmannians and majority logic decoding of Grassmann codes," *Finite Fields Their Appl.*, vol. 73, Aug. 2021, Art. no. 101843.

[4] M. B. Can, R. Joshua, and G. V. Ravindra, "Higher Grassmann codes," *Finite Fields Their Appl.*, vol. 76, Dec. 2021, Art. no. 101905.

[5] M. B. Can, R. Joshua, and G. V. Ravindra, "Higher Grassmann codes II," *Finite Fields Their Appl.*, vol. 89, Aug. 2023, Art. no. 102211.

[6] M. Can, R. Joshua, and G. V. Ravindra, "Corrigendum to higher Grassmann codes II," *Finite Fields Appl.*, vol. 109, Jan. 2026, Art. no. 102700.

[7] A. R. Calderbank and P. W. Shor, "Good quantum error-correcting codes exist," *Phys. Rev. A, Gen. Phys.*, vol. 54, no. 2, pp. 1098–1105, Aug. 1996.

[8] P. Delsarte, J. M. Goethals, and F. J. Mac Williams, "On generalized Reed–Muller codes and their relatives," *Inf. Control*, vol. 16, pp. 403–442, Apr. 1970.

[9] I. B. Djordjevic, *Quantum Information Processing, Quantum Computing, and Quantum Error Correcting Correction, An Engineering Approach*, 2nd ed., New York, NY, USA: Academic.

[10] D. Gottesman, "The Heisenberg representation of quantum computers," 1998, *arXiv:quant-ph/9807006*.

[11] S. R. Ghorpade and R. Ludhani, "On the minimum distance, minimum weight codewords, and the dimension of projective Reed–Muller codes," *Adv. Math. Commun.*, vol. 18, no. 2, pp. 360–382, 2024.

[12] W. C. Huffman and V. Pless, *Fundamentals of Error Correcting Codes*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[13] H. B. Eggers, "A new family of fault tolerant quantum Reed–Muller codes," M.S. thesis, Dept. Math., Clemson University, Clemson, SC, USA, 2020.

[14] J. C. Jantzen, *Representations of Algebraic Groups* (Mathematical Surveys and Monographs), vol. 107, 2nd ed., Providence, RI, USA: American Mathematical Society, 2003.

[15] A. Ketkar, A. Klappenecker, S. Kumar, and P. K. Sarvepalli, "Nonbinary stabilizer codes pver finite fields," *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 4892–4912, Nov. 2006.

[16] J. Lavauzelle, "Lifted projective Reed–Solomon codes," *Designs, Codes Cryptography*, vol. 87, no. 7, pp. 1541–1575, Jul. 2019.

[17] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes* (North Holland Mathematical Library). Amsterdam, The Netherlands: North Holland, 1977.

[18] N. Nakashima and H. Matsui, "Decoding of projective Reed–Muller codes by dividing a projective space into affine spaces," *IEICE Trans. Fundamentals Electron., Commun. Comput. Sci.*, vol. 99, no. 3, pp. 733–741, Mar. 2016, doi: 10.1587/transfun.e99.a.733.

[19] D. Y. Nogin, "Codes associated to Grassmannians," in *Arithmetic, Geometry and Coding Theory, Luminy*. Berlin, Germany: De Gruyter, 1993, pp. 145–154.

[20] J. Preskill, "Lectures on quantum computation," Quantum Information Computation, Dept. Physics, California InstituteTechnology, Google Books, Jan. 2015.

[21] E. M. Rains, "Nonbinary quantum codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 1827–1832, 1999.

[22] N. Rengaswamy, R. Calderbank, M. Newman, and H. D. Pfister, "Classical coding problem from transversal t gates," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2020, pp. 1891–1896.

[23] N. Rengaswamy, R. Calderbank, M. Newman, and H. D. Pfister, "On optimality of CSS codes for transversal T," *IEEE J. Select. Areas Inf. Theory*, vol. 1, no. 2, pp. 499–514, Feb. 2020.

[24] D. Ruano and R. San-José, "The hull variation problem for projective Reed–Müller codes and quantum error-correcting codes," Math. Inst., Univ. Valladolid, Valladolid, Spain, Tech. Rep., 2013.

[25] C. S. Seshadri, *Introduction to the Theory of Standard Monomials* (Texts and Readings in Mathematics). Cham, Switzerland: Springer, 2016.

[26] A. B. Sorensen, "Projective Reed–Müller codes," *IEEE Trans. Inf. Theory*, vol. 37, no. 6, pp. 1567–1576, 1991.

[27] R. Stanley, "Enumerative combinatorics," in *Cambridge Studies in Advanced Mathematics*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 2000.

[28] A. M. Steane, "Simple quantum error correcting codes," *Phys. Rev. Lett.*, vol. 77, pp. 793–797, Feb. 1996.