Intersection Cohomology of reductive varieties

Outline

- The (generalized) h-vectors of Stanley.

- Brief review of equivariant intersection cohomology

- The basic framework and the key theorems

- An outline of the proof of the key theorems

- Reductive varieties and group compactifications

- Examples

The h-vectors of Stanley

Intersection cohomology and the generalized h-vectors

- A few key properties of intersection cohomology

- The generalized h-vectors as the intersection cohomology Betti numbers

- Equivariant intersection cohomology

The basic frame work

- Slices for algebraic group actions

- Attractive slices, links

- The Poincaré series in equivariant intersection cohomology

Key Theorem 1 (B-   , 2001)

X a G-variety with finitely many orbits, each having an attractive slice.

Then:

i) The $H^*(BG)$-module $IH^*_G(X)$ admits a filtration with subquotients $IH^*_{\mathcal{O},G}(X)$

ii) For $\mathcal{O} = Gx$, $IH^*_{\mathcal{O},G}(X) \cong H^{*+2dim(\mathcal{O})}(BG_x) \otimes IH^*_x(X)$ if G_x connected

Corollary X a projective spherical variety with connected stabilizers. Then $IH^i(X) = 0$ and $\mathcal{H}^i(IC(X)) = 0$ for all *odd i*.

Key Theorem 2 (B-   , 2004) (The algorithm)

X projective, as in the last theorem and with connected stabilizers

$IP_X(t)$: the Poincaré polynomial for $IH^*(X)$

$IP_{X,x}(t)$: the Poincaré polynomial for $\mathcal{H}^*(i_x^* IC(X))$

$P_{G/T}(t)$, $P_{G_x/T_x}(t)$: the Poincaré polynomial for G/T, G_x/T_x

Then: (i) $IP_X(t)$

$$= \Sigma_x (1-t^2)^{r-r_x} \frac{P_{G/T}(t)}{P_{G_x/T_x}(t)} t^{2d_x} IP_{X,x}(1/t)$$

where $r = dim(T)$, $r_x = dim(T_x)$.

(ii) $IP_{X,x}(t) = \tau_{\leq d_x - 1}((1-t^2) IP_{\mathbb{P}(\mathcal{S}_x)}(t))$

Proofs of Key Theorems:outline

• Proof of Key Theorem 2 assuming Key Theorem 1

• **Proof of Key Theorem 1**

Step 1: $\cdots \to IH^n_{G,\mathcal{O}}(X) \to IH^n_G(X) \to IH^n_G(X - \mathcal{O}) \to \cdots$ breaks up after localizing at the prime $\mathfrak{p} = ker(H^*(BG) \to H^*(BG_x))$

Step 2: reduction to showing $IH^*_{G,\mathcal{O}}(X) \to IH^*_{G,\mathcal{O}}(X)_{\mathfrak{p}}$ is injective

Step 3: $IH^*_{G,\mathcal{O}}(X) \cong \mathbb{H}^*(BT_x, Ri^!_x IC^{T_x}(X))^{W_x}$

Step 4: The latter $\cong (H^{*+2dim(\mathcal{O})}(BT_x) \otimes IH^*_x(X))^{W_x}$

Reductive varieties

• G: connected reductive group, B, B^-: opposite Borels, $T = B \cap B^-$

• $\Lambda = \chi(T)$, $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$, W: the Weyl group of (G, T)

• Definition: X is *reductive* if (i) X normal and spherical for $G \times G$, (ii) there exists an $x \varepsilon X$ fixed by $diag(T)$ with $(B^- \times B)x$ dense and (iii)$(G \times G)_x$ connected

• Simple examples

(W-admissible rational convex polyhedral cones in $\Lambda_{\mathbb{R}}$) $\simeq$ (affine reductive varieties for G)

(W-admissible convex polytopes in $\Lambda_{\mathbb{R}}$) $\simeq$ (projective reductive varieties for G)

Theorem(B-   , 2004) The class of projective reductive varieties is closed under links. The same holds for projective compactifications of reductive groups.

Corollary (B-   , 2004) The algorithm considered earlier applies to the class of projective reductive varieties and to the class of projective compactifications of reductive groups.

A few examples

• **Toric varieties**

$X \to$ a rational convex polytope $\delta \subseteq \Lambda_{\mathbb{R}}$

T-orbits$\to$ faces of δ

The T-orbit $\mathcal{O}_\phi$ has an open T-invariant nbd $\simeq \mathcal{O}_\phi \times \mathcal{S}_\phi$. $\mathcal{S}_\phi$= the slice and $\mathbb{P}(\mathcal{S}_\phi)$= the link

The formula in Key Theorem 2:

$IP_X(t) = \Sigma_\phi (1-t^2)^{dim(\phi)} t^{2d_x} IP_{X,x_\phi}(1/t)$ - known formula for toric varieties

• **Rank 1 case**

The polytope: a line segment $[\lambda, \mu]$, $\lambda, \mu \varepsilon \Lambda$

Wlog λ dominant : λ corresponds to a closed $G \times G$-orbit $\mathcal{O}_\lambda = G/P_{I(\lambda)} \times G/P^-_{I(\lambda)}$, $I(\lambda) =$ all simple roots orthogonal to λ

The slice $\mathcal{S}_\lambda =$ the affine reductive variety for $L_{I(\lambda)}$ and the link is $P_{I(\lambda)}/P_{I(\delta)} \times P^-_{I(\lambda)}/P^-_{I(\delta)}$, $I(\delta) = I(\mu) \cap I(\lambda)$.

Further cases: μ also dominant: three orbits. In case μ not dominant, only two orbits

• **Compactifications of** GL_2 $\Lambda = \mathbb{Z}^2$, B=upper triangular matrices, B^-= lower triangular matrices. The unique simple root $\alpha = (1, -1)$. $W = \mathbb{Z}/2$:reflections about the diagonal. $\Lambda^+_{\mathbb{R}}$= the pos. Weyl chamber = points below the diagonal

(projective compactifications) $\simeq$ (convex rational polytopes in $\Lambda_{\mathbb{R}}$ symmetric with respect to the diagonal)

Case 1: ϕ an edge below the diagonal. Slice: $\mathbb{A}^1$ and link =$Spec \quad k$

Case 2:ϕ an edge symmetric with respect to the diagonal. Again slice:$\mathbb{A}^1$ and link = $Spec \quad k$

Case 3:ϕ a vertex below the diagonal. Slice of dimension 2 and link = $\mathbb{P}^1$

Case 4: ϕ a vertex on the diagonal. Link is $\mathbb{P}^3$.

References

M. Brion and R. Joshua: Vanishing of odd dimensional intersection cohomology II, Math. Annalen, **321**, 399-437, (2001)

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http://www.math.ohio-state.edu/~ joshua/pub.html