K-Theory and G-Theory of DG-stacks

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Abstract

In this paper, we establish results of a basic nature for the the K-theory and G-theory of algebraic stacks, i.e. Artin stacks. At the same time, we enlarge the framework a bit more so that these results not only hold for stacks, but also for what are called *dg-stacks*, i.e. algebraic stacks where the usual structure sheaf is replaced by a sheaf of dgas.

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*Supported by the MPI(Bonn), IHES and a grant from the NSA.

1 Introduction.

The purpose of this paper is to present basic material on the K-theory and G-theory of algebraic stacks so as to serve as a reference. The following should serve as ample justification for a work like the present one:

- Algebraic stacks have evolved in recent years into an area that has several important applications. The K-theory of algebraic stacks plays an important role in the Riemann-Roch formula and the study of vector bundles and coherent sheaves on algebraic stacks.

- There has been a recent reworking of the foundations based on issues with the smooth site and the *lisse-étale site*. In view of this, it seems important to carefully rework some of the basic results on the K-theory and G-theory of Artin stacks in this somewhat modified framework.

- At the same time, we have tried to extend the basic framework to include *dg-stacks*, which are Artin stacks where the structure sheaf has been replaced by a sheaf of commutative dgas. Such dg-stacks have proven quite useful in the study of *virtual phenomena*, for example, and are being studied actively in the context of *DG-Algebraic Geometry.*

It needs to be perhaps also pointed out that we have chosen deliberately not to extend these results to the true new frontiers in derived algebraic geometry where even the basic notions like schemes have to be replaced by derived schemes which are built by gluing derived affine schemes using quasi-isomorphisms. Several reasons justify this decision: (i) The background material needed to develop the foundations of K-theory and G-theory in this context seems formidable. (ii) Dg-stacks and dg-schemes as considered in this paper are already sufficient for many applications (and already sufficiently esoteric enough). (iii) The main goal of the present paper, as pointed out above, is more modest: it is simply to provide a suitable reference for basic results in the area which is currently lacking in the literature.

Several applications of the results in this paper are discussed in forthcoming papers of the author: see [J4] and [J5]. For example, in [J4] and [J5], following upon the work in [J2], we have established various push-forward and localization formulae for the virtual structure sheaves and virtual fundamental classes on certain stacks.

Here is an outline of the paper. We begin section 2 by considering the K-theory and G-theory of Artin stacks, i.e. where the structure sheaf is the usual structure sheaf. After discussing a few key results in this context, we proceed to consider the basics of DG-stacks in section 3. We begin this section by recalling some of the key results on dg-stacks already established in [J2, section 2]. The main new results in this section start with Proposition 3.13 where we discuss several equivalent characterizations of perfect and pseudo-coherent complexes on dg-stacks. The next section (section 4) is devoted to a detailed proof of the projective space bundle theorem for the K-theory of dg-stacks. While the proof is based on Thomason's proof of a corresponding result for schemes (see [T]), it needs to be pointed out that our proof works mainly because of the carefully chosen definition of the K-theory of a dg-stack.

Section 5 is a detailed discussion of the G-theory of dg-stacks, establishing key properties like, devissage, localization and a form of homotopy property. It needs to be pointed out that this G-theory is not the K-theory of any abelian category, but the K-theory of the Waldhausen category of quasi-coherent $\mathcal{A}$-modules over the sheaf of dgas $\mathcal{A}$ with cartesian coherent cohomology sheaves. Therefore, it is far from obvious (though perhaps reasonable to expect) that the techniques of Quillen (see [Q]) can indeed be extended to establish these results. We conclude the paper by discussing forms of cohomology (homology) theories on dg-stacks that come equipped with a multiplicative Chern-character map (a Riemann-Roch transformation, respectively).

As has been already observed in [J3, section 2], the K-theory and G-theory of algebraic stacks are much better studied in the framework of Waldhausen K-theory. This justifies our focus on Waldhausen style K-theory and G-theory. Appendix A summarizes the definitions and a few basic results on Waldhausen K-theory, so as to serve as a reference for the rest of the paper while Appendix B discusses briefly injective resolutions of dg-modules.

Acknowledgments. The author wishes to thank the referee for a careful reading of an earlier draft of the paper and asking several important questions and comments that have improved the paper. In particular, we have added a discussion on the behavior of G-theory under quasi-isomorphisms of dg-stacks: this appears in 3.2.

2 Fundamentals of the K-theory and G-theory of algebraic stacks

Definition 2.1. An algebraic stack $\mathcal{S}$ will mean an algebraic stack (of Artin type) as in [LM, Definition (4.1)] which is Noetherian and finitely presented over a Noetherian base scheme. In particular, algebraic stacks will be quasi-compact and quasi-separated. An *action* of a group scheme G on a stack $\mathcal{S}$ will mean morphisms $\mu, pr_2 : G \times \mathcal{S} \rightarrow \mathcal{S}$ and $e : \mathcal{S} \rightarrow G \times \mathcal{S}$ satisfying the usual relations. (This implies, that we are only considering left-actions.)

It may be shown readily that if G is a smooth group scheme acting on an algebraic stack $\mathcal{S}$, a quotient stack $[\mathcal{S}/G]$ exists as an algebraic stack. To see this one may proceed as follows. First one defines the stack $[\mathcal{S}/G]$ as follows. Given a scheme T, the objects of the category $[\mathcal{S}/G](T)$ are pairs $(\tilde{T} \rightarrow T, \psi : \tilde{T} \rightarrow \mathcal{S})$ where $\tilde{T} \rightarrow T$ is a principal G-bundle and ψ is a G-equivariant map. The morphisms are isomorphisms of such pairs. It is straight-forward to verify that this a stack. Let η be an object of $\mathcal{S}(T)$. Then η corresponds to a map $T \rightarrow \mathcal{S}$ and defines a unique G-equivariant map $\eta' : G \times T \rightarrow \mathcal{S}$, where G acts by multiplication on the left-factor G. Sending η to the pair $(G \times T \rightarrow T, \eta' : G \times T \rightarrow \mathcal{S})$ defines a map of stacks $\mathcal{S} \rightarrow [\mathcal{S}/G]$. Since $\tilde{T}$ is a principal G-bundle, after pull-back to some smooth covering T' of T, $\tilde{T}$ becomes trivial, i.e. will be isomorphic to $G \times T'$. Any G-equivariant map $G \times T' \rightarrow \mathcal{S}$ is determined by the composite map $e \times T' \rightarrow G \times T' \rightarrow \mathcal{S}$. Using these observations, one may show that if $X \rightarrow \mathcal{S}$ is an atlas for $\mathcal{S}$, then the composite map $X \rightarrow \mathcal{S} \rightarrow [\mathcal{S}/G]$ is a smooth map which is surjective, thereby proving that $[\mathcal{S}/G]$ is an algebraic stack. In this case, there is an equivalence between the category of G-equivariant $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}$ and the category of $\mathcal{O}_{[\mathcal{S}/G]}$-modules. (See [J1, Appendix].) Therefore, one may incorporate the equivariant situation into the following discussion by considering quotient stacks of the form $[\mathcal{S}/G]$.

Starting with [LM, Chapter 12], the lisse-étale site has become more commonly used than the smooth site. Therefore, we will also use this site throughout the paper. Observe that if $\mathcal{S}$ is an algebraic stack, the underlying category of $\mathcal{S}_{lis-et}$ is the same as the underlying category of the smooth site $\mathcal{S}_{smt}$ which are smooth maps $u : U \rightarrow \mathcal{S}$, with U an algebraic space. The coverings of an object $u : U \rightarrow \mathcal{S}$ are étale surjective maps $\{u_i : U_i \rightarrow U | i\}$. We will provide $\mathcal{S}_{lis-et}$ with the structure sheaf $\mathcal{O}_{\mathcal{S}}$. One defines a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules M on $\mathcal{S}_{lis-et}$ to be *cartesian* as in [LM, Definition 12.3], i.e. for each map $\phi : U \rightarrow V$ in $\mathcal{S}_{lis-et}$, the induced map $\phi^{-1}(M_{|V_{et}}) \rightarrow M_{|U_{et}}$ is an isomorphism. In fact it suffices to have this property for all smooth maps ϕ. *In this paper, we will restrict to complexes of $\mathcal{O}_{\mathcal{S}}$-modules M whose cohomology sheaves are all cartesian.*

Definition 2.2. A bounded complex of $\mathcal{O}_{\mathcal{S}}$-modules M is *strictly perfect* if locally on the site $\mathcal{S}_{lis-et}$, M is a bounded complex of locally-free coherent $\mathcal{O}_{\mathcal{S}}$-modules and the cohomology sheaves are all cartesian. M is *perfect* if locally on the site $\mathcal{S}_{lis-et}$, M is quasi-isomorphic to a strictly perfect $\mathcal{O}_{\mathcal{S}}$-module and the cohomology sheaves are all cartesian. M is *pseudo-coherent* if it is locally quasi-isomorphic to a bounded above complex of locally free $\mathcal{O}_{\mathcal{S}}$-modules with bounded coherent cohomology sheaves which are cartesian. (One may readily prove, in view of the hypotheses, that if M is perfect, it is pseudo-coherent. Observe that, the usual definition of pseudo-coherence as in [SGA6] does not require the cohomology sheaves to be bounded; we have included this hypothesis in the definition of pseudo-coherence mainly for convenience.)

Let $\mathcal{S}'$ denote a closed algebraic sub-stack of $\mathcal{S}$. Then the category of all perfect (pseudo-coherent, strictly perfect) complexes with supports contained in $\mathcal{S}'$ along with quasi-isomorphisms forms a *bi-Waldhausen category* (see [TT, 1.2.4 Definition]): the *cofibrations* are those maps that are *degree-wise split monomorphisms*, the *fibrations* are those maps that are *degree-wise split epimorphisms* and the *weak-equivalences* are those maps that are quasi-isomorphisms.

Definition 2.3. The category of all perfect (pseudo-coherent, strictly perfect) complexes with supports contained in $\mathcal{S}'$ provided with the above structure of a bi-Waldhausen category will be denoted $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$, $\mathrm{StPerf}_{\mathcal{S}'}(\mathcal{S})$, respectively). Given a Waldhausen category $\mathbf{A}$, $N_\bullet wS_\bullet \mathbf{A}$ will denote the nerve in each degree of the simplicial category $wS_\bullet \mathbf{A}$. One lets the K-theory space of $\mathbf{A}$ to be $\Omega|N_\bullet wS_\bullet \mathbf{A}|$. We will denote this by $K(\mathbf{A})$ or $K(\mathbf{A}, w)$ when the choice of the subcategory $w(\mathbf{A})$ needs to be specified. The *K-theory space (G-theory space) of $\mathcal{S}$ with supports in $\mathcal{S}'$* will be defined to be the K-theory space of the Waldhausen category $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$, respectively) and denoted $K_{\mathcal{S}'}(\mathcal{S})$ ($G_{\mathcal{S}'}(\mathcal{S})$, respectively).

We also let $\mathrm{Perf}_{fl,\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{fl,\mathcal{S}'}(\mathcal{S})$) denote the full sub-category of $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$) consisting of complexes of flat $\mathcal{O}_{\mathcal{S}}$-modules in each degree. (The existence of functorial flat resolutions is shown in [J2, Appendix 10.2].)

Remarks 2.4. (i) Observe from [Ill, Chapitre I, Théorème 4.2.1.1] that flat $\mathcal{O}_{\mathcal{S}}$-modules have the *additional property that they are direct limits of finitely generated flat (in fact free) submodules at each stalk.* (Observe that the existence of flat resolutions and the Waldhausen approximation theorem (see Theorem 7.2) imply that one obtains a weak-equivalence $K(\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})) \simeq K(\mathrm{Perf}_{fl,\mathcal{S}'}(\mathcal{S}))$.)

(ii) There are several equivalent bi-Waldhausen category structures that one may put on the category of all perfect (pseudo-coherent, strictly perfect) complexes with supports contained in $\mathcal{S}'$. In all of these the weak-equivalences will be given by quasi-isomorphisms. One could take fibrations to be degree-wise epimorphisms instead of degree-wise split epimorphisms and similarly one could take cofibrations to be degree-wise monomorphisms as we chose above. One could also choose a Waldhausen category structure by omitting either the subcategory of cofibrations or the subcategory of fibrations. These all give the same K-theory spectra as shown in [TT, 1.9.2 Theorem] and [TT, 1.5.5].

Corollary 2.5. *Let $\mathcal{S}$ denote a smooth algebraic stack. Then the natural map $K(\mathcal{S}) \rightarrow G(\mathcal{S})$ is a weak-equivalence. In case $\mathcal{S}'$ is a closed algebraic sub-stack of $\mathcal{S}$, the natural map $K_{\mathcal{S}'}(\mathcal{S}) \rightarrow G(\mathcal{S}')$ is a weak-equivalence.*

Proof. The hypothesis that $\mathcal{S}$ is smooth, implies every complex of $\mathcal{O}_{\mathcal{S}}$-modules which is pseudo-coherent is perfect. (See [J3, section 2] for details; it is shown there the same proof as in [TT, 3.21 Theorem] for schemes works.) This proves the first assertion and the second follows similarly. □

Often it is convenient to compute $G(\mathcal{S})$ using the category of coherent sheaves which are defined as follows.

Definition 2.6. We define a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}_{lis-et}$ to be *quasi-coherent* if its restriction to the étale sites of *all* atlases for $\mathcal{S}$ are quasi-coherent. Coherent sheaves and locally free coherent sheaves are defined similarly. (Observe that this is slightly different from the usage in [LM], where a quasi-coherent sheaf is also assumed to be cartesian as in [LM] Definition 12.3. However, such a definition would then make it difficult to define a quasi-coherator that converts a complex of $\mathcal{O}_{\mathcal{S}}$-modules to a complex of quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules. This justifies our choice.) An $\mathcal{O}_{\mathcal{S}}$-module will always mean a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}_{lis-et}$. $Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ (or $Mod(\mathcal{S}_{lis-et}, \mathcal{O}_{\mathcal{S}})$ to be more precise) will denote the category of $\mathcal{O}_{\mathcal{S}}$-modules.

Let $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ ($\mathrm{QCoh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, $\mathrm{Coh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$) denote the category of all $\mathcal{O}_{\mathcal{S}}$-modules (all quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules, all coherent $\mathcal{O}_{\mathcal{S}}$-modules, respectively). Given any of the above abelian categories $\mathbf{A}$, let $\mathrm{C}^b_{cc}(\mathbf{A})$ ($\mathrm{C}^b_{cart}(\mathbf{A})$) denote the category of all bounded complexes of objects in $\mathbf{A}$ with cohomology sheaves that are cartesian and coherent (cartesian, respectively). Similarly we will $\mathrm{C}_{bcc}(\mathbf{A})$ denote the full sub-category of complexes in $\mathbf{A}$ with cohomology sheaves that are cartesian and coherent and vanish in all but finitely many degrees. These are all bi-Waldhausen categories with same structure as above, i.e. with co-fibrations (fibrations) being maps of complexes that are degree-wise split monomorphisms (degree-wise split epimorphisms, respectively) and weak-equivalences being maps that are quasi-isomorphisms.

Proposition 2.7. *The obvious inclusion functors $\mathrm{C}^b_{cart}(\mathrm{Coh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})) \rightarrow \mathrm{C}^b_{cc}(\mathbf{A}) \rightarrow \mathrm{C}_{bcc}(\mathbf{A}) \rightarrow \mathrm{Pseudo}(\mathcal{S})$ induce weak-equivalence on taking the corresponding K-theory spaces.*

Proof. Recall that all the stacks we consider are Noetherian. Therefore, this is clear in view of the Waldhausen approximation theorem (see Theorem 7.2 in the appendix) and the results of [J2, Appendix B]. □

We let $K_{naive}(\mathcal{S}) = K(\mathrm{StPerf}(\mathcal{S}))$. (Recall $\mathrm{StPerf}(\mathcal{S})$ is the category of strictly perfect complexes on $\mathcal{S}$.)

Proposition 2.8. *Assume that every coherent sheaf on the algebraic stack $\mathcal{S}$ is the quotient of a vector bundle. Then the obvious map $K_{naive}(\mathcal{S}) \rightarrow K(\mathcal{S})$ is a weak-equivalence.*

Proof. The proof follows along the same lines as in [TT, 3.8 Lemma] and [TT, 2.3.1 Proposition] making repeated use of the Waldhausen approximation theorem: see Theorem 7.2 in the appendix. However, we sketch an outline for the sake of completeness. First one may establish the following:

($*$) given any pseudo-coherent complex $F^\bullet$ and a map $p : P^\bullet \to F^\bullet$, with $P^\bullet$ a bounded above complex of vector bundles, there exists a bounded above complex of vector bundles $Q^\bullet$, and maps $p' : P^\bullet \to Q^\bullet$, $q : Q^\bullet \to F^\bullet$ so that $p = q \circ p'$ and q is a quasi-isomorphism.

Now it remains to show that if $F^\bullet$ is perfect and $P^\bullet$ is a bounded complex of vector bundles, then $Q^\bullet$ can be chosen to be a bounded complex. We may assume that $F^\bullet$ is bounded. Let $Q^\bullet$ be as in (*). It is perfect and let k be so that $\mathcal{H}^n(Q^\bullet) = 0$, $F^n = 0$, $n \leq k$. We will show $Q^p/B^p(Q^\bullet) = Q^p/Im(d : Q^{p-1} \to Q^p) = coker(d^{p-1} : Q^{p-1} \to Q^p)$ is a vector bundle for some $p << 0$ so that $\tau_{\geq p}(Q^\bullet)$ is a bounded complex of vector bundles. (If $K^\bullet$ is a complex, $\tau_{\geq p}(K^\bullet)$ is the complex given by 0 in degrees $\leq p-1$, $Coker(d^{p-1} : K^{p-1} \to K^p)$ in degree p and K^i in degrees $> p$.) Clearly $P^\bullet \to Q^\bullet \to \tau_{\geq p}(Q^\bullet) \to \tau_{\geq p}(F^\bullet) = F^\bullet$. Now, assume $B^k(Q^\bullet)$ has finite tor dimension N. Then the short-exact sequence $0 \to B^n(Q^\bullet) \to Q^n \to B^{n+1}(Q^\bullet) \to 0$, $n \leq k$, shows that $Coker(d^{n-1} : Q^{n-1} \to Q^n) \cong B^{n+1}$ for all $n \leq k$. The same short-exact sequence also shows, by an argument using Tor, that $B^{k-N}(Q^\bullet) \cong Coker(d^{k-N-2} : Q^{k-N-2} \to Q^{k-N-1})$ is flat. It is locally finitely presented and hence locally free, i.e. a vector bundle. Therefore, $\tau_{\geq k-N-1}(Q^\bullet)$ is a bounded complex of vector bundles. Therefore, it suffices to show that $B^k(Q^\bullet)$ is of finite tor dimension:

Consider $\alpha : \sigma_{\geq k+1}(Q^\bullet)(= 0 \to Q^{k+1} \to Q^{k+2} \to \cdots \to Q^n \to \cdots) \to Q^\bullet$. Then $Cone(\alpha)[-1]$ is perfect since $Q^\bullet$ and $\sigma_{\geq k+1}(Q^\bullet)$ are perfect. But $\mathcal{H}^i(Cone(\alpha)[-1]) = 0$ for $i \neq k+1$ and $= Q^k/Im(d^{k-1})$ if $i = k+1$. So $Im(d^{k-1}) = B^k(Q^\bullet)$ is of finite tor dimension. □

Remark 2.9. The results of [J2, Appendix B] show that K-theory is a contravariant functor for arbitrary maps between algebraic stacks. In fact, we have taken considerable effort in our basic framework so that this property holds. In the case of quotient stacks of the form $[X/G]$, one could adopt the simpler set-up in [T] where one uses the Zariski site of X to define equivariant sheaves. In this case, none of the complications due to the smooth site appear.

Example 2.10. Assume the base scheme is a field k and that G is a linear algebraic group over k. On a quotient stack $[X/G]$, when the scheme X is G-quasi-projective (i.e. X admits a G-equivariant locally closed immersion into a projective space $\mathbb{P}^n$ on which G acts linearly), every coherent sheaf is the quotient of a vector bundle. This follows from the work of Thomason: see [T].

3 Fundamentals of DG-stacks

We have already developed much of the basic material in [J2, section 2]: therefore, we will only recall the main definitions and basic results here.

Definition 3.1. (a) A DG-stack is an algebraic stack $\mathcal{S}$ as in Definition 2.1 which is also defined over a field of characteristic 0, and provided with a sheaf of commutative dgas $\mathcal{A}$ in $Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, so that the following conditions are satisfied: (i) $\mathcal{A}^i = 0$ for $i > 0$ or $i << 0$, (ii) each $\mathcal{A}^i$ is a coherent $\mathcal{O}_{\mathcal{S}}$-module and the cohomology sheaves $\mathcal{H}^i(\mathcal{A})$ are all cartesian. (iii). We will further require that $\mathcal{A}^0 = \mathcal{O}_{\mathcal{S}}$.

(b) The above definition is often a bit too restrictive. It will be often convenient to modify the hypotheses on $\mathcal{A}$ so that $\mathcal{A}$ satisfies the following alternate hypotheses in addition to being a sheaf of commutative dgas in $Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$: (i) $\mathcal{A}^i = 0$ for $i > 0$, (ii) each $\mathcal{A}^i$ is a flat $\mathcal{O}_{\mathcal{S}}$-module, the cohomology sheaves $\mathcal{H}^i(\mathcal{A})$ are coherent, cartesian and trivial for $i << 0$ and (iii) $\mathcal{H}^0(\mathcal{A})$ defines a closed sub-stack of $\mathcal{S}$. (The last condition means that $\mathcal{H}^0(\mathcal{A})$ is isomorphic as a sheaf of algebras to a quotient of $\mathcal{O}_{\mathcal{S}}$ by a sheaf of ideals.)

Throughout the paper, a dg-stack by default will mean one in the sense of (b); we will explicitly mention when the dg-stack is in the sense of (a).

3.0.1

One may replace a dga $\mathcal{A}$ in the sense of Definition 3.1(a), by a *flat resolution* $\tilde{\mathcal{A}} \to \mathcal{A}$, i.e. a complex of *flat* $\mathcal{O}_{\mathcal{S}}$-modules, trivial in positive degrees so that $\tilde{\mathcal{A}}$ is a commutative dga and provided with a quasi-isomorphism

$\tilde{\mathcal{A}} \to A$. See [J2, 10.2] where it is shown that one can find such flat resolutions. Starting with a dga as in Definition 3.1(a), this will produce a dga in the sense of Definition 3.1(b): the dg-structure sheaves as in Definition 3.1(b) will be typically obtained this way. In this case, we will also use $\mathcal{A}$ to denote the resolution $\tilde{\mathcal{A}}$ unless we need to be more specific. We will use this only in section 5, while considering devissage for G-theory. Since, we show G-theory is invariant under quasi-isomorphism, it will follow that one could replace $\mathcal{A}$ by $\tilde{\mathcal{A}}$ when considering G-theory. Observe also that our hypotheses imply that $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded Noetherian rings.

The need to consider dg-stacks should be clear in view of the applications to virtual structure sheaves and virtual fundamental classes: see example 3.9.

For the purposes of this paper, we will define a DG-stack $(\mathcal{S}, \mathcal{A})$ to have property P if the associated underlying stack $\mathcal{S}$ has property P: for example, $(\mathcal{S}, \mathcal{A})$ is *smooth* if $\mathcal{S}$ is smooth. Often it is convenient to also include disjoint unions of such algebraic stacks into consideration.

Remark 3.2. The restriction to characteristic 0 while considering *dg*-stacks is simply because commutative dgas are not well-behaved in positive characteristics. It needs to be perhaps pointed out also that, all the main results proved in this paper hold in arbitrary characteristics, when the dg-structure sheaf is the usual structure sheaf, i.e. they hold for general Artin stacks satisfying the hypotheses as in Definition 2.1.

3.0.2 Morphisms and quasi-isomorphisms of dg stacks

A 1-morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of DG-stacks is a morphism of the underlying stacks $\mathcal{S}' \to \mathcal{S}$ together with a map $\mathcal{A} \to f_*(\mathcal{A}')$ of sheaves compatible with the map $\mathcal{O}_{\mathcal{S}} \to f_*(\mathcal{O}_{\mathcal{S}'})$. Such a morphism will have property P if the associated underlying 1-morphism of algebraic stacks has property P. Clearly DG-stacks form a 2-category.

When $(\mathcal{S}, \mathcal{A})$ is a dg-stack as in Definition 3.1(a) or (b), let $\bar{\mathcal{S}}$ denote the closed sub-stack of $\mathcal{S}$ defined by $\mathcal{H}^0(\mathcal{A})$. We let $\bar{i} : \bar{\mathcal{S}} \to \mathcal{S}$ denote the corresponding closed immersion. Observe that $\mathcal{H}^*(\mathcal{A})$ identifies with a sheaf of graded modules on $(\bar{\mathcal{S}}, \mathcal{H}^0(\mathcal{A}))$. A morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of DG-stacks will be called a *quasi-isomorphism* if the following conditions are satisfied: (i) The map $f : \mathcal{S}' \to \mathcal{S}$ induces isomorphisms $\bar{f} : \bar{\mathcal{S}}' \to \bar{\mathcal{S}}$, $\bar{f}^{-1}(\mathcal{H}^*(\mathcal{A})) \overset{\cong}{\to} \mathcal{H}^*(\mathcal{A}')$ and (ii) the square

$$\begin{array}{ccc} \bar{S}' & \xrightarrow{\bar{f}} & \bar{S} \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ \mathcal{S}' & \xrightarrow{f} & \mathcal{S} \end{array} \tag{3.0.3}$$

is cartesian.

Observe as a consequence that the induced map $f^{-1}(i_*(\mathcal{H}^*(\mathcal{A}))) \to i'_* \bar{f}^{-1}(\mathcal{H}^*(\mathcal{A})) \to i'_*(\mathcal{H}^*(\mathcal{A}'))$ is an isomorphism. (The square (3.0.3) being cartesian is necessary for the first map to be an isomorphism in general.)

If $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{S}', \mathcal{A}')$ are two DG-stacks, one defines their *product* to be the product stack $\mathcal{S} \times \mathcal{S}'$ endowed with the sheaf of DGAs $\mathcal{A} \boxtimes \mathcal{A}'$.

3.0.4 Coherent and Perfect $\mathcal{A}$-modules

A left $\mathcal{A}$-module is a complex of sheaves M of $\mathcal{O}_{\mathcal{S}}$-modules, *bounded above* and so that M is a sheaf of left-modules over the sheaf of dgas $\mathcal{A}$ (on $\mathcal{S}_{lis-et}$) and so that the cohomology sheaves $\mathcal{H}^i(M)$ are all cartesian. The category of all left $\mathcal{A}$-modules and morphisms will be denoted $Mod_l(\mathcal{S}, \mathcal{A})$. We define a map $f : M' \to M$ in $Mod_l(\mathcal{S}, \mathcal{A})$ to be a quasi-isomorphism if it is a quasi-isomorphism of $\mathcal{O}_{\mathcal{S}}$-modules: observe that this is equivalent to requiring that $\mathcal{H}^*(Cone(f)) = 0$ in $Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$. This is in view of the fact that the mapping cone of the given map $f : M' \to M$ of $\mathcal{A}$-modules taken in the category of $\mathcal{O}_{\mathcal{S}}$-modules has an induced $\mathcal{A}$-module structure. A diagram $M' \overset{f}{\to} M \to M'' \to M[1]$ in $Mod_l(\mathcal{S}, \mathcal{A})$ is a *distinguished triangle* if there is a map $Cone(f) \to M''$ in $Mod_l(\mathcal{S}, \mathcal{A})$ which is a quasi-isomorphism. Since we assume $\mathcal{A}$ is a sheaf of commutative dgas, there is an equivalence of categories between left and right modules; therefore, henceforth we will simply refer to $\mathcal{A}$-modules rather than left or right $\mathcal{A}$-modules.

Definition 3.3. An $\mathcal{A}$-module M is *perfect* if the following holds: there exists a non-negative integer n and distinguished triangles $F_iM \rightarrow F_{i+1}M \rightarrow F_{i+1}M/F_iM \rightarrow F_iM[1]$ in $Mod(\mathcal{S}, \mathcal{A})$, for all $0 \leq i \leq n-1$, so that $F_0M \simeq \mathcal{A} \overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_0$, $F_{i+1}M/F_iM \simeq \mathcal{A} \overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_{i+1}$ with each P_i a perfect complex of $\mathcal{O}_{\mathcal{S}}$-modules and there is given a quasi-isomorphism $F_nM \rightarrow M$ of $\mathcal{A}$-modules. The morphisms between two such objects will be just morphisms of $\mathcal{A}$-modules. This category will be denoted $Perf(\mathcal{S}, \mathcal{A})$. M is *coherent* if $\mathcal{H}^*(M)$ is bounded and finitely generated as a sheaf of $\mathcal{H}^*(\mathcal{A})$-modules. Again morphisms between two such objects will be morphisms of $\mathcal{A}$-modules. This category will be denoted $Coh(\mathcal{S}, \mathcal{A})$. A left-$\mathcal{A}$-module M is *flat* if $M \underset{\mathcal{A}}{\otimes} - : Mod(\mathcal{S}, \mathcal{A}) \rightarrow Mod(\mathcal{S}, \mathcal{A})$ preserves quasi-isomorphisms. If $\mathcal{S}'$ is a given closed sub-algebraic stack of $\mathcal{S}$, $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ will denote the full sub-category of $Perf(\mathcal{S}, \mathcal{A})$ consisting of objects with supports contained in $\mathcal{S}'$.

Remark 3.4. Recall that all $\mathcal{A}$-modules M we consider are required to have cartesian cohomology sheaves. Therefore, when $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, the category $Coh(\mathcal{S}, \mathcal{A})$ identifies with $C_{bcc}(Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$.

We proceed to define the structure of bi-Waldhausen categories with cofibrations, fibrations and weak-equivalences on $Coh(\mathcal{S}, \mathcal{A})$, $Perf(\mathcal{S}, \mathcal{A})$ and $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$. Let $f : M \rightarrow N$ denote a map of $\mathcal{A}$-modules. A collection of maps $g = \{g^n : N^n \rightarrow M^n | n\}$, not necessarily a chain-map, but which commutes with the $\mathcal{A}$-action (i.e. $g(a.b^n) = ag(b^n), a\varepsilon\Gamma(U, \mathcal{A}),\ b^n\varepsilon\Gamma(U, N^n),\ U\varepsilon\mathcal{S}_{lis-et}$.) will be called an *$\mathcal{A}$-compatible right splitting to* f, if $f^n \circ g^n = id_{N^n}$, for all n. One defines *$\mathcal{A}$-compatible left splitting to* f similarly. Now one defines a map $f : M \rightarrow N$ in $Coh(\mathcal{S}, \mathcal{A})$, $Perf(\mathcal{S}, \mathcal{A})$ or $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ to be a *cofibration* (*fibration*) if f has an $\mathcal{A}$-compatible left splitting (right splitting, respectively). In particular, observe that each cofibration is a degree-wise split monomorphism and each fibration is a degree-wise split epimorphism.

The main examples of such cofibrations (fibrations) arise as follows. Let $f : M \rightarrow N$ denote a map of $\mathcal{A}$-modules and let $Cyl(f)$ ($Cocyl(f)$) denote the mapping cylinder (mapping cocylinder, respectively) defined as follows:

$$Cyl(f)^n = M^{n+1} \oplus N^n \oplus M^n \text{ with the boundary map } \delta(m', n, m) = (-\delta(m'), \delta(n) - f(m'), \delta(m) + m')$$

$$Cocyl(f)^n = N^n \oplus N^{n-1} \oplus M^n \text{ with the boundary map } \delta(n', n, m) = (\delta(n'), -\delta(n) + n' - f(m), \delta(m))$$

Then the induced map $i : M \rightarrow Cyl(f)$ has an $\mathcal{A}$-compatible left-inverse and the induced map $p : Cocyl(f) \rightarrow N$ has an $\mathcal{A}$-compatible right-inverse. (Similarly, the obvious map $N \rightarrow Cone(f)$ has an $\mathcal{A}$-compatible left-inverse.)

Definition 3.5. (K-theory and G-theory of dg-stacks.) We will let the fibrations (cofibrations) be the those maps $M \rightarrow N$ of $\mathcal{A}$-modules that have an $\mathcal{A}$-compatible right-splitting (left-splitting, respectively) and weak-equivalences to be maps of $\mathcal{A}$-modules that are quasi-isomorphisms. To see this defines the structure of a bi-Waldhausen category, see [TT]: observe that it suffices to verify the cofibrations, fibrations and weak-equivalences are stable by compositions and satisfy a few easily verified extra properties as in [TT, Definitions 1.2.1 and 1.2.3]. We will let $Coh(\mathcal{S}, \mathcal{A})$ ($Perf(\mathcal{S}, \mathcal{A})$, $Perf_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$) denote the above category with this bi-Waldhausen structure. The K-theory (G-theory) space of $(\mathcal{S}, \mathcal{A})$ will be defined to be the K-theory space of the Waldhausen category $Perf(\mathcal{S}, \mathcal{A})$ ($Coh(\mathcal{S}, \mathcal{A})$, respectively) and denoted $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$, respectively). When $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$) will be denoted $K(\mathcal{S})$ ($G(\mathcal{S})$, respectively). Let $Perf_{fl}(\mathcal{S}, \mathcal{A})$ denote the full sub-category of $Perf(\mathcal{S}, \mathcal{A})$ consisting of flat $\mathcal{A}$-modules. This sub-category inherits a bi-Waldhausen category structure from the one on $Perf(\mathcal{S}, \mathcal{A})$ where a map $f : M \rightarrow N$ of $\mathcal{A}$-modules is a cofibration (fibration) if and only if it is a cofibration (fibration, respectively) in $Perf(\mathcal{S}, \mathcal{A})$ and $cokernel(f)$ ($kernel(f)$, respectively) is a flat $\mathcal{A}$-module.

Remark 3.6. In view of the existence of the cylinder and co-cylinder functors as above, [TT, 1.9.2 Theorem] shows that one may also define cofibrations (fibrations) to be degree-wise monomorphisms (degree-wise epimorphisms) and that the resulting K-theory will be the same.

Proposition 3.7. *(See [J2, Proposition 2.9].) (i)If M is perfect, it is coherent.*

(ii) Let $M\varepsilon Perf(\mathcal{S}, \mathcal{A})$. Then there exists a flat $\mathcal{A}$-module $\tilde{M}\varepsilon Perf(\mathcal{S}, \mathcal{A})$ together with a quasi-isomorphism $\tilde{M} \rightarrow M$.

(iii) Let $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ denote a distinguished triangle of $\mathcal{A}$-modules. Then if two of the modules M', M and M'' are coherent (perfect) $\mathcal{A}$-modules, then so is the third.

(iv) Let $\phi : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks. Then one obtains an induced functor $\phi^* : Perf_{fl}(\mathcal{S}, \mathcal{A}) \to Perf_{fl}(\mathcal{S}', \mathcal{A}')$ of bi-Waldhausen categories with cofibrations, fibrations and weak-equivalences.

(v) Assume in addition to the situation in (iii) that $\mathcal{S}' = \mathcal{S}$ and that the given map $\phi : \mathcal{A}' \to \mathcal{A}$ is a quasi-isomorphism. Then $\phi_* : Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}, \mathcal{A}')$ defines a functor of bi-Waldhausen categories with cofibrations, fibrations and weak-equivalences. Moreover, the compositions $\phi_* \circ \phi^*$ and $\phi^* \circ \phi_*$ are naturally quasi-isomorphic to the identity.

(vi) There exists natural pairing $(\quad)\overset{L}{\underset{\mathcal{A}}{\otimes}}(\quad) : Perf(\mathcal{S}, \mathcal{A}) \times Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathcal{S}, \mathcal{A})$ so that $\mathcal{A}$ acts as the unit for this pairing.

Proof. This is proved in detail in [J2, Proposition 2.9]. However, there we only considered the structure of Waldhausen categories with fibrations and weak-equivalences, where fibrations were only required to be degree-wise epimorphisms. Therefore, the only differences are in the last 3 statements. We will only consider (iv) as the modifications required for the other should be the same. It is clear that the functors in (iv) and (v) preserve quasi-isomorphisms and that the bi-functor in (vi) preserves quasi-isomorphisms in each argument. Since we have defined cofibrations to be those of maps of $\mathcal{A}$-modules that have an $\mathcal{A}$-compatible left-inverse, it is easy to see that ϕ^* preserves such cofibrations; similarly, since fibrations are those maps of $\mathcal{A}$-modules that have an $\mathcal{A}$-compatible right inverse, it follows readily that ϕ^* preserves fibrations. These prove (iv). Since the functor ϕ_* sends an $\mathcal{A}$-module M to the same object, but viewed as an $\mathcal{A}'$-module, using the map $\phi : \mathcal{A}' \to \mathcal{A}$, it follows readily that ϕ_* preserves cofibrations and fibrations. (Recall the cofibrations (fibrations) are degree-wise monomorphisms (epimorphisms) of $\mathcal{A}$-modules with $\mathcal{A}$-compatible left inverse (right inverse, respectively) in the sense of Definition 3.5.) ∎

Remarks 3.8. 1. Observe that the above K-theory spectra, $K(Perf(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$ and $K(Perf(\mathcal{S}, \mathcal{A}))$ are in fact E^∞-ring spectra and the obvious augmentation $\mathcal{O}_{\mathcal{S}} \to \mathcal{A}$ makes $K(\mathcal{S}, \mathcal{A})$ a $K(\mathcal{S})$-algebra. Given two modules M and N over $\mathcal{A}$, one may compute $\mathcal{H}^*(M\overset{L}{\underset{\mathcal{A}}{\otimes}}N)$ using the spectral sequence:

$$E_2^{s,t} = Tor_{s,t}^{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M), \mathcal{H}^*(N)) \Rightarrow \mathcal{H}^*(M\overset{L}{\underset{\mathcal{A}}{\otimes}}N)$$

If M and N are coherent and one of them is also a perfect $\mathcal{A}$-module, then they both have bounded cohomology sheaves and the above spectral sequence is strongly convergent. It follows that, if M and N are coherent $\mathcal{A}$-modules and one of them is perfect, then $M\overset{L}{\underset{\mathcal{A}}{\otimes}}N$ is coherent. It follows from this observation that $G(\mathcal{S}, \mathcal{A})$ is a module spectrum over $K(\mathcal{S}, \mathcal{A})$.

2. Assume $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ is a *proper map* of *DG*-stacks so that $Rf_* : D_+(Mod(\mathcal{S}', \mathcal{O}_{\mathcal{S}'})) \to D_+(Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$ has finite cohomological dimension. Now Rf_* induces a map $Rf_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$.

3. Assume that the dg-structure sheaf $\mathcal{A}$ is in fact the structure sheaf $\mathcal{O}$ and the stack $\mathcal{S}$ is *smooth*. Then we proved in section 1 that the obvious map $K(\mathcal{S}) \to G(\mathcal{S})$ is a weak-equivalence. If $\mathcal{S}'$ is a closed sub-stack of $\mathcal{S}$, then the obvious map $K_{\mathcal{S}'}(S) \to G(\mathcal{S}')$ is also a weak-equivalence where $K_{\mathcal{S}'}(\mathcal{S})$ denotes the K-theory of the Waldhausen category $Perf_{\mathcal{S}'}(\mathcal{S})$.

Example 3.9. *Algebraic stacks provided with virtual structure sheaves* The basic example of a DG-stack that we consider will be an algebraic stack (typically of the form $\mathfrak{M}_{g,n}(X, \beta)$) provided with a *virtual structure sheaf* provided by a *perfect obstruction theory*. Here X is a projective variety over a field k of characteristic 0, β is a one dimensional cycle and $\mathfrak{M}_{g,n}(X, \beta)$ denotes the stack of stable curves of genus g and n-markings associated to X. The virtual structure sheaf $\mathcal{O}^{virt}$ is the corresponding sheaf of dgas. Since this is the key-example of dg-stacks we consider, we will discuss this example in some detail.

Let $\mathcal{S}$ denote a Deligne-Mumford stack (over k) with $u : U \to \mathcal{S}$ an atlas and let $i : U \to M$ denote a closed immersion into a smooth scheme. Let $C_{U/M}$ ($N_{U/M}$) denote the normal cone (normal bundle, respectively) associated to the closed immersion i. (Recall that if $\mathcal{I}$ denotes the sheaf of ideals associated to the closed immersion i, $C_{U/M} = Spec\underset{n}{\oplus}\mathcal{I}^n/\mathcal{I}^{n+1}$ and $N_{U/M} = SpecSym(\mathcal{I}/\mathcal{I}^2)$. Now $[C_{U/M}/i^*(T_M)]$ ($[N_{U/M}/i^*(T_M)]$) denotes the *intrinsic normal cone* denoted $\mathcal{C}_{\mathcal{S}}$ (the *intrinsic abelian normal cone* denoted $\mathcal{N}_{\mathcal{S}}$, respectively).

Let $E^\bullet$ denote a complex of $\mathcal{O}_{\mathcal{S}}$-modules so that it is trivial in positive degrees and whose cohomology sheaves in degrees 0 and -1 are coherent. Let $L^\bullet_{\mathcal{S}}$ denote the *cotangent complex* of the stack $\mathcal{S}$ over k. A morphism $\phi : E^\bullet \to L^\bullet_{\mathcal{S}}$ in the derived category of complexes of $\mathcal{O}_{\mathcal{S}}$-modules is called an *obstruction theory* if ϕ induces an isomorphism (surjection) on the cohomology sheaves in degree 0 (in degree -1, respectively). We call the obstruction theory $E^\bullet$ *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$ (i.e. locally on the étale site of the stack, it is quasi-isomorphic to a complex of vector bundles concentrated in degrees 0 and -1). In this case, one may define the *virtual dimension* of $\mathcal{S}$ with respect to the obstruction theory $E^\bullet$ as $rank(E^0) - rank(E^{-1})$: this is a locally constant function on $\mathcal{S}$, which we will assume (as customary), is in fact constant. Moreover, in this case, we let $\mathcal{E}_{\mathcal{S}} = h^1/h^0(E^\bullet) = [\mathcal{E}_1/\mathcal{E}_0]$ where $\mathcal{E}_i = SpecSym(E^{-i})$. We will denote $\mathcal{E}_i$ also by $C(E^{-i})$.

Now the morphism ϕ defines a closed immersion $\phi^\vee : \mathcal{N}_{\mathcal{S}} \to \mathcal{E}_{\mathcal{S}}$. Composing with the closed immersion $\mathcal{C}_{\mathcal{S}} \to \mathcal{N}_{\mathcal{S}}$ one observes that $\mathcal{C}_{\mathcal{S}}$ is a closed cone sub-stack of $\mathcal{E}_{\mathcal{S}}$. Let the zero-section of $\mathcal{S}$ in $\mathcal{E}_{\mathcal{S}}$ be denoted $0_{\mathcal{S}}$. Now we define the *virtual structure sheaf* $\mathcal{O}^{virt}_{\mathcal{S}}$ with respect to the given obstruction theory to be $L0^*_{\mathcal{S}}(\mathcal{O}_{\mathcal{C}_{\mathcal{S}}})$. It is shown in [J2, p. 15] that then $(\mathcal{S}, \mathcal{O}^{virt}_{\mathcal{S}})$ is a dg-stack in the sense of Definition 3.1.

Remark 3.10. The dg-structure sheaf $\mathcal{O}^{virt}_{\mathcal{S}}$ may also be defined as $L0^!_{\mathcal{S}}(\mathcal{O}_{\mathcal{C}}) = O_{\mathcal{C}} \underset{\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}}{\otimes} K(\mathcal{O}_{\mathcal{S}})$ where $K(\mathcal{O}_{\mathcal{S}})$ is the canonical Koszul-resolution of $\mathcal{O}_{\mathcal{S}}$ by $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules provided by the obstruction theory. This has the *dis-advantage* that it will not be a complex of $\mathcal{O}_{\mathcal{S}}$-modules but only of $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules. In fact, both definitions provide the same class in the ordinary G-theory of the stack $\mathcal{S}$: see [J4] Theorem 1.2.

In the above example, we only considered the absolute case; there is relative variant of this, where the given stack $\mathcal{S}$, in addition to being Deligne-Mumford over k will be relative Deligne-Mumford over a base which can in fact be an Artin stack. In this case the cotangent complex $L^\bullet_{\mathcal{S}}$ will denote the relative cotangent complex over the given base.

Proposition 3.11. *(See [J2, Proposition 2.13].) Let $(\mathcal{S}, \mathcal{A})$ denote a DG-stack in the above sense and let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks.*

(i) An $\mathcal{A}$-module M is coherent in the above sense if and only if it is pseudo-coherent (i.e. locally on $\mathcal{S}_{lis-et}$ quasi-isomorphic to a bounded above complex of locally free sheaves of $\mathcal{O}_{\mathcal{S}}$-modules) with bounded coherent cohomology sheaves of $\mathcal{O}_{\mathcal{S}}$-modules.

(ii) One has an induced map $f^* : K(\mathcal{S}, \mathcal{A}) \to K(\mathcal{S}', \mathcal{A}')$ and if f is proper and of finite cohomological dimension one also has an induced map $f_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$.

(iii) If $\mathcal{H}^*(\mathcal{A}')$ is of finite tor dimension over $f^{-1}(\mathcal{H}^*(\mathcal{A})$, then one obtains an induced map $f^* : G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}', \mathcal{A}')$.

(iv) If f_* sends $Perf(\mathcal{S}', \mathcal{A}')$ to $Perf(\mathcal{S}, \mathcal{A})$, then it induces a direct image map $f_* : K(\mathcal{S}', \mathcal{A}') \to K(\mathcal{S}, \mathcal{A})$.

Convention 3.12. *DG-stacks whose associated underlying stack is of Deligne-Mumford type will be referred to as Deligne-Mumford DG-stacks.*

We end this discussion on the K-theory of dg-stacks with the following results that give various *avatars* for the K-theory. The first result is on the K-theory of perfect complexes of $\mathcal{O}$-modules quoted from [TT] section 3.

Proposition 3.13. *Let $\mathcal{S}$ denote an algebraic stack as in Definition 2.1. Then the following bi-Waldhausen categories of $\mathcal{O}_{\mathcal{S}}$-modules are equivalent, where the bi-Waldhausen structure is given as follows: weak-equivalences are quasi-isomorphisms of $\mathcal{O}_{\mathcal{S}}$-modules and fibrations (cofibrations) are maps of complexes that are degree-wise split epimorphisms (degree-wise split monomorphisms, respectively).*

- *perfect complexes of $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect strict bounded complexes of $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect bounded above complexes of flat $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect bounded below complexes of injective $\mathcal{O}_{\mathcal{S}}$-modules*

- *perfect complexes of quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect complexes of injective objects in the category of quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect complexes of coherent $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect strict bounded complexes of coherent $\mathcal{O}_{\mathcal{S}}$-modules*

Proof. The equivalences of the first four categories may be obtained as in [TT] section 3, 3.5 Lemma. The equivalence of categories between these and the last four follow making use of the quasi-coherator discussed in [J2, Appendix B]. (See also [TT, 3.6 Lemma and 3.7 Lemma].) Moreover, one may observe that, since the stack is assumed to be Noetherian, the category of quasi-coherent sheaves is a Grothendieck category and hence has enough injectives. (See [J2, Appendix B] again for details.) ☐

Lemma 3.14. *Assume the dg-stack $(\mathcal{S}, \mathcal{A})$ is one in the sense of Definition 3.1(a). Then any $\mathcal{A}$-module M which in each degree is a quasi-coherent $\mathcal{O}_{\mathcal{S}}$-module is the filtered colimit of a direct system of $\mathcal{A}$-modules that are bounded complexes and in each degree a coherent $\mathcal{O}_{\mathcal{S}}$-module.*

Proof. Clearly $M = \lim_{\overrightarrow{\alpha}} M_\alpha$, where each M_α is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules, with each M_α^n an $\mathcal{O}_{\mathcal{S}}$ submodule of M^n in each degree n. Let $\tilde{M}_\alpha =$ the image of the natural map $\mathcal{A} \otimes M_\alpha \rightarrow M$. Recall that $\mathcal{A}$ is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules by assumption. Therefore, each $\tilde{M}_\alpha$ is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules; clearly each $\tilde{M}_\alpha$ is also an $\mathcal{A}$-module. This proves the lemma. ☐

3.1 Coherent Approximation

Throughout this subsection a dg-stack will mean one in the sense of Definition 3.1(a). Next we proceed to show that if M is coherent as an $\mathcal{A}$-module, then M is quasi-isomorphic to an $\mathcal{A}$-module (not necessarily an $\mathcal{A}$-submodule of M) which is a bounded complex and in each degree a coherent $\mathcal{O}_{\mathcal{S}}$-module. This is proved in Theorem 3.16.

We next let M denote an $\mathcal{A}$-module and n an integer. We let $\tau_{\leq n-1}M$ denote the subcomplex of M defined by

$$\begin{aligned}(\tau_{\leq n-1}M)^i &= M^i, i < n, \\ &= Im(d^{n-1} : M^{n-1} \rightarrow M^n), i = n, \\ &= 0, i > n.\end{aligned} \tag{3.1.1}$$

The observation that $\mathcal{A}^i = 0$ for $i > 0$ and $= \mathcal{O}_{\mathcal{S}}$ for $i = 0$ shows that $\tau_{\leq n-1}M$ gets an induced structure as an $\mathcal{A}$-module. (To see this observe that $\mathcal{A}^i.M^j \rightarrow M^{i+j}$. In view of the above hypotheses on $\mathcal{A}$, $i \leq 0$, so that $i+j \leq j$. The differentials of the complex M are $\mathcal{O}_{\mathcal{S}}$-linear maps, so that $\mathcal{A}^0.(Im(d : M^{n-1} \rightarrow M^n)) \subseteq (Im(d : M^{n-1} \rightarrow M^n))$.) Therefore, $\tau_{\geq n}M = M/(\tau_{\leq n-1}M)$ also gets the structure of an $\mathcal{A}$-module. Observe that $\tau_{\geq n}$ is the functor that kills all the cohomology in degrees $< n$. These observations prove the following lemma.

Lemma 3.15. *Next suppose M is an $\mathcal{A}$-module where M has bounded coherent cohomology sheaves and let n denote an integer so that $\mathcal{H}^i(M) = 0$ for all $i < n$. Then the map $M \rightarrow \tau_{\geq n}M$ is a quasi-isomorphism of $\mathcal{A}$-modules and $\tau_{\geq n}M$ is bounded below. If M is bounded above, then $\tau_{\geq n}M$ is also bounded.*

Next we consider *killing cohomology classes* for $\mathcal{A}$-modules by attaching $\mathcal{A}$-cells: basically we show that the usual construction carries over to the setting of $\mathcal{A}$-modules. Therefore let M denote an $\mathcal{A}$-module and let $\alpha = \alpha_N : C^N \rightarrow Z^N(M) \subseteq M^N$ denote a map where C^N is an $\mathcal{O}_{\mathcal{S}}$-module and $Z^N(M)$ denotes the N-cycles of M. We let $\mathcal{A}.\alpha : \mathcal{A} \otimes C^N[-N] \rightarrow M$ denote the induced map of $\mathcal{A}$-modules, where $\mathcal{A} \otimes C^N[-N]$ has the obvious $\mathcal{A}$-module structure where $\mathcal{A}$ acts on the left. We let $Cone(\mathcal{A}.\alpha)$ denote the corresponding mapping cone. Observe that this is the $\mathcal{A}$-module defined by

$$Cone(\mathcal{A}.\alpha)^n = \mathcal{A}^{n+1-N} \otimes C^N \oplus M^n \tag{3.1.2}$$

with the differential $\delta(a\otimes c, m^n) = (\delta(a)\otimes c, \delta(m^n) - a.\alpha(c)), \quad m\varepsilon M^n$. Then it follows from the above definition that $Cone(\mathcal{A}.\alpha)^n = Cone(\alpha)^n$ for all $n \geq N-1$. (Recall $Cone(\alpha)^n = M^n, n \neq N-1$ and $= C^N \oplus M^{N-1}$, if $n = N-1$.) Therefore

$$\mathcal{H}^i(Cone(\mathcal{A}.\alpha)) \cong \mathcal{H}^i(Cone(\alpha)), \text{ for all } i \geq N. \tag{3.1.3}$$

In particular if α is such that the composition $C^N \to Z^N(M) \to \mathcal{H}^N(M)$ is an epimorphism, then the long exact sequence in cohomology sheaves for $C^N[-N] \to M \to Cone(\alpha)$ shows that $\mathcal{H}^N(Cone(\alpha)) = 0$. Therefore, $\mathcal{H}^N(Cone(\mathcal{A}.\alpha)) = 0$, i.e. we have killed the cohomology in degree N of the $\mathcal{A}$-module M by attaching the $\mathcal{A}$-cell, $\mathcal{A} \otimes C^N[-N]$. Observe also that $Cone(\mathcal{A}.\alpha)^n = M^n$ for all $n \geq N$, so that

$$\mathcal{H}^n(Cone(\mathcal{A}.\alpha)) \cong \mathcal{H}^n(M), \text{for all } n > N. \tag{3.1.4}$$

Theorem 3.16. *Let M be an $\mathcal{A}$-module with bounded coherent cohomology sheaves. Then there exists an $\mathcal{A}$-module $P(M)$, which is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules together with a quasi-isomorphism $P(M) \to M$.*

Proof. In view of the quasi-coherator discussed in [J2, Appendix B], we may assume without loss of generality that M consists of quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree. Assume that N_0 is an integer so that $\mathcal{H}^i(M) = 0$ for all $i > N_0$. Denoting by $Z^{N_0}(M) = ker(d : M^{N_0} \to M^{N_0+1})$, the map $Z^{N_0}(M) \to \mathcal{H}^{N_0}(M)$ is an epimorphism. Then, there is a coherent $\mathcal{O}_{\mathcal{S}}$-submodule, C^{N_0}, of $Z^{N_0}(M)$, so that the composite map $C^{N_0} \to Z^{N_0}(M) \to \mathcal{H}^{N_0}(M)$ is also an epimorphism. We let $\alpha_{N_0} : C^{N_0}[-N_0] \to M$ denote the corresponding map, where $C^{N_0}[-N_0]$ is the complex of $\mathcal{O}_{\mathcal{S}}$-modules concentrated in degree N where it is C^{N_0}. Next we obtain an induced map of $\mathcal{A}$-modules:

$$\mathcal{A}.\alpha_{N_0} : \mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}} C^{N_0}[-N_0] \to M \tag{3.1.5}$$

so that the composite map $C^{N_0}[-N_0] = \mathcal{O}_{\mathcal{S}} \otimes_{\mathcal{O}_{\mathcal{S}}} C^{N_0}[-N_0] \to \mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}} C^{N_0}[-N_0] \to M$ is the map corresponding to $C^{N_0} \to Z^{N_0}(M) \to M^{N_0}$. By our choice of C^{N_0}, the map $C^{N_0} = H^{N_0}(C^{N_0}[-N_0]) \to H^{N_0}(M)$ is an epimorphism. Therefore, so is the induced map $\mathcal{H}^{N_0}(\mathcal{A}.\alpha_{N_0}) : \mathcal{H}^{N_0}(\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}} C^{N_0}[-N_0]) \to \mathcal{H}^{N_0}(M)$. Observe also that $\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{S}}} C^{N_0}[-N_0]$ is *a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree.* We will denote this complex by $P_{N_0}(M)$ and let $p_{N_0} = \mathcal{A}.\alpha_{N_0}$.

Observe that one has the distinguished triangle $P_{N_0}(M) \overset{p_{N_0}}{\to} M \to Cone(p_{N_0}) \to P_{N_0}(M)[1]$ which results in the long-exact sequence:

$$\cdots \to \mathcal{H}^i(P_{N_0}(M)) \to \mathcal{H}^i(M) \to \mathcal{H}^i(Cone(p_{N_0})) \to \mathcal{H}^{i+1}(P_{N_0}(M)) \to \cdots \tag{3.1.6}$$

Since $\mathcal{H}^{N_0+k}(P_{N_0}(M)) = 0$ for all $k > 0$ and $\mathcal{H}^{N_0}(P_{N_0}(M)) \to \mathcal{H}^{N_0}(M)$ is a surjection by our choice of $P_{N_0}(M)$, it follows that

$$\mathcal{H}^i(Cone(p_{N_0})) = 0, i \geq N_0. \tag{3.1.7}$$

i.e. The map $p_{N_0} : P_{N_0}(M) \to M$ is an N_0-quasi-isomorphism.

3.1.8

We will construct a sequence of complexes $P_k(M)$, $k \leq N_0$, which are $\mathcal{A}$-modules, consisting of coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree, are bounded complexes, trivial in degrees $> N_0$, and are provided with compatible maps $p_k : P_k(M) \to M$ which are k-quasi-isomorphisms, i.e. induce an isomorphism on $\mathcal{H}^i$ for $i > k$ and an epimorphism on $\mathcal{H}^k$. In order to construct these inductively, we will assume that N is an integer for which such a $P_N(M)$ has been already constructed. To start the induction, we may let $N = N_0$ and let $P_N(M)$ denote the complex constructed above. Observe that $Cone(p_N)$ has bounded coherent cohomology sheaves and that $\mathcal{H}^i(Cone(p_N)) = 0$ for all $i \geq N$. Therefore, we will now replace M by $Cone(p_N)$ and find a coherent $\mathcal{O}_{\mathcal{S}}$-submodule C^{N-1} of $Z^{N-1}(Cone(\alpha_N)) = Z^{N-1}(Cone(p_N))$ so that the composite map $C^{N-1} \to Z^{N-1}(Cone(p_N)) \to \mathcal{H}^{N-1}(Cone(p_N))$ is an epimorphism. This provides us with a map $C^{N-1}[-N+1] \to Cone(\alpha_N) \to Cone(\mathcal{A}.\alpha_N) = Cone(p_N) \to P_N(M)[1]$, i.e. a map $\alpha_{N-1} : C^{N-1}[-N] \to P_N(M)$.

We let $P_{N-1}(M) = Cone(\mathcal{A}.\alpha_{N-1})$. We now observe that the induced map $q_{N-1} = \mathcal{A}.\alpha_{N-1} : \mathcal{A} \otimes C^{N-1}[-N] \to P_N(M)$ also factors through $Cone(\mathcal{A}.\alpha_N)[-1] = Cone(p_N)[-1]$, which is the homotopy fiber of the obvious map $p_N : P_N(M) \to M$. This shows that the composition $p_N \circ q_{N-1}$ is chain homotopically trivial. Therefore, one obtains an induced map $p_{N-1} : P_{N-1}(M) = Cone(\mathcal{A}.\alpha_{N-1}) \to M$ making the triangle

$$\begin{array}{ccc} P_N(M) & \xrightarrow{p_N} & M \\ \downarrow & \nearrow_{p_{N-1}} & \\ P_{N-1}(M) = Cone(\mathcal{A}.\alpha_{N-1}) & & \end{array}$$

commute. Observe also that the induced map

$$\mathcal{H}^{N-1}(\mathcal{A} \otimes C^{N-1}[-N+1]) \to \mathcal{H}^{N-1}(Cone(p_N)) \qquad (3.1.9)$$

is an epimorphism by the assumptions on C^{N-1}.

Since $\mathcal{A} \otimes C^{N-1}[-N]$ is the homotopy fiber of the map $P_N(M) \to P_{N-1}(M)$, a comparison of the long exact sequences in cohomology associated to the distinguished triangles $P_N(M) \overset{p_N}{\to} M \to Cone(p_N) \to P_N(M)[1]$ and $P_{N-1}(M) = Cone(q_{N-1}) \overset{p_{N-1}}{\to} M \to Cone(p_{N-1}) \to P_{N-1}(M)[1]$ shows that the homotopy fiber of the induced map $Cone(p_N) \to Cone(p_{N-1})$ identifies with $\mathcal{A}.C^{N-1}[-N+1]$. In view of (3.1.9) and the observation that $H^i(\mathcal{A} \otimes C^{N-1}[-N+1]) = 0$ for all $i > N-1$, it follows that $\mathcal{H}^{N-1}(Cone(p_{N-1})) = 0$. Therefore, $\mathcal{H}^i(p_{N-1})$ is an epimorphism for $i = N-1$. By construction, one may readily see that $\mathcal{H}^i(p_{N-1})$ is an isomorphism for $i \geq N$. Therefore, p_{N-1} is an $N-1$-quasi-isomorphism. By construction $P_{N-1}(M)$ is an $\mathcal{A}$-module which in each degree is a coherent $\mathcal{O}_{\mathcal{S}}$-module and is trivial in degrees $> N_0$.

We may therefore, continue the inductive construction and define $P_k(M)$ as an $\mathcal{A}$-module, consisting of coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree and provided with a map $p_k(M) : P_k(M) \to M$, $k \leq N_0$ which is a k-quasi-isomorphism, i.e. where $\mathcal{H}^i(p_k(M))$ is an isomorphism for $i > k$ and an epimoprhism for $i = k$. Finally one lets $P(M) = \lim_{k \to \infty} P_k(M)$ along with the map $p(M) : P(M) \to M$ defined as $\lim_{k \to \infty} p_k(M)$. One verifies immediately using (3.1.4) that $p(M)$ is a quasi-isomorphism: clearly $P(M)$ is an $\mathcal{A}$-module. Since $\mathcal{A}^i = 0$ for $i > 0$, the construction shows that $P_{k-2}(M)$ and $P_k(M)$ differ only in degrees $< k$. Therefore, it follows that $P(M)$ consists of coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree. Since each $P_k(M)$ is trivial in degrees above N_0, $P(M)$ is bounded above. But, $P(M)$ is possibly unbounded below. However, one may apply Lemma 3.15 to replace $P(M)$ by a quasi-isomorphic $\mathcal{A}$-module which is also bounded below and consists of coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree. This proves the theorem. □

Corollary 3.17. *Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in Definition 3.1(a). Then the following bi-Waldhausen categories define weakly-equivalent K-theory spaces (where the bi-Waldhausen structure is given as follows: weak-equivalences (fibrations, cofibrations) are quasi-isomorphisms of $\mathcal{A}$-modules (degree-wise epimorphisms of $\mathcal{A}$-modules with $\mathcal{A}$-compatible right-inverses, degree-wise monomorphisms of $\mathcal{A}$-modules with $\mathcal{A}$-compatible left-inverses respectively).*

- *perfect $\mathcal{A}$-modules*
- *perfect $\mathcal{A}$-modules that are strict bounded complexes of $\mathcal{O}_{\mathcal{S}}$-modules*
- *perfect and flat $\mathcal{A}$-modules that are bounded above complexes of $\mathcal{O}_{\mathcal{S}}$-modules, where an $\mathcal{A}$-module M is flat if the functor $M \underset{\mathcal{A}}{\otimes} - : Mod(\mathcal{S}, \mathcal{A}) \to Mod(\mathcal{S}, \mathcal{A})$ preserves quasi-isomorphisms*
- *perfect $\mathcal{A}$-modules that are quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules in each degree*
- *perfect $\mathcal{A}$-modules that are bounded complexes of coherent $\mathcal{O}$-modules*

Proof. Recall that $\mathcal{A}$ is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules. The definition of the quasi-coherator as in [J2, Appendix B: 10.0.4] shows that it sends $\mathcal{A}$-modules to quasi-coherent $\mathcal{A}$-modules. This readily proves that the first and fourth categories produce weakly-equivalent K-theory spaces. Next any perfect $\mathcal{A}$-module is coherent as an $\mathcal{A}$-module. Therefore Theorem 3.16 shows that it is quasi-isomorphic to an $\mathcal{A}$-module which consists of coherent $\mathcal{O}$-modules in each degree. These observations readily prove that the the first, second and fifth categories produce weakly-equivalent K-theory spaces. The weak-equivalence of the K-theory spaces produced by the first and third categories is clear in view of the functorial flat resolutions as in [J2, Appendix B]. □

3.2 Behavior under quasi-isomorphisms

In this section we consider briefly the behavior of G-theory of dg-stacks under 1-morphisms that are quasi-isomorphisms in the sense of 3.0.2. Here we may let the dg-structure sheaves be defined as in Definition 3.1(a) or (b).

Proposition 3.18. *Let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks that is a quasi-isomorphism. Then the following hold:*

(i) *$f_* : Mod(\mathcal{S}', \mathcal{A}') \to Mod(\mathcal{S}, \mathcal{A})$ is exact in the sense it preserves quasi-isomorphisms and therefore identifies with Rf_*.*

(ii) *$Lf^* : Mod(\mathcal{S}', \mathcal{A}') \to Mod(\mathcal{S}, \mathcal{A})$ defined by $M \mapsto \mathcal{A}' \overset{L}{\underset{f^{-1}(\mathcal{A})}{\otimes}} f^{-1}(M)$ identifies with f^{-1}.*

(iii) *Moreover the natural transformations $M \to Rf_* f^{-1}(M)$ and $f^{-1}Rf_*(N) \to N$ are quasi-isomorphisms for $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$ and $N \varepsilon Mod(\mathcal{S}', \mathcal{A}')$. Therefore $f_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$ induces a weak-equivalence with inverse defined by $Lf^* = f^{-1}$, i.e. G-theory for dg-stacks depends on the dg-stack only up to quasi-isomorphism.*

Proof. We will make use of the cartesian square in (3.0.3) throughout this proof. Let $M \varepsilon Mod(\mathcal{S}', \mathcal{A}')$. Then $\mathcal{H}^*(M)$ identifies with a sheaf of graded modules over $\mathcal{H}^0(\mathcal{A}')$. Therefore $Rf_*\mathcal{H}^t(M) \simeq Rf_* i'_*(\mathcal{H}^t(M)) \simeq i_* R\bar{f}_*(\mathcal{H}^t(M)) \simeq i_* \bar{f}_*(\mathcal{H}^t(M))$ since $\bar{f}$ is an isomorphism. It follows that the spectral sequence $E_2^{s,t} = R^s f_* \mathcal{H}^t(M) \Rightarrow R^{s+t} f_*(M)$ degenerates providing the identification $Rf_*(M) \simeq f_*(M)$.

Under these hypotheses, the map $f^{-1}(\mathcal{H}^*(\mathcal{A})) \to \mathcal{H}^*(\mathcal{A}')$ identifies with the map $f^{-1}(i_*(\mathcal{H}^*(\mathcal{A}))) = i'_* \bar{f}^{-1}(\mathcal{H}^*(\mathcal{A})) \to i'_* \mathcal{H}^*(\mathcal{A}')$ which is clearly an isomorphism. Therefore, we obtain the identification up to quasi-isomorphisms, for $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$:

$$f^{-1}(\mathcal{A}) \simeq \mathcal{A}',\ Lf^*(M) = \mathcal{A}' \overset{L}{\underset{f^{-1}(\mathcal{A})}{\otimes}} f^{-1}(M) \simeq f^{-1}(M).$$

This follows from a corresponding spectral sequence for Lf^*, $E_{s,t}^2 = Tor_{s,t}^{\mathcal{H}^*(f^{-1}(\mathcal{A}))}(\mathcal{H}^*(\mathcal{A}'), \mathcal{H}^*(f^{-1}(M))) \Rightarrow \mathcal{H}^{s+t}(Lf^*(M))$, which degenerates providing the identification $Lf^*(M) \simeq f^{-1}(M)$.

Under these circumstances, it is straightforward to check that the natural maps $M \to Rf_* f^{-1}(M)$ and $f^{-1}Rf_*(N) \to N$ are quasi-isomorphisms for $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$ and $N \varepsilon Mod(\mathcal{S}', \mathcal{A}')$. Therefore $f_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$ induces a weak-equivalence with inverse defined by $Lf^* = f^{-1}$. These prove the proposition. □

Remark 3.19. It seems unlikely that K-theory is invariant under quasi-isomorphisms of dg-stacks in general.

4 K-Theory of projective space bundles

Throughout this section a dg-stack will mean one in the sense of Definition 3.1(a). Let $(\mathcal{S}, \mathcal{A})$ denote a given dg-stack and let $\mathcal{E}$ denote a vector bundle of rank r on $\mathcal{S}$. Let $\pi : Proj(\mathcal{E}) \to \mathcal{S}$ denote the obvious projection. Let $\mathbb{P} = Proj(\mathcal{E})$ provided with the dg-structure sheaf $\pi^*(\mathcal{A})$. Clearly the functor $\pi^*(\quad) \otimes \mathcal{O}_{\mathbb{P}}(-i)$ defines a functor $Perf(\mathcal{S}, \mathcal{A}) \to Perf(\mathbb{P}, \pi^*(\mathcal{A}))$, for each i. The aim of this section is to prove the following result.

Theorem 4.1. *(Projective space bundle theorem for dg-stacks) Assume the above situation. Then the map*

$$(x_0, x_1, ..., x_{r-1}) \mapsto \Sigma_{i=0}^{r-1} \pi^*(x_i) \otimes [\mathcal{O}_{\mathbb{P}}(-i)]$$

induces a weak-equivalence of K-theory spectra: $\Pi_{i=0}^{r-1} \mathbf{K}(\mathcal{S}, \mathcal{A}) \to \mathbf{K}(\mathbb{P}, \pi^*(\mathcal{A}))$.

The case $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$ follows essentially as in Thomason - see [T] section 4. The proof we provide for the case of dg-stacks below will be an extension of this: as it will become clear in the proof, the definition of perfection for $\mathcal{A}$-modules, we have adopted in 3.0.4 seems appropriate enough for this proof to work.

Proposition 4.2. *(Basic properties of the derived functor $R\pi_*$ (for $\mathcal{O}$-modules))*

The functor $R\pi_* : D_+(Mod(\mathbb{P}, \mathcal{O}_{\mathbb{P}})) \rightarrow D_+(Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$ has the following properties:

(1) $R\pi_*$ preserves perfection. For all integers $q \geq 0$, $R^q\pi_*$ preserves quasi-coherence and coherence. If $M \varepsilon Mod(\mathbb{P}, \mathcal{O}_{\mathbb{P}})$ is cartesian so is each $R^q\pi_*(M)$.

(2) For $q > rank(\mathcal{E})$, $R^q\pi_*(\mathcal{F}) = 0$, where $\mathcal{F}$ is any quasi-coherent sheaf on $Proj(\mathcal{E})$.

(3) For any coherent $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$ on $Proj(\mathcal{E})$, there is an integer $n_0(\mathcal{F}) = n_0$ so that for all $n \geq n_0$ and all $q \geq 1$, $R^q\pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(n)) = 0$.

(4) For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_{\mathbb{P}}$-module on $Proj(\mathcal{E})$ and for $\mathcal{M}$ a complex of quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules, there is a canonical isomorphism:

$R^q\pi_*(\mathcal{F} \otimes \pi^*M) \cong R^q\pi_*(\mathcal{F}) \otimes M.$

(5) For all integers n, there are natural isomorphisms

$$\begin{aligned} R^q\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) &= 0, q \neq 0, r-1 \\ &= S^n(\mathcal{E}), q = 0, \\ R^q\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) &= (S^{-r-n}\mathcal{E})^{\vee} \otimes (\Lambda^r\mathcal{E})^{\vee}, q = r-1, \end{aligned}$$

where $S^k\mathcal{E}$ is the k-th symmetric power of $\mathcal{E}$, considered to be 0 for $k \leq -1$, $\Lambda^r(\mathcal{E})$ is the maximal exterior power of $\mathcal{E}$ and $(\quad)^{\vee}$ sends a vector bundle to its dual, $(\quad)^{\vee} = \mathcal{H}om(\quad, \mathcal{O}_{\mathcal{S}})$.

(6) On $Proj(\mathcal{E})$, there is a canonical map $\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ that is an epimorphism.

(7) The Koszul sequence $0 \rightarrow \pi^*(\Lambda^r\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-r) \rightarrow \pi^*(\Lambda^{r-1}\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(1-r) \rightarrow \cdots \rightarrow \pi^*(\Lambda^2(\mathcal{E})) \otimes \mathcal{O}_{\mathbb{P}}(-2) \rightarrow \pi^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}}(-1) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow 0$ is exact.

(8) The dual sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \pi^*(\mathcal{E}^{\vee}) \otimes \mathcal{O}_{\mathbb{P}}(1) \rightarrow \pi^*(\Lambda^2(\mathcal{E}^{\vee})) \otimes \mathcal{O}_{\mathbb{P}}(2) \rightarrow \cdots \rightarrow \pi^*(\Lambda^r\mathcal{E}^{\vee}) \otimes \mathcal{O}_{\mathbb{P}}(r) \rightarrow 0$ is also exact.

Proof. In order to prove the assertion that $R^q\pi_*$ sends cartesian $\mathcal{O}_{\mathbb{P}}$-modules to cartesian $\mathcal{O}_{\mathcal{S}}$-modules, it suffices to show that for each *smooth* map $\phi : U \rightarrow V$ in $\mathcal{S}_{lis-et}$, $\phi^*(R^q\pi_*(M)_{|V_{et}}) \simeq R^q\pi_*(M)_{|U_{et}}$: see [LM, Chapter 12]. This now follows readily using flat base change. Since the map π is representable, the remaining assertions follow readily from the corresponding assertions proven for schemes as in [TT, 4.5 and 4.6]. ☐

One may define the derived functor $R\pi_* : D_b(Mod(\mathbb{P}, \pi^*(\mathcal{A}))) \rightarrow D_b(Mod(\mathcal{S}, \mathcal{A}))$ making use of the injective resolutions discussed in Appendix B. One defines the derived categories $D_b(Perf(\mathcal{S}, \mathcal{A}))$ ($D_b(Perf(\mathbb{P}, \pi^*(\mathcal{A})))$) by inverting maps that are quasi-isomorphisms in the category $Perf(\mathcal{S}, \mathcal{A})$ ($Perf(\mathbb{P}, \pi^*(\mathcal{A}))$, respectively). (Observe that each perfect complex has bounded cohomology sheaves, which accounts for the subscript b.) Next one observes that $R\pi_*$ induces a functor $R\pi_* : D_b(Perf(\mathbb{P}, \pi^*(\mathcal{A}))) \rightarrow D_b(Perf(\mathcal{S}, \mathcal{A}))$. To see this, first consider $R\pi_*$ applied to a complex of the form $\pi^*(\mathcal{A}) \otimes \mathcal{F}^{\bullet}$, where $\mathcal{F}^{\bullet} \varepsilon D_b(Perf(\mathbb{P}, \mathcal{O}_{\mathbb{P}}))$. Now the projection formula and the observation that $R\pi_*$ sends perfect complexes of $\mathcal{O}_{\mathbb{P}}$-modules to perfect complexes of $\mathcal{O}_{\mathcal{S}}$-modules shows that $R\pi_*(\pi^*(\mathcal{A}) \otimes \mathcal{F}^{\bullet}) \varepsilon D_b(Perf(\mathcal{S}, \mathcal{A}))$. Recall from Definition 3.3 that, in general, an object M in $Perf(\mathbb{P}, \pi^*(\mathcal{A}))$ is quasi-isomorphic to a $\pi^*(\mathcal{A})$-module provided with a finite sequence of maps $F_iM \rightarrow F_{i+1}M$ so that each successive mapping cone and F_0M is of the form $\pi^*(\mathcal{A}) \otimes \mathcal{F}^{\bullet}$ as above. Therefore, it follows that $R\pi_*$ induces a functor $D_b(Perf(\mathbb{P}, \pi^*(\mathcal{A}))) \rightarrow D_b(Perf(\mathcal{S}, \mathcal{A}))$. Let $IPerf(\mathbb{P}, \pi^*(\mathcal{A}))$ ($IPerf(\mathcal{S}, \mathcal{A})$) denote the Waldhausen category of perfect $\pi^*(\mathcal{A})$ -modules consisting of injective $\mathcal{O}_{\mathbb{P}}$-modules in each degree (perfect $\mathcal{A}$-modules consisting of injective $\mathcal{O}_{\mathcal{S}}$-modules in each degree, respectively). Here the fibrations and cofibrations are defined as in Definition 3.5. Then it is easy to see that π_* induces an exact functor $\pi_* : IPerf(\mathbb{P}, \pi^*(\mathcal{A})) \rightarrow IPerf(\mathcal{S}, \mathcal{A})$.

Next we recall the notion of regularity defined by Mumford : see [M, Lecture 14] and [Q, Section 8]. Recall that a quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is said to be *m-regular* in the sense of Mumford, if $R^q\pi_*(\mathcal{F}(m-q)) = 0$ for all $q \geq 1$. Observe that, if $\mathcal{F}$ is m-regular, then $\mathcal{F}(n)$ is $(m-n)$-regular. Moreover, if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_{\mathbb{P}}$-modules on $\mathbb{P}$, then there exists an integer m_0 so that $\mathcal{F}$ is n-regular for all $n \geq m_0$. *We will say a complex of $\mathcal{O}_{\mathbb{P}}$-modules $M^\bullet$ is m-regular if each M^i is m-regular as before.* A $\pi^*(\mathcal{A})$-module M will be called *m-regular* if it is a complex of quasi-coherent sheaves of $\mathcal{O}_{\mathbb{P}}$-modules which is m-regular in the above sense.

Proposition 4.3. *(See [TT] 4.7.1.) Let $0 \to F' \to F \overset{\beta}{\to} F'' \to 0$ denote an exact sequence of quasi-coherent $\mathcal{O}_{\mathbb{P}}$-modules. Then the following hold:*

(i) If F is m-regular, $F(k)$ is $m-k$ regular.

(ii) If F' and F'' are n-regular, then so is F.

(iii) If F is n-regular and F' is $(n+1)$-regular, then F'' is n-regular.

(iv) If F is $n+1$-regular and F'' is n-regular and if $\pi_*F(n) \to \pi_*F''(n)$ is an epimorphism, then F' is $n+1$-regular.

(v) Given any object $F \varepsilon Perf(\mathbb{P}, \pi^*(\mathcal{O}_{\mathcal{S}}))$ that is a strictly bounded complex of coherent $\mathcal{O}$-modules, there exists an integer n so that F is n-regular.

(vi) If F is n regular, then F is m-regular for all $m \geq n$.

(vii) The product map $\pi_*(F(k)) \otimes \mathcal{E} = \pi_*(F(k)) \otimes \pi_*(\mathcal{O}_{\mathbb{P}}(1)) \to \pi_*(F(k+1))$ is an epimorphism for each $k \geq m$ if F is m-regular.

(viii) $\pi^*\pi_*(F(k)) \to F(k)$ is an epimorphism for each $k \geq m$, if F is m-regular.

(ix) If F is 0-regular, then $R^q\pi_*(F) = 0$ for all $q \geq 1$.

(x) If $M \varepsilon Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, then $\pi^*(M)$ is 0-regular.

Proof. (i) is clear from the definition while (ii), (iii) and (iv) may be proved readily using the long-exact sequence obtained on applying the derived functors $\{R^q\pi_* | q \geq 0\}$. Our definition of regularity above along with the observation that any coherent sheaf is n-regular for some $n >> 0$ along with (vi) (proven below) readily proves (v). (vi) may proven using ascending induction on $m-n$, making use of the exact sequence in Proposition 4.2(7): see [Q, section 8]. It suffices to show this for $m = n+1$. We tensor the exact sequence in Proposition 4.2(7) with $F(n)$ to obtain

$$0 \to \pi^*(\Lambda^r\mathcal{E}) \otimes F(n-r) \to \pi^*(\Lambda^{r-1}\mathcal{E}) \otimes F(n+1-r) \to \cdots \to \pi^*(\Lambda^2(\mathcal{E})) \otimes F(n-2) \to \pi^*(\mathcal{E}) \otimes F(n-1) \to F(n) \to 0.$$

Then break this into short-exact sequences: $0 \to Z_p \to \pi^*(\Lambda^p\mathcal{E}) \otimes F(n-p) \to Z_{p-1} \to 0$ where the Z_p are the kernels of the maps in the long-exact sequence above. Now an application of the projection formula and the hypothesis that F is $n-$regular, show that $\pi^*(\Lambda^p\mathcal{E}) \otimes F$ is $n-$ regular. Therefore, $\pi^*(\Lambda^p\mathcal{E}) \otimes F(n-p)$ is p-regular. By using descending induction on p, starting with the $r+1$-regular object $0 = Z_r$ and applying Proposition 4.3(iii) to the short-exact sequence above, one sees that Z_{p-1} is $p-$regular. In particular, $Z_0 = F(n)$ is $1-$regular, i.e. F is $n+1$-regular. This completes the proof of (vi).

Clearly the maps in (vii) and (viii) are natural and hence extend to algebraic stacks. To check that these maps are epimorphisms, one may reduce to schemes where these two assertions are proven in [TT, 4.7.2 Lemma]. To prove (ix), we use (vi) to conclude that F is 0-regular implies F is 1-regular and hence that $R^1\pi_*F = R^1\pi_*F(1-1) = 0$. Similarly F is 0-regular implies that it is m-regular for all $m \geq 1$, and hence (with $m = q$) that $R^q\pi_*(F(0)) = R^q\pi_*(F(q-q)) = 0$ for all $q \geq 1$.

To prove (x), observe that by Proposition 4.2(4), with $F = \mathcal{O}_{\mathbb{P}}(-q)$, $R^q(\pi_*(\pi^*(M))(-q)) = R^q\pi_*(\mathcal{O}_{\mathbb{P}}(-q) \otimes \pi^*(M)) = R^q\pi_*(\mathcal{O}_{\mathbb{P}}(-q)) \otimes M$. By Proposition 4.2(5), it suffices to consider the case $q = r-1$ if it is positive. In this case we may let n in Proposition 4.2(5) $= -q$ so that $n = -r+1 > -r$ so that $-r-n < 0$ and hence $R^q\pi_*(\mathcal{O}_{\mathbb{P}}(-q)) = 0$ thereby proving (x). ∎

Given a $\pi^*(\mathcal{A})$-module M, we let $M(s) = M \underset{\mathcal{O}_{\mathbb{P}}}{\otimes} \mathcal{O}_{\mathbb{P}}(s)$. Observe this identifies with $M \underset{\pi^*(\mathcal{A})}{\otimes} \pi^*(\mathcal{A})(s)$.

Quillen (see [Q] section 8.1.11) defined certain functors $T_n : Mod(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \to Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ that send coherent sheaves to coherent sheaves, in the case of schemes. Thomason showed the existence of corresponding functors for perfect complexes of $\mathcal{O}$-modules. We will show the same general strategy works to prove the projective space bundle theorem for dg-stacks. First we recall the definition of these functors, which are defined using ascending induction on n making use of certain auxiliary functors denoted Z_n. We let $Z_{-1} = id : Perf(\mathbb{P}, \mathcal{O}) \to Perf(\mathbb{P}, \mathcal{O})$. Having defined the functor Z_{n-1}, we will let T_n be defined by $T_n(M) = \pi_*(Z_{n-1}(M)(n))$ and let $Z_n(M) = kernel(\pi^*T_n(M)(-n) \to Z_{n-1}(M))$ where the map $\pi^*T_n(M)(-n) \to Z_{n-1}(M)$ is the obvious map. Clearly the Z_n and T_n are additive functors and they preserve coherence. The following result is an extension to algebraic stacks and dg-stacks of results proven in [TT] 4.8.4 and 4.8.5 for schemes:

Proposition 4.4. *(i) There exist functors T_i: (0-regular $\mathcal{O}_{\mathbb{P}}$-modules) $\to$ ($O_{\mathcal{S}}$-modules), $i = 0, \cdots, r-1$. For every 0-regular coherent sheaf F on $\mathbb{P}$ there is an exact sequence:*

$$0 \to \mathcal{O}(-r+1) \otimes \pi^*T_{r-1}(F) \to \cdots \to \mathcal{O} \otimes \pi^*T_0(F) \to F \to 0.$$

The functors T_i, $i = 0, \cdots, r-1$ are exact in the sense they preserve quasi-isomorphism between complexes of $\mathcal{O}_{\mathbb{P}}$-modules that are 0-regular. They also send short exact sequences of 0-regular $\mathcal{O}_{\mathbb{P}}$-modules to short exact sequences of $\mathcal{O}_{\mathcal{S}}$-modules.

(ii) If $F^\bullet$ is a bounded complex of 0-regular coherent sheaves on $\mathbb{P}$, there is an exact sequence of complexes for each $k = 0, \cdots, r-1$(i.e. it is exact for each fixed F^i):

$$0 \to T_k(F^\bullet) \to \mathcal{E} \otimes T_{k-1}(F^\bullet) \to S^2\mathcal{E} \otimes T_{k-2}(F^\bullet) \to \cdots \to S^k\mathcal{E} \otimes T_0(F^\bullet) \to \pi_*(F^\bullet) \to 0.$$

(iii) Each of the functors Z_n sends $\pi^*(\mathcal{A})$-modules to $\pi^*(\mathcal{A})$-modules. Similarly each of the functors T_n sends $\pi^*(\mathcal{A})$-modules to $\mathcal{A}$-modules.

(iv) If M is an $\mathcal{A}$-module and K is an $\mathcal{O}_{\mathcal{S}}$-module, then $M \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} K$ is an $\mathcal{A}$-module. Similarly, if M is a $\pi^*(\mathcal{A})$-module and K is an $\mathcal{O}_{\mathbb{P}}$-module, then $M \underset{\mathcal{O}_{\mathbb{P}}}{\otimes} K$ is a $\pi^*(\mathcal{A})$-module.

Proof. The first step is to prove inductively that if F is a 0-regular coherent $\mathcal{O}_{\mathbb{P}}$-module on $\mathbb{P}$, then $Z_{n-1}(F)(n)$ is also 0-regular, or equivalently that $Z_{n-1}(F)$ is n-regular. This will be proven using ascending induction on n, the case $n = 0$ being clear since $Z_{-1} = id$. The inductive step follows from the exact sequence:

$$0 \to (Z_n(F))(n) \to \pi^*T_n(F) \to Z_{n-1}(F)(n) \to 0 \tag{4.0.1}$$

whenever F is a 0-regular coherent $\mathcal{O}_{\mathbb{P}}$-module on $\mathbb{P}$. The surjectivity follows from the assertion (viii) in Proposition 4.3 using the assumption that $Z_{n-1}(F)$ is n-regular. The exactness of the above sequence at other places follows readily from the definition of $Z_n(F)$. One may also readily observe that $\pi_*\pi^*T_n(F) \to \pi_*Z_{n-1}(F)(n)$ is an epimorphism (in fact an isomorphism since $\pi_*\pi^* = id$) so that Proposition 4.3(iv) applies to prove $Z_n(F)(n)$ is 1-regular, or equivalently that $Z_n(F)(n+1)$ is 0-regular. This completes the inductive step and shows that $Z_{n-1}(F)(n)$ is 0-regular for all $n \geq 0$ whenever F is 0-regular. Now Proposition 4.3(ix) shows that $R\pi_*(Z_{n-1}(F)(n)) = \pi_*(Z_{n-1}(F)(n))$ if F is 0-regular and hence that T_n is an exact functor on 0-regular coherent $\mathcal{O}_{\mathbb{P}}$-modules on $\mathbb{P}$.

The exact sequence in (i) is obtained by tensoring the exact sequences in (4.0.1) with $\mathcal{O}_{\mathbb{P}}(-n)$ and splicing together. The exact sequence in (ii) is obtained by tensoring the exact sequence in (i) with $\mathcal{O}_{\mathbb{P}}(k)$, $0 \leq k \leq r-1$ and applying π_*. Therefore the maps in the complex in (ii) are defined for algebraic stacks; that it is exact follows now by working locally on the stack and hence reducing to the case of schemes.

Next we consider (iii). Clearly Z_{-1} being the identity functor sends $\pi^*(\mathcal{A})$-modules to $\pi^*(\mathcal{A})$-modules. Now the definition of $T_0 = \pi_*$ shows it sends $\pi^*(\mathcal{A})$-modules to $\mathcal{A}$-modules. One may now complete the proof of the third assertion by ascending induction on n.

The last assertion is clear since $\mathcal{A}$ is a sheaf of commutative dgas. $\square$

Remark 4.5. Assume $M\varepsilon Mod(\mathbb{P}, \pi^*(\mathcal{A}))$ which is a complex of coherent 0-regular $\mathcal{O}_{\mathbb{P}}$-modules on $\mathbb{P}$. Then the exact sequence in (i) in the last Proposition provides a quasi-isomorphism

$$\{\mathcal{O}(-r+1) \otimes \pi^* T_{r-1}(M) \to \cdots \to \mathcal{O} \otimes \pi^* T_0(M)\} \xrightarrow{\simeq} M$$

of $\pi^*(\mathcal{A})$-modules.

Let $\mathbf{A}$ denote the category of perfect $\pi^*(\mathcal{A})$-modules which are strict bounded complexes of 0-regular coherent $\mathcal{O}_{\mathbb{P}}$-modules on $\mathbb{P}$.

Let $\mathbf{B}$ denote the category of perfect $\pi^*(\mathcal{A})$-modules which are strict bounded complexes of coherent $\mathcal{O}_{\mathbb{P}}$-modules on $\mathbb{P}$. We provide both these categories with the structure of bi-Waldhausen categories in the obvious manner: weak-equivalences are weak-equivalences in $Perf(\mathbb{P}, \pi^*(\mathcal{A}))$ and cofibrations (fibrations) are those maps whose cokernel (kernel, respectively) also belongs to the same sub-category. There is an obvious inclusion functor $I : \mathbf{A} \to \mathbf{B}$ of Waldhausen categories.

Proposition 4.6. *The functor I induces a weak-equivalence on K-theory spectra. More specifically, for every $B\varepsilon\mathbf{B}$, there is an $A\varepsilon\mathbf{A}$ and a quasi-isomorphism (of $\pi^*(\mathcal{A})$-modules) $B \xrightarrow{\simeq} A$.*

Proof. Let $B\varepsilon\mathbf{B}$. By Proposition 4.3(v), we may assume that there is an integer n so that B is n-regular. If $n < 0$, then B is 0-regular by Proposition 4.3(vi). Next suppose $n > 0$. In this case we will use ascending induction on n. To do the induction step, we may assume the result is known for $B\varepsilon\mathbf{B}$ that is $(n-1)$-regular. Then, for any $k \geq 1$, $B(k)$ is $n-1$-regular by Proposition 4.3(i) and (vi). Tensoring the (locally split) exact Koszul sequence in Proposition 4.2(8) with B gives an exact sequence of complexes. One may view this as a quasi-isomorphism of B with the rest of the complex, i.e.

$$B \to Tot(\pi^*(\mathcal{E})^\vee \otimes B(1) \to \cdots \to \pi^*(\overset{r}{\wedge}\mathcal{E}^\vee) \otimes B(r)) = B'$$

Now we observe: (i) the complex on the right is $n-1$-regular, (ii) the maps there are maps of $\pi^*(\mathcal{A})$-modules, so that the Total complex produces a $\pi^*(\mathcal{A})$-module. (This follows from the following observation. Let the complex of $\mathcal{O}_{\mathbb{P}}$-modules provided by all-but the starting term of the exact Koszul sequence in Proposition 4.2(8) be denoted M. Then we took the double complex $B \underset{\mathcal{O}_S}{\otimes} M$, and then its total complex. The formation of the total complex, Tot is associative, so that we obtain induced pairings: $Tot(\pi^*(\mathcal{A}) \underset{\mathcal{O}_S}{\otimes} Tot(B \underset{\mathcal{O}_S}{\otimes} M)) \cong Tot(Tot(\pi^*(\mathcal{A}) \underset{\mathcal{O}_S}{\otimes} B) \underset{\mathcal{O}_S}{\otimes} M) \to Tot(B \underset{\mathcal{O}_S}{\otimes} M)$, where the last map is obtained using the observation that B is a $\pi^*(\mathcal{A})$-module.) By the same observations the map from B to the Total complex of this complex is also a map of $\pi^*(\mathcal{A})$-modules. Now we apply the inductive hypothesis, to conclude there is a complex $A\varepsilon\mathbf{A}$ and a quasi-isomorphism $B' \xrightarrow{\simeq} A$ of $\pi^*(\mathcal{A})$-modules . Composing this with the quasi-isomorphism $B \xrightarrow{\simeq} B'$ provides the required quasi-isomorphism. $\square$

Proposition 4.7. *The functors T_k, for $k = 0, \cdots, r-1$, preserve perfection for $\pi^*(\mathcal{A})$-modules that are 0-regular.*

Proof. These statements are proved using ascending induction on k. Since $T_0 = \pi_*$, we need to first show π_* sends perfect $\pi^*(\mathcal{A})$-modules that are 0-regular to perfect $\mathcal{A}$-modules. Let $F_0 M \to \tilde{F}_0 M$ denote a quasi-isomorphism to a complex of 0-regular $\mathcal{A}$-modules chosen as above. Let P_1 denote the canonical homotopy pushout defined by the square:

$$\begin{array}{ccc} F_0 M & \longrightarrow & F_1 M \\ \downarrow & & \downarrow \\ \tilde{F}_0 M & \longrightarrow & P_1 \end{array}$$

Since the maps in the top row and left column are maps of $\pi^*(\mathcal{A})$-modules, the induced maps forming the other sides of the square are also maps of $\pi^*(\mathcal{A})$-modules. Now we let $P_1 \to \tilde{F}_1 M$ denote a quasi-isomorphism to a $\pi^*(\mathcal{A})$-module which is 0-regular. There is an induced map from $Cone(F_0 M \to F_1 M) \to Cone(\tilde{F}_0 M \to \tilde{F}_1 M)$: this is a quasi-isomorphism. Moreover, the definition of the mapping cone shows that the complex

$Cone(\tilde{F}_0M \to \tilde{F}_1M)$ is also 0-regular. Therefore $\pi_*(Cone(\tilde{F}_0M \to \tilde{F}_1M)) \simeq R\pi_*(Cone(\tilde{F}_0M \to \tilde{F}_1M)) \simeq R\pi_*(Cone(F_0M \to F_1M)) \cong R\pi_*(\pi^*(\mathcal{A})\overset{L}{\underset{\mathcal{O}}{\otimes}}Q_1) \simeq \mathcal{A}\overset{L}{\underset{\mathcal{O}_\mathcal{S}}{\otimes}} R\pi_*(Q_1)$ for some perfect complex of $\mathcal{O}_\mathbb{P}$-modules Q_1. Similarly, $\pi_*(\tilde{F}_0M) \simeq R\pi_*(F_0M) \simeq R\pi_*((\pi^*(\mathcal{A})\overset{L}{\underset{\mathcal{O}}{\otimes}}Q_0) \simeq \mathcal{A}\overset{L}{\underset{\mathcal{O}_\mathcal{S}}{\otimes}} R\pi_*(Q_0)$ for some perfect complex of $\mathcal{O}_\mathbb{P}$-modules Q_0. Since $R\pi_*$ sends perfect complexes of $\mathcal{O}_\mathbb{P}$-modules to perfect complexes of $O_\mathcal{S}$-modules, it follows that $\pi_*(\tilde{F}_1M)$ is perfect.

Now one may repeat the above construction to obtain a commutative diagram:

$$\begin{array}{ccccccc}
F_0M & \longrightarrow & F_1M & \longrightarrow \cdots \longrightarrow & F_{n-1}M & \longrightarrow & F_nM = M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{F}_0M & \longrightarrow & \tilde{F}_1M & \longrightarrow \cdots \longrightarrow & \tilde{F}_{n-1}M & \longrightarrow & \tilde{F}_nM
\end{array}$$

where the vertical maps are all quasi-isomorphisms of $\pi^*(\mathcal{A})$-modules and the bottom row consists of $\pi^*(\mathcal{A})$-modules that are 0-regular. Since M itself is 0-regular, the induced map $\pi_*(M) \to \pi_*(\tilde{F}_nM)$ is a quasi-isomorphism. Since each $\tilde{F}_iM$ is 0-regular, π_* applied to the bottom row provides distinguished triangles $\pi_*(\tilde{F}_{i-1}M) \to \pi_*(\tilde{F}_iM) \to \pi_*(Cone(\tilde{F}_{i-1}M \to \tilde{F}_iM))$, $i = 1, \cdots, r$. Moreover, the argument for the case $i = 1$ above shows that $\pi_*(Cone(\tilde{F}_{i-1}M \to \tilde{F}_iM)) \simeq R\pi * (\pi^*(\mathcal{A})\overset{L}{\underset{\mathcal{O}_\mathbb{P}}{\otimes}}Q_i) \cong \mathcal{A}\overset{L}{\underset{\mathcal{O}_\mathcal{S}}{\otimes}} R\pi_*(Q_i)$ for some perfect complex of $\mathcal{O}_\mathcal{S}$-modules Q_i. These arguments, therefore show that $T_0 = \pi_*$ sends perfect $\pi^*(\mathcal{A})$-modules that are 0-regular to perfect $\mathcal{A}$-modules.

To prove the corresponding assertion for the higher T_is, we will use ascending induction on n and the following exact-sequence considered in Proposition 4.4(ii):

$$0 \to T_k(M) \to \mathcal{E} \otimes T_{k-1}(M) \to S^2\mathcal{E} \otimes T_{k-2}(M) \to \cdots \to S^k\mathcal{E} \otimes T_0(M) \to \pi_*(M) \to 0. \tag{4.0.2}$$

This sequence exists for all $0 \leq k \leq r-1$. Assuming we have proved that T_0,... T_{k-1} preserve perfection, we may show T_k preserves perfection as follows. Recall from Proposition (3.7)(iii) that if two terms in a distinguished triangle are perfect, so is the remaining term. Therefore, it suffices to show the image of the map $\mathcal{E} \otimes T_{k-1}(M) \to S^2\mathcal{E} \otimes T_{k-2}(M)$ is perfect. Since the sequence in (4.0.2) is exact, one may identify this with the kernel of the map $S^2\mathcal{E} \otimes T_{k-2}(M) \to S^3\mathcal{E} \otimes T_{k-3}(M)$. By Proposition (3.7), it again suffices to show that the image of the same map is perfect. One may now identify this with the kernel of the next map in the exact sequence (4.0.2) and continue. Eventually we reduce to the fact that both $S^k\mathcal{E} \otimes T_0(M)$ and $\pi_*(M) = T_0(M)$ are perfect. □

With the above propositions at our disposal, the proof in [TT] 4.9- 4.11 may be modified as follows to complete the proof of theorem 4.1.

Let **C** denote the category of perfect $\mathcal{A}$-modules which are strict bounded complexes of coherent $\mathcal{O}_\mathcal{S}$-modules provided with the obvious bi-Waldhausen structure. We have established the following results in the above discussion:

- There is an obvious inclusion functor $I : \mathbf{A} \to \mathbf{B}$ of bi-Waldhausen categories.
- There are functors $T_k : \mathbf{A} \to \mathbf{C}$ of bi-Waldhausen categories, $k = 0, 1, ..., r-1$.
- There are functors of bi-Waldhausen categories $\mathcal{O}(-k) \otimes \pi^*(\quad) : \mathbf{C} \to \mathbf{B}$ for $k0, 1, ..., r-1$.
- There is a natural quasi-isomorphism in **B**

$$I(M) \overset{\simeq}{\leftarrow} Tot(\mathcal{O}(-r+1) \otimes \pi^*T_{r-1}(M) \to \cdots \to \mathcal{O} \otimes \pi^*T_0(M)), \quad M \varepsilon \mathbf{A}. \tag{4.0.3}$$

Now the proof of the theorem reduces to showing that the functor

$$\oplus_{k=0}^{r-1}\mathcal{O}(-k) \otimes \pi^*(\quad) : \Pi_{i=0}^{r-1}\mathbf{C} \to \mathbf{B} \tag{4.0.4}$$

induces a weak-equivalence on K-theory spectra. In fact we will show it induces a map that is a split monomorphism and a split epimorphism in the homotopy category of spectra.

Lemma 4.8. *The map in (4.0.4) is a split monomorphism up to homotopy.*

Proof. Here the proof in [TT] 4.10 applies verbatim. The key observations are contained Proposition 4.2. However, here are some details for the sake of completeness. The formulae in Proposition 4.2 show that $R\pi_*(\mathcal{O}(n-k) \otimes \pi^* M) = 0$ for all $0 \leq n < k \leq r-1$ and that $R\pi_*(\mathcal{O} \otimes \pi^* M) = R\pi_* \pi^*(M) = M$ for any quasi-coherent complex of $\mathcal{O}$-modules M on $\mathcal{S}$. Now consider the map $K(\mathbf{B}) \to \Pi_{i=0}^{r-1} K(\mathbf{C})$ induced by $M \to (R\pi_* M, R\pi_* M(1), \cdots, R\pi_* M(r-1))$. Pre-composing this map with the map $\Pi_{i=0}^{r-1} K(\mathbf{C}) \to K(\mathbf{B})$ induced by the map in (4.0.4), we obtain an endomorphism of $\Pi_{i=0}^{r-1} K(\mathbf{C})$ represented by an $(r \times r)$-matrix of maps $\Pi_{i=0}^{r-1} K(\mathbf{C}) \to \Pi_{i=0}^{r-1} K(\mathbf{C})$ which has 1s along the diagonal and 0s above the diagonal. Therefore this matrix is invertible and therefore the composite endomorphism of $\Pi_{i=0}^{r-1} K(\mathbf{C})$ is a weak-equivalence. This proves the lemma. □

To show the map $\Pi_{i=0}^{r-1} K(\mathbf{C}) \to K(\mathbf{B})$ induced by the map in (4.0.4) is a split epimorphism on taking homotopy groups, we use the functors T_ns. By (4.0.3) above, The map $I : K(\mathbf{A}) \to K(\mathbf{B})$ is homotopic to the map induced by the functor $M \mapsto I(M) = Tot(\mathcal{O}(-r+1) \otimes \pi^* T_{r-1}(M) \to \cdots \to \mathcal{O} \otimes \pi^* T_0(M))$. Using the additivity theorem in K-theory, one sees that this map is homotopic to the map $\Sigma_{k=0}^{r-1} (-1)^k \mathcal{O}(-k) \otimes \pi^* T_k$. This shows the map induced by the functor I factors through the map on K-theory induced by the map in (4.0.4) via the map $K(\mathbf{A}) \to \Pi_{i=0}^{r-1} K(\mathbf{C})$ given by $(K(T_0), -K(T_1), \cdots, (-1)^{r-1} K(T_{r-1}))$.

Recall that we have already showed the map induced by the functor I is a weak-equivalence on K-theory spectra: see Proposition 4.6. Therefore, this proves the map $\Pi_{i=0}^{r-1} K(\mathbf{C}) \to K(\mathbf{B})$ induced by the map in (4.0.4) is a split epimorphism on taking homotopy groups. This completes the proof of Theorem 4.1.

5 G-theory of dg-stacks: devissage, localization and homotopy property

In this section, we will extend all the basic results on G-theory, for e.g. devissage, localization and homotopy property, to the G-theory of algebraic dg-stacks.

5.1 The basic situation

Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in Definition 3.1(a) or (b). We will need to consider three distinct situations in this context, for all of which we will prove a devissage theorem. To simplify the discussion, we will consider these situations together. In all three cases we have a sheaf of graded Noetherian rings $R = \underset{n \geq 0}{\oplus} R_n$ on $\mathcal{S}_{lis-et}$ with $R_0 = \mathcal{O}_{\mathcal{S}}$ and each R_n a coherent $\mathcal{O}_{\mathcal{S}}$-module. We will let n be called the *weight* of R_n.

(1) The *first case* is when $R_i = 0$ for all $i > 0$.

(2) The *second case* is when $R = \mathcal{O}_{\mathcal{S}}[t]$, with t an indeterminate. In this case each $R_i = \mathcal{O}_{\mathcal{S}} t^i$.

(3) To state the *third case* we proceed as follows. Let $\mathcal{W}$ denote the graded ring R as in (2). We let $\mathcal{W}$ be provided with the filtration $F_n(\mathcal{W}) = \{p(t) | deg(p(t)) \leq n, p(t) \varepsilon \mathcal{O}_{\mathcal{S}}[t]\}$. Now we *let R in the third case* denote the graded ring defined by $R = \oplus_{n \geq 0} R_n$, where $R_n = F_n(\mathcal{W}) z^n$ and z is another indeterminate.

Let $B = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} R$. Now B has a natural grading induced by the grading on R, so that $B = \underset{n \geq 0}{\oplus} B_n$. One may observe that B is a sheaf of graded dgas, i.e. if $B_n = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} R_n$, then there is an induced pairing $B_n \otimes B_m \to B_{n+m}$ of chain complexes that is associative. Moreover since $\mathcal{A}$ was assumed to be a sheaf of commutative dgas, and $U(R)$ is strictly commutative in all of the above cases, $U(B) = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} U(R)$ is a commutative dga with the degree 0 part, $B^0 =$ the sheaf of graded rings R. (Here U denotes the functor forgetting the gradation. The superscript on B is induced from the superscript on $\mathcal{A}$, which denotes the part of the complex in degree 0.)

5.1.1 Notation

In the above situation, since R (and therefore, B) have a non-trivial gradation only in the last two cases, we will distinguish these two by denoting the graded objects there in boldface. i.e. Henceforth $\mathbf{R} = \underset{n\geq 0}{\oplus} \mathbf{R}_n$ and $\mathbf{B} = \underset{n\geq 0}{\oplus} \mathbf{B}_n$ will denote the objects considered in the last two situations in 5.1.

Definition 5.1. A *graded B-module* M will mean a $U(B)$-module M, so that the following hypotheses are satisfied: M is provided with a gradation so that $M = \underset{n\geq 0}{\oplus} M_n$, (i) each M_n is an $\mathcal{A}$-module so that the underlying complex of $\mathcal{O}_{\mathcal{S}}$-module has bounded quasi-coherent cohomology uniformly in n (i.e. M has bounded cohomology and each $\mathcal{H}^i(M_n)$ is quasi-coherent) and (ii) the above gradation on M is compatible with a graded B-module structure. i.e. The multiplication $B \otimes M \to M$ sends $B_n \otimes M_m \to M_{n+m}$. (iii) Moreover we require that each M_n has *cartesian* cohomology sheaves. We will let n be called the *weight* of M_n.

The category of all such graded modules over B will be denoted $\mathrm{QCoh}_{gr}(\mathcal{S}, B)$: a morphism $f : M' = \oplus_n M'_n \to M = \oplus_n M_n$ of such graded modules means that (i) f preserves the gradation and induces a map $M'_n \to M_n$ of $\mathcal{A}$-modules for each $n \geq 0$ and that (ii) f is compatible with the graded multiplications $B_n \otimes M'_m \to M'_{n+m}$ and $B_n \otimes M_m \to M_{n+m}$. In this situation we will let

$$Hom_{gr-B}(M', M) = \{f \varepsilon Hom_{U(B)}(U(M'), U(M)) \mid f(M'_n) \subseteq M_n, \text{ for all } n\} \tag{5.1.2}$$

This is the external Hom in the category $\mathrm{QCoh}_{gr}(\mathcal{S}, B)$ while $Hom_{U(B)}(U(M'), U(M))$ denotes the external hom in the category of $U(B)$-modules.

For a graded B-module M as above, we let $F_p(M)$ denote the graded B-submodule generated by the M_n, $0 \leq n \leq p$.

Definition 5.2. We will say a graded B-module is *coherent* if (i) each $\mathcal{H}^*(M_n)$ is a finitely generated $\mathcal{H}^*(\mathcal{A})$-module, which is in fact bounded and *cartesian* and (ii) the inclusion $F_p(M) \to M$ induces a quasi-isomorphism of $U(B)$-modules for some $p \geq 0$.

When $\mathcal{A}$ is a dga as in Definition 3.1(a), then we may adopt a stronger definition where a graded B-module is coherent if (i) each M_n is a coherent $\mathcal{A}$-module, which is in fact bounded (see Theorem 3.16), with cartesian cohomology sheaves and (ii) the inclusion $F_p(M) \to M$ induces a quasi-isomorphism of $U(B)$-modules for some $p \geq 0$.

When $\mathcal{A}$ is a dga as in Definition 3.1(a), we obtain the following graded version of Theorem 3.16 which may be proved in exactly the same manner with the dga $\mathcal{A}$ replaced by the graded dga $\mathbf{B}$.

Proposition 5.3. *Assume the dga $\mathcal{A}$ is as in Definition 3.1(a) and assume the situation in 5.1.1. Let M be a complex of graded $\mathbf{B}$-modules with bounded cohomology sheaves that are coherent as graded modules over the graded ring $\mathbf{R}$. Then there exists a graded $\mathbf{B}$-module $P(M)$, so that $P(M)$ is a bounded complex which in each degree is coherent as a graded module over the graded ring $\mathbf{R}$ together with a quasi-isomorphism $P(M) \to M$ of graded $\mathbf{B}$-modules.*

The sub-category of coherent B-modules will be denoted $\mathrm{Coh}_{gr}(\mathcal{S}, B)$. The category $\mathrm{Coh}_{gr}(\mathcal{S}, B))$ has the obvious structure of a bi-Waldhausen category with weak-equivalences being maps of complexes of graded $\mathbf{B}$-modules that are quasi-isomorphisms and cofibrations (fibrations) $f : M \to N$ being maps of complexes of graded B-modules so that $U(f)$ is a monomorphism with a $U(B)$-compatible left-inverse (epimorphism with a $U(B)$-compatible right inverse, respectively in the sense of Definition 3.5). The corresponding K-theory spectrum will be denoted $\mathbf{K}(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})))$.

Example 5.4. Assume $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$. Now $B = R$. We may assume first that the B-module M is a single graded R-module concentrated in degree 0 (as a complex). In this case the M is coherent as a B-module if and only if each M_n is a coherent $\mathcal{O}_{\mathcal{S}}$-module which is cartesian and the inclusion $F_p(M) \to M$ is an isomorphism for some $n \geq 0$. Therefore, in this case, our notion of coherent B-modules agrees with the usual notion of coherent graded B-modules as in [Q, section 6] adapted to algebraic stacks. Next one may consider the situation where $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$ still, but the B-module M is a bounded complex of graded $\mathcal{R}$-modules. In this case M is coherent as a graded B-module if and only if each M_n is a bounded complex of $\mathcal{O}_{\mathcal{S}}$-modules with coherent cartesian cohomology sheaves and the obvious inclusion $F_p(M) \to M$ is a quasi-isomorphism for some $p \geq 0$.

Remarks 5.5. For the most part all our graded rings and modules will be graded by non-negative integers. The only exception to this appears in the proof of Theorem 5.17, where we need to consider graded rings of the form $C[z, z^{-1}]$ which is the ring of Laurent series over a ring or dga C and graded by powers of z. In this case the graded modules over such a ring will also be graded by $\mathbb{Z}$: however, we do not consider the K-theory or the G-theory of such graded modules.

Assume the situation of (5.1.2). Then we define

$$\mathcal{H}om_B(M', M)_n = \{f \varepsilon \mathcal{H}om_{U(B)}(U(M'), U(M)) | f(M'_p) \subseteq M_{n+p}, \text{ for all } p\} \tag{5.1.3}$$

where $\mathcal{H}om_{U(B)}(\quad,\quad)$ denotes the sheaf-hom for $U(B)$-modules. This defines a graded object $\mathcal{H}om_B(M', M) = \oplus_{n\varepsilon\mathbb{Z}}\mathcal{H}om_B(M', M)_n$. Next assume $B = R$, i.e. let $\mathcal{A} = R_0$. Then we make the following observations (that follow readily from [N-O, 2.4.5 and 2.4.7 Corollaries]):

- Suppose M' is a complex concentrated in one degree (i.e. when M' is viewed as a complex, it is given by $\bar{M}'[n]$ for a coherent graded R-module $\bar{M}'$ in the usual sense). Then M' is coherent as a graded R-module if and only if $U(M')$ is coherent as a $U(R)$-module.
- If M' is a coherent graded R-module in the above sense, then $\mathcal{H}om_R(M', M) = \mathcal{H}om_{U(R)}(U(M'), U(M))$. A corresponding identification upto quasi-isomorphism holds for the the derived functors of the above $\mathcal{H}om$-functors.
- If M' is a graded R-bi-module, (i.e. $U(M')$ is a $U(R)$-bi-module with the left and right multiplication by R_n taking M'_m to M'_{n+m}), then $\mathcal{H}om_R(M', M)$ has the structure of a graded module over R.
- $\mathcal{H}om_R(R/I, M)_n = 0$ for all $n < 0$. (This is because the unit $1\varepsilon R_0$ and therefore if f sends 1 to $m\varepsilon M_m$, f will send R_p to M_{m+p}. By our assumption we consider only graded modules M for which $M_q = 0$ for $q < 0$, so that m above must be non-negative.)

5.2 Coherent sheaves with supports in a sub-dg-stack and devissage

All through this subsection, except in Proposition 5.12 and Corollary 5.13, we will assume that the dga $\mathcal{A}$ is as in Definition 3.1(b). There are two distinct situations that fit into this context. The *first* is when $R = \mathcal{O}_{\mathcal{S}}$ with the trivial gradation, i.e. $R_i = \mathcal{O}_{\mathcal{S}}$ if $i = 0$ and $= 0$ if $i > 0$. Then $B_i = \mathcal{A}$ if $i = 0$ and $= 0$ if $i \neq 0$. *We will assume that in this case, each $M_n = 0$ for $n > 0$ as well, so that in this case, we will be considering simply $\mathcal{A}$-modules.* In this case we let $\mathcal{I}$ denote a sheaf of ideals in $\mathcal{O}_{\mathcal{S}}$ and $\mathcal{S}'$ will denote the corresponding closed algebraic sub-stack. We let $\mathcal{A}' = i^*(\mathcal{A}) = \mathcal{O}/\mathcal{I} \underset{\mathcal{O}}{\otimes} \mathcal{A}$. *We will need to assume that the closed immersion i and the dga $\mathcal{A}$ are such that $\mathcal{A}'$ is a dga satisfying the hypotheses in Definition 3.1(b) on the stack $\mathcal{S}'$.* (For example, this is always true if $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, in which case $i^*(\mathcal{A}) = \mathcal{O}_{\mathcal{S}'}$ for any closed immersion i.)

We say that a bounded complex of sheaves F of $\mathcal{A}$-modules on $\mathcal{S}$ has supports in $\mathcal{S}'$, if the cohomology sheaves, $\mathcal{H}^*(F)$, have supports in $\mathcal{S}'$. We let $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ denote the K-theory of the Waldhausen category of bounded complexes of $\mathcal{A}$-modules with cohomology sheaves that have supports in $\mathcal{S}'$. This situation will then enable us to prove the devissage weak-equivalence: $G(\mathcal{S}') \simeq G_{\mathcal{S}'}(\mathcal{S})$ as in Corollary 5.11.

The *second* situation is where $\mathbf{R}$ is the sheaf of graded rings defined above in 5.1(3). In this case $\mathcal{I}$ will denote the principal ideal (z). This situation will be considered in Theorem 5.17 and will enable us to prove the homotopy property for G-theory of dg-stacks. However, Theorem 5.17 will still need to invoke Theorem 5.10, but in this new context. Therefore, in order to avoid having to prove two different versions of the results in Lemma 5.6 through Theorem 5.10, we state the results up to Corollary 5.11 in a general setting so that they are applicable in both contexts. In particular, a graded B-module will mean one in the first sense through Corollary 5.11 and will mean one in the second sense thereafter in Proposition 5.15 and Theorem 5.17: in fact, as pointed out above, we will use $\mathbf{B}$ to denote B in this context so that the difference in meaning should be clear.

We will also consider the *second case* in 5.1 in the proof of Proposition 5.15: here too $\mathbf{B}$ will be used to denote B.

Lemma 5.6. *Let R denote one of the sheaves of graded rings on $\mathcal{S}_{lis-et}$ considered in 5.1. Let I denote a sheaf of ideals in R as above, i.e. either a sheaf of ideals in $R_0 = \mathcal{O}_{\mathcal{S}}$ or the ideal (z). Let $F\varepsilon \mathrm{QCoh}_{gr}(\mathcal{S}, \mathcal{R})$ so that all $\mathcal{H}^n(F)$ are killed by I^k for some fixed integer k. Then the natural map $\mathcal{RH}om_R(R/I^k, F) \to F$ is a quasi-isomorphism, i.e. $\mathcal{E}xt^n_{\mathcal{R}}(R/I^k, F) \cong \mathcal{H}^n(F)$ for all n. Moreover for any $F\varepsilon \mathrm{QCoh}_{gr}(\mathcal{S}, R)$ with bounded cohomology sheaves that are each killed by some power of I, there exists an integer k for which the above isomorphism holds for all n.*

Proof. Recall that

$$\mathcal{H}om_R(M,N) = Equalizer(\ \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(M,N) \underset{n_*}{\overset{m^*}{\rightrightarrows}} \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(R \otimes M, N)\ )$$

where m^* and n_* are the obvious maps induced by $\lambda_M : R \otimes M \to M$ and $\lambda_N : R \otimes N \to N$ which are the given module structures.

Working locally on $\mathcal{S}_{lis-et}$ we reduce to the case where the stack is an affine Noetherian scheme. By our hypothesis, R, R/I^k are quasi-coherent and F has quasi-coherent cohomology sheaves. Therefore, we reduce to proving the corresponding assertion where R is a graded ring, I a graded ideal in R and F is a complex of R-modules whose cohomology groups are killed by some power k of the ideal I. Now one has the spectral sequence

$$E_2^{s,t} = \mathcal{E}xt^s_R(R/I^k, \mathcal{H}^t(F)) \Rightarrow \mathcal{E}xt^{s+t}_R(R/I^k, F)$$

with $E_2^{s,t} = 0$ for all $s > 0$ and $\cong \mathcal{H}^t(F)$ for $s = 0$.

See, for example, [Hart, proof of Theorem 2.8]. Therefore, $\mathcal{E}xt^n_R(R/I^k, F) \cong \mathcal{H}om_R(R/I^k, \mathcal{H}^n(F))$ for all n. Since $\mathcal{H}^*(F)$ has bounded cohomology sheaves and $\mathcal{S}$ is quasi-compact, the last statement is clear. This proves the lemma. ☐

Corollary 5.7. *Let $F\varepsilon \mathrm{QCoh}_{gr}(\mathcal{S}, B)$ and let I denote a sheaf of ideals as in the Lemma. Assume that all $\mathcal{H}^n(F)$ are killed by I^k for some fixed integer k. Then the natural map $\mathcal{RH}om_B(B/I^k, F) \to F$ is a quasi-isomorphism.*

Proof. In view of the last lemma, it suffices to observe the isomorphism

$$\mathcal{RH}om_B(B/I^k, F) = \mathcal{RH}om_B(B \underset{R}{\otimes} R/I^k, F) \cong \mathcal{RH}om_R(R/I^k, For(F))$$

where $For : Mod(\mathcal{S}, B) \to Mod(\mathcal{S}, R)$ is the obvious forgetful functor. The $\mathcal{RH}om$s denote the derived functors of the internal $\mathcal{H}om$ for graded modules. Here the derived functor $\mathcal{RH}om_B$ is defined using the injective resolutions in Appendix B. The last identification follows from the observation that since the dga $\mathcal{A}$ is assumed to be flat over $R_0 = \mathcal{O}_{\mathcal{S}}$, B is flat over R and therefore, the forgetful functor $For : Mod(\mathcal{S}, B) \to Mod(\mathcal{S}, R)$ preserves injectives. (It is precisely here that we need to use the hypothesis that $\mathcal{A}$ is flat over $\mathcal{O}_{\mathcal{S}}$.) ☐

Remark 5.8. One can interpret the conclusions of the above corollary in the two distinct cases considered above. Observe that when one considers the *first* situation above, then the last corollary shows the following: let F denote a bounded complex of $\mathcal{A}$-modules with coherent cohomology sheaves that have supports in $\mathcal{S}'$. Then the obvious map $\lim_{k\to\infty} \mathcal{RH}om_{\mathcal{A}}(\mathcal{A}/\mathcal{I}^k, F) \to F$ is a quasi-isomorphism. In the second situation, the last corollary shows the following instead: let $F\varepsilon \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})$ so that the cohomology sheaves of F are killed by some power of $z\varepsilon \mathcal{W}[z]$. Then the obvious map $\lim_{k\to\infty} \mathcal{RH}om_{\mathbf{B}}(\mathbf{B}/I^k, F) \to F$ is a quasi-isomorphism.

For each integer $k \geq 0$, let $\mathrm{Coh}_{gr}(\mathcal{S}, B/I^k)$ denote the full sub-category of $\mathrm{Coh}_{gr}(\mathcal{S}, B)$ which are killed by I^k. This category gets the obvious induced structure of a Waldhausen category from the one on $\mathrm{Coh}_{gr}(\mathcal{S}, B)$. Let $\mathrm{Coh}_{gr,B/I^k}(\mathcal{S}, B)$ denote the full sub-category of $\mathrm{Coh}_{gr}(\mathcal{S}, B)$ of complexes whose cohomology sheaves are killed by I^k.

Lemma 5.9. *The obvious map $K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^k)) \to K(\mathrm{Coh}_{gr,B/I^k}(\mathcal{S}, B))$ is a weak-equivalence for all k.*

Proof. For each fixed k, clearly there is a natural functor $\mathrm{Coh}_{gr}(\mathcal{S}, B/I^k) \to \mathrm{Coh}_{gr,B/I^k}(\mathcal{S}, B))$. It suffices to show this induces a weak-equivalence on K-theory spaces.

Let $\mathrm{ICoh}_{gr,B/I^k}(\mathcal{S}, B)$ denote the full sub-category of $\mathrm{Coh}_{gr,\mathcal{B}/I^k}(\mathcal{S}, B)$ consisting of complexes of graded injective B-modules. Similarly, let $\mathrm{ICoh}_{gr}(\mathcal{S}, B/I^k)$ denote the full sub-category of $\mathrm{Coh}_{gr}(\mathcal{S}, B/I^k)$ consisting of complexes of graded *injective B/I^k-modules.* Now $\mathcal{RH}om_B(B/I^k, \quad)$ may be replaced by the functor $\mathcal{H}om_B(B/I^k, \quad)$ on $\mathrm{ICoh}_{gr,B/I^k}(\mathcal{S}, B)$ so that it is strictly functorial. The last corollary shows that, for any $F\varepsilon\mathrm{ICoh}_{gr,B/I^k}(\mathcal{S}, B)$, the natural map $\mathcal{H}om_B(B/I^k, F) \to F$ is a quasi-isomorphism. (One may also readily see that the functor $\mathcal{H}om_B(B/I^k, \quad)$ preserves cofibrations and fibrations. Therefore, the Waldhausen approximation theorem (see Theorem 7.2) applies readily to the obvious inclusion functor

$$\mathrm{ICoh}_{gr}(\mathcal{S}, B/I^k) \to \mathrm{ICoh}_{gr,B/I^k}(\mathcal{S}, B)$$

to prove it provides a weak-equivalence on taking the associated K-theory spaces. This proves the lemma. □

Theorem 5.10. *(Devissage) Assume the above situation. Then the obvious inclusion*

$$\mathrm{Coh}_{gr}(\mathcal{S}, B/I) \to \lim_{k\to\infty} \mathrm{Coh}_{gr}(\mathcal{S}, B/I^k))$$

of Waldhausen categories induces a weak-equivalence on taking the associated K-theory spaces.

Proof. Now we will fix an integer $k_0 > 0$ and consider the functor

$$F_{k_0} = \mathcal{H}om_B(B/I^{k_0}, \quad) : \mathrm{ICoh}_{gr}(\mathcal{S}, B/I^{k_0}) \to \mathrm{ICoh}_{gr}(\mathcal{S}, B/I^{k_0}).$$

Clearly, the above functor induces the identity on the associated derived categories. One may now observe that, the functors $F_j = \mathcal{H}om_B(B/I^j, \quad)$, $1 \le j \le k_0$ define functors $\mathrm{ICoh}_{gr}(\mathcal{S}, B/I^{k_0}) \to \mathrm{ICoh}_{gr}(\mathcal{S}, B/I^{k_0})$; since they preserve weak-equivalences as well as cofibrations and fibrations, they induce maps of the corresponding K-theory spaces. Moreover one has a distinguished triangle $F_{i-1}(M) \to F_i(M) \to F_i(M)/F_{i-1}(M) = \mathcal{H}om_B(I^{i-1}/I^i, M)$, $M\varepsilon\mathrm{ICoh}_{gr}(\mathcal{S}, B/I^{k_0})$. One may in fact replace the above distinguished triangle with another where the middle term is the mapping cylinder of $F_{i-1}(M) \to F_i(M)$ and the third term is its mapping cone. Therefore, the map $F_{i-1}(M) \to F_i(M)$ will be a cofibration.

One may show that each F_jM has bounded cohomology sheaves that are cartesian as follows. Let For : $\mathrm{Coh}_{gr}(\mathcal{S}, B/I^j) \to \mathrm{Coh}_{gr}(\mathcal{S}, R/I^j)$ denote the obvious forgetful functor. Then one observes that $For \circ \mathcal{RH}om_B(B/I^j, \quad) = \mathcal{RH}om_R(R/I^j, \quad) \circ For$ and therefore, it suffices to prove that $\bar{F}_jM = \mathcal{RH}om_R(R/I^j, For(M))$ has bounded cohomology sheaves that are cartesian. Given a graded R-module M, one may filter it by $M_k = {}_{I^k}M$ = all elements in M killed by I^k. In case $j \ge k$, $\bar{F}_j(M_k) \simeq M_k$. One may utilize the distinguished triangle $\bar{F}_j(M_{k-1}) \to \bar{F}_j(M_k) \to \bar{F}_j(M_k/M_{k-1}) \to \bar{F}_j(M_{k-1})[1]$ and ascending induction on k to prove $\bar{F}_j(M)$ has bounded cohomology for all j. Recall that to prove the cohomology sheaves of $\bar{F}_j(M)$ are all cartesian, it suffices to show the map $\phi^*(\bar{F}_j(M)_{|U_{et}})) \to \bar{F}_j(M)_{|V_{et}}$ is a quasi-isomorphism for each smooth map $\phi : V \to U$ of affine schemes in $\mathcal{S}_{lis-et}$. We do this by considering the two cases in 5.2 separately. In the first case $R = \mathcal{O}_\mathcal{S}$ and I is a sheaf of ideal defining a closed sub-stack $\mathcal{S}'$ in $\mathcal{S}$. Since ϕ is smooth, one reduces now to showing that $\phi^*(\mathcal{RH}om(\mathcal{O}_\mathcal{S}/I^j, M)_{|U_{et}}) \to \mathcal{RH}om(O_\mathcal{S}/I^j, M)_{|V_{et}}$ is a quasi-isomorphism if M is a coherent $\mathcal{O}_\mathcal{S}$-module which is cartesian. By taking a resolution of $(O_\mathcal{S}/I^j)_{|U_{et}}$ by a complex of locally free $\mathcal{O}_U$-modules, one readily shows that $\phi^*(\mathcal{RH}om(\mathcal{O}_\mathcal{S}/I^j, M)_{|U_{et}}) \simeq \mathcal{RH}om(\phi^*((\mathcal{O}_\mathcal{S}/I^j)_{|U_{et}}), \phi^*(M_{|U_{et}})) \simeq \mathcal{RH}om((\mathcal{O}_\mathcal{S}/I^j)_{|V_{et}}, M_{|V_{et}}) = \mathcal{RH}om(\mathcal{O}_\mathcal{S}/I^j, M)_{|V_{et}}$. In the second case R is the graded ring $\oplus_{n\ge 0}F_n(\mathcal{W})z^n$ and $I = (z)$, so one may use the resolution $0 \to R \xrightarrow{z^j} R \to R/(z^j) \to 0$ and the resulting distinguished triangle on applying $\mathcal{RH}om_R(\quad, M)$ to reduce to the statement $\phi^*(M_{|U_{et}}) \simeq M_{|V_{et}}$.

By additivity (see Theorem 7.4 in the appendix), it follows that the identity map of $K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0}))$ factors as

$$\Sigma_i F_i/F_{i-1} : K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0})) \to K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I))$$

followed by the obvious map of the latter into $K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0}))$. Moreover, the composition $K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I)) \to K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0})) \to K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I))$, where the first map is induced by the obvious inclusion $\mathrm{Coh}_{gr}(\mathcal{S}, B/I) \to \mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0})$ and the last map is $\Sigma_i F_i/F_{i-1}$, is the identity. It follows, therefore, that the obvious map $K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I)) \to K(\mathrm{Coh}_{gr}(\mathcal{S}, B/I^{k_0}))$ is a weak-equivalence. Taking the direct limit as $k_0 \to \infty$, one obtains the required weak-equivalence. $\square$

Corollary 5.11. *Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in Definition 3.1(b) and let $i : \mathcal{S}' \to \mathcal{S}$ denote the closed immersion of an algebraic sub-stack so that $\mathcal{A}' = i^*(\mathcal{A})$ is a dga on $\mathcal{S}'$ satisfying the hypotheses in Definition 3.1(b). Then the obvious map $G(\mathcal{S}', \mathcal{A}') = K(\mathrm{Coh}(\mathcal{S}', \mathcal{A}')) \to K(\mathrm{Coh}_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})) = G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ is a weak-equivalence where $\mathrm{Coh}_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ denotes the full sub-category of $\mathrm{Coh}(\mathcal{S}, \mathcal{A})$ of complexes whose cohomology sheaves have supports in $\mathcal{S}'$. In particular, the obvious map $G(\mathcal{S}') = K(\mathrm{Coh}(\mathcal{S}')) \to K(\mathrm{Coh}_{\mathcal{S}'}(\mathcal{S})) = G_{\mathcal{S}'}(\mathcal{S})$ is a weak-equivalence.*

Proof. Here we let $B = \mathcal{A}$ with the trivial gradation, i.e., $B_i = \mathcal{A}$ if $i = 0$ and $= 0$ if $i > 0$. Since, by our convention, a graded module $M = \oplus_{n \geq 0} M_n$ in this case means one with $M_n = 0$ for all $n > 0$, the category $\mathrm{Coh}_{gr}(\mathcal{S}, B/I)$ identifies with $\mathrm{Coh}(\mathcal{S}', \mathcal{A}')$ while $\mathrm{Coh}_{gr, B/I^k}(\mathcal{S}, B)$ identifies with $\mathrm{Coh}_{\mathcal{A}/I^k}(\mathcal{S}, \mathcal{A})$. Moreover the category $\lim\limits_{k \to \infty} \mathrm{Coh}_{gr, B/I^k}(\mathcal{S}, B)$ identifies with $\mathrm{Coh}_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$. $\square$

Proposition 5.12. *Let $\phi : (S, \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a quasi-isomorphism of dg-stacks as in Definition 3.1(a) or (b) with the same underlying stack. Let R denote a sheaf of graded rings as in 5.1 and let $B' = \mathcal{A}' \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} R$, $B = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} R$ and let I denote a sheaf of ideals in R as in 5.2. Then the obvious maps $K(Coh_{gr}(\mathcal{S}, B')) \to K(Coh_{gr}(\mathcal{S}, B))$, $K(Coh_{gr, B'/I^k}(\mathcal{S}, B')) \to K(Coh_{gr, B/I^k}(\mathcal{S}, B)$ and therefore the induced map*

$$K(\lim_{k \to \infty} Coh_{gr, B'/I^k}(\mathcal{S}, B')) \to K(\lim_{k \to \infty} Coh_{gr, B/I^k}(\mathcal{S}, B)$$

are all weak-equivalences.

Proof. Recall that the functor $\phi_* : Coh_{gr}(\mathcal{S}, B') \to Coh_{gr}(\mathcal{S}, B)$ is just sending a graded B'-module to a graded B-module using the map $B \to \phi_*(B')$. This clearly sends graded modules whose cohomology sheaves are killed by a power of I to graded modules killed by the same power of I. The inverse image functor $L\phi^*$ also does the same. It was already observed earlier that the natural maps $M \to \phi_* L\phi^*(M)$ and $L\phi^* \phi_*(N) \to N$ are quasi-isomorphisms. These observations prove the proposition. $\square$

Corollary 5.13. *Let $\phi : (S, \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a quasi-isomorphism of dg-stacks as in Definition 3.1(a) or (b) with the same underlying stack. Let I denote a sheaf of ideals in $\mathcal{O}_{\mathcal{S}}$ defining a closed substack $\mathcal{S}'$. Then the obvious maps $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A}') \to G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ and $G(\mathcal{S}, \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$ are weak-equivalences.*

Proof. This follows readily from the last proposition by taking the sheaf of graded rings R to be trivially graded as in 5.1(1) with $R_i = 0$ for $i \neq 0$ and $R_0 = \mathcal{O}_{\mathcal{S}}$. $\square$

5.3 Localization

Theorem 5.14. *(Localization for G-theory) Let $i : \mathcal{S}' \to \mathcal{S}$ denote a closed immersion of algebraic stacks with open complement $j : \mathcal{S}'' \to \mathcal{S}$, where $(\mathcal{S}, \mathcal{A})$ is a dg-stack as in Definition 3.1(a).*

(i) Then one obtains the fibration sequence $\Omega G(\mathcal{S}'', \mathcal{A}'') \to G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}'', \mathcal{A}'')$ where $\mathcal{A}'' = j^*(\mathcal{A})$.

(ii) Let $\tilde{\mathcal{A}} \to \mathcal{A}$ denote a flat resolution as in 3.0.1 and assume that the closed immersion i is such that $i^*(\tilde{\mathcal{A}})$ satisfies the hypotheses of a dga as in Definition 3.1(b). Then one also obtains the weak-equivalence: $G(\mathcal{S}', i^*(\tilde{\mathcal{A}})) \simeq G_{\mathcal{S}'}(\mathcal{S}, \tilde{\mathcal{A}}) \simeq G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$.

Proof. (i) follows from Waldhausen's localization theorem (see Theorem 7.3). In more detail, one lets v denote the category of weak-equivalences on $\mathrm{Coh}(\mathcal{S}, \mathcal{A})$ defined by quasi-isomorphisms, while one lets w denote the coarser category of weak-equivalences on $\mathrm{Coh}(\mathcal{S}, \mathcal{A})$ given by maps of complexes that are quasi-isomorphisms after restriction to $\mathcal{S}''$. In order to apply the approximation theorem of Waldhausen (see [Wald, Theorem 1.6.7]) to produce a weak-equivalence $K(\mathrm{Coh}(\mathcal{S}, \mathcal{A}), w) \simeq G(\mathcal{S}'', \mathcal{A}'')$, it suffices to show the following: any map $\alpha : j^*(F) \to F''$, $F \varepsilon \mathrm{Coh}(\mathcal{S}, \mathcal{A})$, $F'' \varepsilon \mathrm{Coh}(S'', \mathcal{A}'')$, may be factored as the composition of $j^*(c) : j^*(F) \to j^*(\tilde{F})$ and a quasi-isomorphism $j^*(\tilde{F}) \to F''$. One may show this as follows.

First we make use of Theorem 3.16 to assume that both F and F'' are bounded complexes consisting of coherent $\mathcal{O}$-modules in each degree. Let For denote the forgetful functor sending an $\mathcal{A}$-module to the corresponding underlying complex of $\mathcal{O}$-modules. Then $For(j_*(F'')) = j_*(For(F'')) = \underset{\alpha}{\mathrm{colim}} N_\alpha$ where each N_α is a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$ sub-modules of $j_*(For(F''))$. Since F'' is bounded, we may assume it is trivial outside a finite interval $[a,b]$, $a, b \varepsilon \mathbb{Z}$. Therefore, each $N^i_\alpha = 0$, for $i \notin [a,b]$. Since each N_α is a subcomplex of $j_*(For(F''))$, it follows that there exists an α_0 so that $For(F'') = j^*j_*(For(F'')) = j^*(N_{\alpha_0})$.

For each $N^i_{\alpha_0}$, let $\mathcal{A}.N^i_{\alpha_0}$ denote the $\mathcal{A}$-submodule of $j_*(F'')$ generated by $N^i_{\alpha_0}$. Clearly this is a coherent $\mathcal{A}$-module. Now let $\tilde{F} = \boxplus_{i=a}^b \mathcal{A}.N^i_{\alpha_0}$ denote the $\mathcal{A}$ submodule of $j_*(F'')$ generated by $\mathcal{A}.N^i_{\alpha_0}$, $i = a, \cdots, b$. Clearly this is also a coherent $\mathcal{A}$-submodule of $j_*(F'')$. Since $\mathcal{A}^0.N^i_{\alpha_0} = \mathcal{O}_{\mathcal{S}}.N^i_{\alpha_0} = N^i_{\alpha_0}$ and $j^*(N_{\alpha_0}) = j^*j_*(For(F'')) = For(F'')$, it follows that $j^*(\tilde{F}) = j^*j_*(F'') = F''$. If $\alpha' : F \to j_*(F'')$ is the adjoint of the map α, then one may replace $\tilde{F}$ by the $\mathcal{A}$-submodule generated by $\tilde{F}$ and $Im(\alpha')$, and assume that $Im(\alpha') \subseteq \tilde{F}$. These arguments provide the weak-equivalence $K(\mathrm{Coh}(\mathcal{S}, \mathcal{A}), w) \simeq G(\mathcal{S}'', \mathcal{A}'')$. Therefore, the localization theorem of Waldhausen (see Theorem 7.3 in Appendix A) then provides the fibration sequence $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}'', \mathcal{A}'')$. Finally Corollary 5.11 provides the weak-equivalence $G(\mathcal{S}', i^*(\tilde{\mathcal{A}})) \simeq G_{\mathcal{S}'}(\mathcal{S}, \tilde{\mathcal{A}})$ while Corollary 5.13 provides the weak-equivalence $G_{\mathcal{S}'}(\mathcal{S}, \tilde{\mathcal{A}}) \simeq G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$. This completes the proof. □

5.4 Homotopy Property

Next we consider the homotopy property for G-theory. This will follow by suitable modifications of Quillen's arguments. We need to invoke Proposition 5.15 in two different contexts during the course of the proof of Theorem 5.17. Therefore we will state it in rather general terms. We will let $\mathbf{R} = \underset{n \geq 0}{\oplus} \mathbf{R}_n$ denote a graded ring so that

$$\begin{aligned} &\mathbf{R}_0 = \mathcal{O}_{\mathcal{S}}, \text{ where } U(\mathbf{R}) \text{ is commutative,} \\ &U(\mathbf{R}) \text{ is flat over } \mathbf{R}_0 \text{ and} \\ &\mathbf{R}_0 \text{ has finite tor dimension over } U(\mathbf{R}). \end{aligned} \tag{5.4.1}$$

Clearly these hypotheses are satisfied in the second and third situations in 5.1. Observe that, in the second situation in 5.1, $U(\mathbf{R}) = \mathcal{O}_{\mathcal{S}}[t]$ and that $\mathbf{R}_0 = \mathcal{O}_{\mathcal{S}}$, with $\mathbf{R}_i = \mathcal{O}_{\mathcal{S}} t^i$. Therefore, $U(\mathbf{R})$ is free over $\mathbf{R}_0$ and $\mathbf{R}_0$ has tor dimension 1 as a $U(\mathbf{R})$-module. In the third situation in 5.1, $\mathbf{R}_n = F_n(\mathcal{W}) z^n$, and $F_n(\mathcal{W}) = \{p(t) | deg(p(t)) \leq n, p(t) \varepsilon \mathcal{O}_{\mathcal{S}}[t]\}$, so that $U(\mathbf{R}) = \mathcal{O}_S[t][z]$. Clearly this is flat over $\mathbf{R}_0 = \mathcal{O}_{\mathcal{S}}$ and $\mathbf{R}_0$ has finite tor dimension over $U(\mathbf{R})$.

Proposition 5.15. *Assume in addition to the hypotheses in* (5.4.1) *that* $\mathbf{B} = \mathbf{R} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{A}$. *Then one obtains an isomorphism:* $\mathbb{Z}[t] \underset{\mathbb{Z}}{\otimes} \pi_*(G(\mathcal{S}, \mathcal{A})) \to \pi_*(K(Coh_{gr}(\mathcal{S}, \mathbf{B})))$.

Proof. Recall $\mathbf{B} = \mathbf{R} \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{A}$. Therefore if $F(\mathbf{R}_0) \to \mathbf{R}_0$ is a resolution of $\mathbf{R}_0$ by graded flat $\mathbf{R}$-modules, $\mathbf{B}_0 \overset{L}{\underset{U(\mathbf{B})}{\otimes}} M \simeq (F(\mathbf{R}_0) \otimes \mathcal{A}) \underset{(U(\mathbf{R}) \otimes \mathcal{A})}{\otimes} M \simeq F(\mathbf{R}_0) \underset{U(\mathbf{R})}{\otimes} M$ for any $M \varepsilon \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})$.

Let $N = \oplus_{n \geq 0} N_n \varepsilon \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})$. Then each N_n is an $\mathcal{A}$-module. Since $U(\mathbf{B})$ is a commutative dga with $\mathbf{B}_0 = \mathcal{A}$, it follows the complex of graded $\mathbf{R}$-submodules of N generated by N_n identifies with the graded $\mathbf{B}$-submodule of N generated by N_n. Let $F_p(N)$ denote the $\mathbf{B}$-submodule of N generated by $N_0, \cdots, N_p$ and let $T_0 : \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B}) \to Mod(\mathcal{S}, \mathcal{A})$ denote the functor defined by $T_0(N) = \mathbf{B}_0 \underset{U(\mathbf{B})}{\otimes} N \cong \mathbf{R}_0 \underset{U(\mathbf{R})}{\otimes} N$. (This is graded by $T_0(N)_p = N_p / (\mathbf{B}_1 N_{p-1} + \cdots + \mathbf{B}_p N_0)$.) The last isomorphism shows one may identify $T_0 \circ For : \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B}) \to \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{R}) \to \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{R}_0)$ where the last functor is given by $N \mapsto \mathbf{R}_0 \underset{U(\mathbf{R})}{\otimes} N$ with the composite functor $For \circ T_0 : \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B}) \to \mathrm{Coh}_{gr}(\mathcal{S}, \mathcal{A}) \to \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{R}_0)$. (Here For denotes the obvious forgetful functors.)

Now [Q, Lemma 1, section 6], shows that if $Tor_1^{U(\mathbf{R})}(\mathbf{R}_0, U(N)) = 0$ for $N \varepsilon \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{R})$, the natural map $\mathbf{R}(-p) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} T_0(N)_p \to F_p(N)/F_{p-1}(N)$ is an isomorphism of graded $\mathbf{R}$-modules. (Here $\mathbf{R}(-p)_n = \mathbf{R}_{n-p}$ and $F_i(N)$ denotes the graded sub- $\mathbf{R}$-module generated by $N_0, \cdots, N_i$. We also let $\mathbf{B}(-p) = \mathbf{R}(-p) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \mathcal{A}$.) In case $N \varepsilon \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})$, the natural map $\mathbf{B}(-p) \underset{\mathcal{A}}{\otimes} T_0(N)_p \cong \mathbf{R}(-p) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} T_0(N)_p \to F_p(N)/F_{p-1}(N)$ is seen to be a map of graded $\mathbf{B}$-modules. In view of the above observations, if $Tor_1^{U(\mathbf{R})}(\mathbf{R}_0, U(N^m)) = 0$ for all m, then the above map will be an isomorphism of graded $\mathbf{B}$-modules.

Next let

$$\mathbf{N} = \{N\varepsilon\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})|Tor_i^{U(\mathbf{R})}(\mathbf{R}_0, U(N^m)) = 0, \text{ for all } i > 0 \text{ and all } m\}.$$

This full sub-category inherits the structure of a bi-Waldhausen category with cofibrations, fibrations and weak-equivalence. Let $M\varepsilon\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})$. We will show that one can find a quasi-isomorphism $F(M) \to M$ of graded $\mathbf{B}$-modules with each term in the complex $F(M)$ a flat graded $\mathbf{R}$-module and with $F(M)$ a bounded above complex. Let $\tilde{F}(M)_0 \to M$ denote a quasi-isomorphism of complexes of graded $\mathbf{R}$-modules with each term of the complex $\tilde{F}(M)_0$ a flat graded $\mathbf{R}$-module. Now the obvious map $d_{-1} : \mathcal{A} \otimes \tilde{F}(M)_0 \to M$ induces a surjection on taking cohomology sheaves. Clearly $F(M)_0 = \mathcal{A} \otimes \tilde{F}(M)_0$ is a *flat* graded-$\mathbf{B}$-module, in the sense that $F(M)_0 \underset{\mathbf{B}}{\otimes} -$ preserves quasi-isomorphisms in the second argument. Let $K_0 = Ker(d_{-1})$. This is clearly a graded $\mathbf{B}$-module and one may repeat the same construction with M replaced by K_0 to define $F(M)_1$ together with a map $F(M)_1 \to K_0$ which will induce a surjective map on cohomology sheaves. Let d_0 denote the composite map $F(M)_1 \to K_0 \to F(M)_0$. Now one may readily see that the composition $d_0 \circ d_{-1} = 0$. One may, therefore repeat the above construction to obtain a sequence of graded $\mathbf{B}$-modules $\{F(M)_n | n \geq 0\}$ together with maps $d_n : F(M)_n \to F(M)_{n-1}$ so that the composition $d_i \circ d_{i-1} = 0$ for all $i \geq 0$. i.e. $\{d_n : F(M)_n \to F(M)_{n-1} | n\}$ is a complex. Since each $F(M)_n$ is also a complex of $\mathbf{R}_0$-modules, this is in fact a double complex of $\mathbf{R}_0$-modules. In view of our basic assumptions in Definition 5.2 we may assume M is bounded as a graded $\mathbf{B}$-module. One may therefore take the total complex of the above complex to obtain a flat $\mathbf{B}$-module $F(M)$ together with a quasi-isomorphism $F(M) \to M$. Moreover, by the construction, each term in $F(M)$ is a flat graded $\mathbf{R}$-module.

The finite tor dimension hypothesis on $\mathbf{R}_0$ as a $U(\mathbf{R})$-module along with the hypothesis that M has bounded cohomology show that $\mathbf{R}_0 \underset{U(\mathbf{R})}{\otimes} U(F(M)) = \mathbf{R}_0 \overset{L}{\underset{U(\mathbf{R})}{\otimes}} U(M)$ has bounded cohomology sheaves. Let $\phi : V \to U$ denote a map between two affine schemes in $\mathcal{S}_{lis-et}$. Then ϕ^* commutes with tensor-products so that $\phi^*(\mathbf{R}_{0|U_{et}} \underset{U(\mathbf{R}_{|U_{et}})}{\otimes} U(F(M)_{|U_{et}})) \simeq \phi^*(\mathbf{R}_{0|U_{et}}) \underset{\phi^*(U(\mathbf{R}_{|U_{et}}))}{\otimes} (\phi^*(U(F(M)_{|U_{et}}))) \simeq \mathbf{R}_{0|V_{et}} \underset{U(\mathbf{R}_{|V_{et}})}{\otimes} U(F(M)_{|V_{et}})$. Moreover, since the differentials of the complex M preserve the gradations, these quasi-isomorphisms also preserve the gradations, thereby showing that each $T_0(F(M))_p$ has cartesian cohomology sheaves.

Let $p \geq 0$ denote a fixed integer. Let $\mathbf{N}_p$ denote the full subcategory of $\mathbf{N}$ consisting of flat graded $\mathbf{B}$-modules N so that the obvious map $F_p(N) \to N$ is quasi-isomorphism. This inherits the structure of a bi-Waldhausen category from $\mathbf{N}$ in the obvious manner. Observe, in view of the results above, that the natural map $\mathbf{B}(-i) \underset{\mathcal{A}}{\otimes} T_0(N)_i \to F_i(N)/F_{i-1}(N)$ is in fact an isomorphism of complexes for all $1 \leq i \leq p$. Let $Coh(\mathcal{S}, \mathcal{A})^{p+1} = \sqcup_{i=0}^{p} Coh(\mathcal{S}, \mathcal{A})$ denote the sum of $p+1$ copies of the bi-Waldhausen category $Coh(\mathcal{S}, \mathcal{A})$. On taking the K-theory of the above bi-Waldhausen categories, one may define a map $b : \pi_i(K(Coh(\mathcal{S}, \mathcal{A}))^{p+1} \to \pi_i(K(\mathbf{N}_p))$ by $b(F_0, \cdots, F_p) = \oplus_{j=0}^{p} \mathbf{B}(-p) \underset{\mathcal{A}}{\otimes} F_j$. Similarly, one may define a map $c : \pi_i(K(\mathbf{N}_p)) \to \pi_i(K(Coh(\mathcal{S}, \mathcal{A})))^{p+1}$ by $N \mapsto ((T_0(N)_j))$, $0 \leq j \leq p$. Clearly $c \circ b = id$. Using the additivity for exact functors (see Theorem 7.4 of Appendix A)), one may also show that $b \circ c = id$ which shows b is an isomorphism. Taking the direct limit over all $p \to \infty$, one may complete the proof of the proposition. ☐

Lemma 5.16. *Let $(S, \mathcal{A})$ denote a dg-stack as in Definition 3.1(a) and let $\mathbf{R}$ denote a sheaf of graded rings as in 5.1(3). Let $\tilde{A} \to \mathcal{A}$ denote a quasi-isomorphism from a dga $\tilde{A}$ as in Definition 3.1(b) and let $B = \mathcal{A} \underset{\mathcal{O}_\mathcal{S}}{\otimes} \mathbf{R}/z\mathbf{R}$, $\tilde{B} = \tilde{A} \underset{\mathcal{O}_\mathcal{S}}{\otimes} \mathbf{R}/z\mathbf{R}$. Then the obvious map $K(Coh_{gr}(\mathcal{S}, \tilde{B})) \to K(Coh_{gr}(\mathcal{S}, B))$ is a weak-equivalence.*

Proof. It suffices to observe that $\mathbf{R}/z\mathbf{R} \cong \oplus_{n \geq 0} \mathcal{O}_\mathcal{S} t^n$ is *flat* over $\mathcal{O}_\mathcal{S}$. ☐

We proceed to prove the homotopy property for the G-theory of dg-stacks.

Theorem 5.17. *(Homotopy property of G-theory) Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in Definition 3.1(a) and let $\pi : \mathcal{S} \times \mathbb{A}^1 \to \mathcal{S}$ denote the obvious projection. Then $\pi^* : G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S} \times \mathbb{A}^1, \pi^*(\mathcal{A}))$ is a weak-equivalence.*

Proof. Now $\mathbf{R} = \mathcal{O}_{\mathcal{S}}[t] = \oplus_{n \geq 0} \mathcal{O}_{\mathcal{S}} t^n$. Let $\mathcal{C} = \pi_* \pi^*(\mathcal{A}) = \mathcal{A}[t]$. The associated graded dga $\mathbf{B} = \mathbf{R} \otimes \mathcal{A} = \oplus_{n \geq 0} \mathcal{A} t^n$. Clearly $(\mathcal{S}, \mathbf{B})$ satisfies the hypotheses of the last proposition.

Consider the filtration of $\mathcal{O}_{\mathcal{S}}[t]$ by $F_p(\mathcal{O}_{\mathcal{S}}[t])$ = the polynomials of degree $\leq p$. Clearly $F_0(\mathcal{O}_{\mathcal{S}}[t]) = \mathbf{R}_0 = \mathcal{O}_{\mathcal{S}}$. $\mathcal{C}$ gets an induced filtration with $F_n(\mathcal{C}) = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} F_n(\mathcal{O}_{\mathcal{S}}[t])$. Let $\mathbf{V}$ denote the graded ring $\oplus_{n \geq 0} F_n(\mathcal{O}_{\mathcal{S}}[t]) z^n$ graded by the degree of the indeterminate z. Then $\mathbf{V}$ is flat over $\mathbf{V}_0 = \mathbf{R}_0 = \mathcal{O}_{\mathcal{S}}$ and $\mathbf{V}/z\mathbf{V} \cong \oplus_{n \geq 0} F_n(\mathcal{O}_{\mathcal{S}}[t])/F_{n-1}(\mathcal{O}_{\mathcal{S}}[t])$ $= \oplus_{n \geq 0} \mathcal{O}_{\mathcal{S}} t^n$ which is $\mathcal{O}_{\mathcal{S}}[t]$ viewed as a graded ring, graded by the degree of the polynomials. $\mathbf{V}/z\mathbf{V}$ has tor dimension 1 over $\mathbf{V}$ and $\mathcal{O}_{\mathcal{S}}$ has tor-dimension 1 over $\mathbf{V}/z\mathbf{V}$, so that $\mathcal{O}_{\mathcal{S}}$ has finite tor dimension over $\mathbf{V}$. Next let $\mathbf{C}' = \mathbf{V} \underset{\mathbf{R}_0}{\otimes} \mathcal{A} = \oplus_{n \geq 0} F_n(\mathcal{A}[t]) z^n$. Then $\mathbf{C}'$ is a graded dga so that $(\mathcal{S}, \mathbf{C}')$ satisfies the hypotheses of the last proposition if we let $\mathbf{R}$ ($\mathbf{R}_0$) there be given by $\mathbf{V}$ ($\mathcal{O}_{\mathcal{S}}$, respectively). Observe also that $\mathbf{C}'/z.\mathbf{C}' \cong \oplus_{n \geq 0} \mathcal{A} t^n = \mathbf{B}$.

Therefore, the obvious pull-back maps induce isomorphisms:

$$\mathbb{Z}[t] \underset{\mathbb{Z}}{\otimes} \pi_i(G(\mathcal{S}, \mathcal{A})) \xrightarrow{\simeq} \pi_i(K(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})))$$
$$\mathbb{Z}[t] \underset{\mathbb{Z}}{\otimes} \pi_i(G(\mathcal{S}, \mathcal{A})) \xrightarrow{\simeq} \pi_i(K(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))) \tag{5.4.2}$$

Observe that localizing the sheaf of algebras $\mathbf{C}'$ at $S = \{z^n | n\}$ provides a dga that is isomorphic to the Laurent polynomials over $\mathcal{C}$, i.e. $\mathcal{C}[z, z^{-1}]$; there is an equivalence of categories between $\mathrm{Mod}_{gr}(\mathcal{S}, \mathcal{C}[z, z^{-1}])$ = the category of all graded modules over $\mathcal{C}[z, z^{-1}]$ and $\mathrm{Mod}(\mathcal{S}, \mathcal{C})$ given by the functor $M \mapsto M/(z-1)M = \mathcal{C}[z, z^{-1}]/((z-1)\mathcal{C}[z, z^{-1}]) \otimes M$.

Next let $\mathrm{Coh}_{gr,fl}(\mathcal{S}, \mathbf{C}')$ denote the full-subcategory of $\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}')$ consisting of objects that are *flat* and provided with two sub-categories of weak-equivalences:

$v(\mathrm{Coh}_{gr,fl}(\mathcal{S}, \mathbf{C}'))$ denoting the usual quasi-isomorphisms of complexes of sheaves of graded modules over the sheaf of graded rings $\mathbf{C}'$ and

$w(\mathrm{Coh}_{gr,fl}(\mathcal{S}, \mathbf{C}'))$ denoting maps that become quasi-isomorphisms after inverting z.

Clearly the first is a sub-category of the second. Now consider the map $\mathrm{Coh}_{gr,fl}(\mathcal{S}, \mathbf{C}') \rightarrow \mathrm{Coh}(\mathcal{S}, \mathcal{C})$ sending a complex of sheaves of graded $\mathbf{C}'$-modules $M^\bullet$ first to $M^\bullet_{(z^n)} = \mathcal{C}[z, z^{-1}] \underset{\mathbf{C}'}{\otimes} M^\bullet$ and then to $M^\bullet_{(z^n)}/(z-1)M^\bullet_{(z^n)}$. In view of Proposition 5.3, one may invoke the Waldhausen approximation theorem (see Theorem 7.2 of Appendix A) just as in the proof of Theorem 5.14 : here the forgetful functor For as in the proof of Theorem 5.14 will be the functor $\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}') \rightarrow \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{R})$, where $\mathbf{R} = \mathbf{V} = \oplus_{n \geq 0} F_n(\mathcal{O}_{\mathcal{S}}[t]) z^n$, sending a graded object $M = \oplus M_n$ to the same object but viewed as a complex of graded modules over the graded ring $\mathbf{R}$. Observe that $\mathbf{C}' = \mathcal{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \mathbf{R}$. The above observations along with the the equivalence between $\mathrm{Mod}_{gr}(\mathcal{S}, \mathcal{C}[z, z^{-1}])$ and $\mathrm{Mod}(\mathcal{S}, \mathcal{C})$ enables one to conclude that this functor induces a weak-equivalence $wS_\bullet(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}')) \rightarrow w_{\mathcal{C}} S_\bullet(\mathrm{Coh}(\mathcal{S}, \mathcal{C}))$ where $w_{\mathcal{C}}$ denotes the maps that are quasi-isomorphisms of sheaves of $\mathcal{C}$-modules. (Similarly, another application of the Waldhausen approximation theorem shows the obvious functor $\mathrm{Coh}_{gr,fl}(\mathcal{S}, \mathbf{C}') \rightarrow \mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}')$ induces a weak-equivalence on K-theory spectra.)

Therefore one may apply Waldhausen's localization theorem with the weak-equivalences v (w) denoting the weak-equivalences defined above. This localization theorem provides the long-exact-sequence:

$$\rightarrow \pi_i(K({}_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))) \xrightarrow{i_*} \pi_i(K(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))) \xrightarrow{j^*} \pi_i(G(\mathcal{S}, \mathcal{C})) \rightarrow \tag{5.4.3}$$

where ${}_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))$ denotes the full subcategory of $\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}')$ of graded $\mathbf{C}'$-modules whose cohomology sheaves are killed by some power of z.

Let $\tilde{\mathbf{C}}' = \tilde{A} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \mathbf{R}$ where $\tilde{A} \rightarrow \mathcal{A}$ is a quasi-isomorphism from a dga $\tilde{A}$ as in Definition 3.1(b). Let ${}_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \tilde{\mathbf{C}}'))$ denotes the full subcategory of $\mathrm{Coh}_{gr}(\mathcal{S}, \tilde{\mathbf{C}}')$ of graded $\tilde{\mathbf{C}}'$-modules whose cohomology sheaves are killed by some power of z. Then Proposition 5.12 provides the weak-equivalences:

$K({}_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))) \simeq K({}_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \tilde{\mathbf{C}}')))$ and $K(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}'))) \simeq K(\mathrm{Coh}_{gr}(\mathcal{S}, \tilde{\mathbf{C}}')))$.

Then Theorem 5.10 with the ideal $I = (z)$ provides the weak-equivalence $K(_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \tilde{\mathbf{C}}')) \simeq K(Coh_{gr}(\mathcal{S}, \tilde{\mathbf{B}}))$ where $\tilde{\mathbf{B}} = \tilde{\mathbf{C}}'/z\tilde{\mathbf{C}}'$. Finally Lemma 5.16 provides the weak-equivalence $K(Coh_{gr}(\mathcal{S}, \tilde{\mathbf{B}})) \simeq K(Coh_{gr}(\mathcal{S}, \mathbf{B}))$. (Recall $\mathbf{B} = \mathbf{C}'/z\mathbf{C}'$.) Therefore, we may replace the term $\pi_i(K(_{(z)}\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{C}')))$ in the localization sequence (5.4.3) with $\pi_i(K(\mathrm{Coh}_{gr}(\mathcal{S}, \mathbf{B})))$.

Let $\mathbf{C}'(-1)$ denote the graded dga defined by $\mathbf{C}'(-1)_n = \mathbf{C}'_{n-1}$. Recall that $F_0(\mathcal{C}) = \mathcal{A}$. Therefore, one may compute i_* exactly as in [Q, Theorem 7, section 6] : one observes the existence of a short exact sequence $0 \rightarrow \mathbf{C}'(-1)\underset{\mathcal{A}}{\otimes}F \rightarrow \mathbf{C}'\underset{\mathcal{A}}{\otimes}F \rightarrow \mathbf{B}\underset{\mathcal{A}}{\otimes}F \rightarrow 0$, $F\varepsilon\mathrm{Coh}(\mathcal{S}, \mathcal{A})$ and conclude (using the additivity theorem: see Theorem 7.4) that i_* corresponds under the isomorphisms in (5.4.2) to multiplying by $1 - t$. Clearly multiplying by $1 - t$ is injective, so that i_* is injective and the above long exact sequence breaks up into short exact sequences. The cokernel of this map therefore identifies with $\pi_* G(\mathcal{S}, \mathcal{A})$. Therefore, one may conclude that the functor $F \mapsto \mathbf{C}' \underset{F_0(\mathcal{C})}{\otimes} F \mapsto \mathcal{A}[t]\underset{\mathcal{A}}{\otimes}F$ induces a weak-equivalence $G(\mathcal{S}, \mathcal{A}) = G(\mathcal{S}, F_0(\mathcal{C})) \rightarrow G(\mathcal{S}, \mathcal{C}) = G(\mathcal{S}, \mathcal{A}[t])$. □

6 Cohomology and Homology theories for Dg-stacks

The material in this section is added mostly as a (rather simple-minded) application of the K-theory and G-theory of stacks. We let (*stacks*) denote a full subcategory of the category of algebraic stacks considered in section 2, i.e. Noetherian stacks which are finitely presented over a given Noetherian base scheme S. For example, (*stacks*) could denote all Noetherian Deligne-Mumford stacks finitely presented over the base scheme S. We will consider different category structures on (*stacks*) by putting restrictions on the morphisms: for example, we may consider all morphisms of finite type, all morphisms that are proper, all representable morphisms etc. Let $(bigraded - rings)$ ($(bigraded - Ab)$) denote the category of bigraded commutative rings with 1 (the category of bigraded abelian groups, respectively). We will assume that we are given

(a) a contravariant functor $H^*(\quad, \bullet) : (stacks) \rightarrow (bigraded - rings)$, contravariant for arbitrary maps of finite type with each $H^i(\quad, j)$ a vector-space over a field of characteristic 0 and

(b) a covariant functor for proper maps $H_*(\quad, \bullet) : (stacks) \rightarrow (bigraded - Ab)$, covariant for all proper morphisms (or for all proper representable morphisms), with each $H_i(\quad, j)$ a vector-space over a field of characteristic 0. We will further assume that homology is a (graded) module over the (graded) cohomology ring.

We will presently show how to define cohomology theories and homology theories for dg-stacks starting with these theories and satisfying the following properties. Clearly one can define cohomology and homology theories for dg-stacks by completely forgetting the dg-structure sheaves. More often, the cohomology of dg-stacks is defined as the corresponding cohomology of the closed sub-stack $\bar{\mathcal{S}}$ (defined as in 3.0.2): see [TV], for example. Our general approach is different from both of these and we define cohomology and homology theories for dg-stacks that also take into account *all* of the dg-structure. *This depends strongly on the existence of a Chern-character map as the definitions below show.* We will provide a comparison of the resulting theories with the other variants in the remarks below.

Let $(dg - stacks)$ denote the category of dg-stacks associated to (*stacks*), i.e. the underlying stacks are the same as those in the chosen full sub-category (*stacks*), but we have replaced their structure sheaves with dg-structure sheaves as in section 3. Then we define

(a) a contravariant functor $H^*_{dg}(\quad, \bullet) : (dg - stacks) \rightarrow (bigraded - rings)$, contravariant for arbitrary maps of finite type with $H^*_{dg}(\quad, \bullet)$ a graded vector-space over a field of characteristic 0 and

(b) a covariant functor $H^{dg}_*(\quad, \bullet) : (dg - stacks) \rightarrow (bigraded - Ab)$, covariant for the same class of proper maps of dg-stacks with each $H^{dg}_*(\quad, \bullet)$ a graded vector-space over a field of characteristic 0 so that the following properties hold:

(i) When the dg-structure sheaf is the usual structure sheaf, $H^*_{dg}(\quad, \bullet)$ identifies with $H^*(\quad, \bullet)$ and $H^{dg}_*(\quad, \bullet)$ identifies with $H_*(\quad, \bullet)$.

(ii) If $Ch : \pi_*(K(\quad)) \rightarrow H^*(\quad, \bullet)$ is a multiplicative Chern-character, then Ch extends to a multiplicative ring homomorphism on $(dg - stacks)$.

(iii) If $\tau : \pi_*(G(\quad)) \rightarrow H_*(\quad, \bullet)$ is a Riemann-Roch transformation, then τ extends to a Riemann-Roch transformation on $(dg - stacks)$.

Definition 6.1. (i) $H^*_{dg}(\mathcal{S}, \bullet) = \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A})) \underset{\pi_*(\mathbf{K}(\mathcal{S},\mathcal{O}))}{\otimes} H^*(\mathcal{S}, \bullet)$ and

(ii) $H^{dg}_*(\mathcal{S}, \bullet) = Hom_{\pi_*(\mathbf{K}(\mathcal{S},\mathcal{O}))}(\pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A})), H_*(\mathcal{S}, \bullet))$

The multiplicative Chern-character $Ch : \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{O})) \rightarrow H^*(\mathcal{S}, \bullet)$, the multiplicative map $\pi_*(\mathbf{K}(\mathcal{S}, \mathcal{O})) \rightarrow \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A}))$ and the pairing $H^*(\mathcal{S}, \bullet) \otimes H_*(\mathcal{S}, \bullet) \rightarrow H_*(\mathcal{S}, \bullet)$ are used in forming the tensor product and Hom above.

When the dg-structure sheaf $\mathcal{A} = \mathcal{O}$, these theories reduce to the usual ones. The map sending $r \varepsilon \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A}))$ to $r \otimes 1$ defines the required extension of the Chern-character. The extension of the Riemann-Roch transformation $\tau : \pi_*(\mathbf{G}(\mathcal{S}, \mathcal{A})) \rightarrow H^{dg}_*(\mathcal{S}, \mathcal{A})$ is defined as follows. First one starts with the obvious map

$$\pi_*(\mathbf{G}(\mathcal{S}, \mathcal{A})) \underset{\pi_*(\mathbf{K}(\mathcal{S},\mathcal{O}))}{\otimes} \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A})) \rightarrow \pi_*(\mathbf{G}(\mathcal{S}, \mathcal{A})).$$

Next one composes this with the forgetful map $\pi_*(\mathbf{G}(\mathcal{S}, \mathcal{A})) \rightarrow \pi_*(\mathbf{G}(\mathcal{S}, \mathcal{O}))$. Finally one composes with the given Riemann-Roch transformation $\tau : \pi_*(\mathbf{G}(\mathcal{S}, \mathcal{O})) \rightarrow H_*(\mathcal{S}, \bullet)$. By adjunction, this corresponds to defining a map $\tau_{dg} : \pi_*(\mathbf{G}(\mathcal{S}, \mathcal{A})) \rightarrow H^{dg}_*(\mathcal{S}, \bullet)$.

Remarks 6.2. The theories defined above are not necessarily invariant under quasi-isomorphism. However, one may obtain a variant of the theory which is invariant under quasi-isomorphism by replacing the stack $(\mathcal{S}, \mathcal{O})$ with $(\bar{\mathcal{S}}, \bar{\mathcal{O}}_{\mathcal{S}} = \mathcal{H}^0(\mathcal{A}))$ and by replacing the dg-stack $(\mathcal{S}, \mathcal{A})$ with $(\bar{\mathcal{S}}, \mathcal{H}^*(\mathcal{A}))$ in Definition 6.1. Here $\bar{\mathcal{S}}$ is the closed sub-stack of $\mathcal{S}$ defined as $Spec\, \mathcal{H}^0(\mathcal{A})$: see 3.0.2.

We provide a quick comparison with the cohomology theories defined, for example, as in [TV]. In [TV], the l-adic étale cohomology of the dg-stack $(\mathcal{S}, \mathcal{A})$ would be defined as $H^*_{et}(\bar{\mathcal{S}}, \mathbb{Q}_l)$. While this definition depends only on the dg-stack $(\mathcal{S}, \mathcal{A})$ up to quasi-isomorphism, it forgets all of the higher cohomology sheaves $\mathcal{H}^i(\mathcal{A})$ which are part of the dg-structure. Depending on the point of view, for example if the higher cohomology sheaves $\mathcal{H}^i(\mathcal{A})$, $i < 0$, are viewed as *some sort of infinitesimal structure of the dg-stack*, this is a perfectly fine definition.

However, we would like to point out that the cohomology theories we have defined in Definition 6.1 have the advantage that they do *not* forget this extra structure and hence seem to be finer invariants of the dg-stack. But, the cohomology theories that are defined in the above sense are always vector spaces over a field of characteristic 0, in the above case over $\mathbb{Q}_l$. i.e. such cohomology theories *cannot* detect torsion.

7 Appendix A: Key theorems of Waldhausen K-theory

Definition 7.1. (See [TT, 1.2.1].) A *category with cofibrations* $\mathbf{A}$ is a category with a zero object 0, together with a chosen sub-category $co(\mathbf{A})$ satisfying the following axioms: (i) any isomorphism in $\mathbf{A}$ is a morphism in $co(\mathbf{A})$, (ii) for every object $A \varepsilon \mathbf{A}$, the unique map $0 \rightarrow A$ belongs to $co(\mathbf{A})$ and (iii) morphisms in $co(\mathbf{A})$ are closed under co-base change by arbitrary maps in $\mathbf{A}$. The morphisms of $co(\mathbf{A})$ are *cofibrations.* A category with fibrations is a category with a zero -object so that the dual category $\mathbf{A}^o$ is a category with cofibrations. A *Waldhausen category* is a category with cofibrations, $co(\mathbf{A})$ together with a sub-category $w(\mathbf{A})$ (of weak-equivalences) so that the following conditions are satisfied :(i) any isomorphism in $\mathbf{A}$ belongs to $w(\mathbf{A})$, (ii) if

$$\begin{array}{ccccc} B & \leftarrow & A & \rightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \leftarrow & A' & \rightarrow & C' \end{array}$$

is a commutative diagram with the vertical maps all in $w(\mathbf{A})$ and the horizontal maps in the left square are cofibrations, then the induced map $B \underset{A}{\sqcup} C \rightarrow B' \underset{A'}{\sqcup} C'$ also belongs to $w(\mathbf{A})$. (iii) If f, g are two composable morphisms in $w(\mathbf{A})$ and two of the three f, g and $f \circ g$ are in $w(\mathbf{A})$, then so is the third. A functor $F : \mathbf{A} \to \mathbf{B}$ between Waldhausen categories is *exact* if it preserves cofibrations and weak-equivalences.

Given a Waldhausen category $(\mathbf{A}, co(\mathbf{A}), w(\mathbf{A}))$, one associates to it the following simplicial category denoted $wS_\bullet\mathbf{A}$: see [TT, 1.5.1 Definition]. The objects of the category $wS_n\mathbf{A}$ are sequences of cofibrations $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ in $co(\mathbf{A})$ together with the choice of a quotient $A_{i,j}$ for each $i < j$ above. (The understanding is that $wS_0\mathbf{A}$ is the category consisting of just the zero object 0.) The morphisms between two such objects $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ and $B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n$ are compatible collections of maps $A_{i,j} \to B_{i,j}$ in $w\mathbf{A}$. Varying n, one obtains the simplicial category $wS_\bullet\mathbf{A}$ as discussed in [TT, 1.5.1 Definition].

The only Waldhausen categories considered in this paper are *complicial* Waldhausen categories in the sense of [TT, 1.2.11]: in this situation the category $\mathbf{A}$ will be a full additive sub-category of the category of chain complexes with values in some abelian category. The cofibrations will be assumed to be maps of chain complexes that split degree-wise and weak-equivalences will contain all quasi-isomorphisms. All the complicial Waldhausen categories we consider will be closed under the formation of the canonical homotopy pushouts and homotopy pull-backs as in [TT, 1.1.2].

Theorem 7.2. *(The Waldhausen approximation theorem :see [TT, 1.9.8].) Let $F : \mathbf{A} \to \mathbf{B}$ denote an exact functor between two complicial Waldhausen categories. Suppose F induces an equivalence of the derived categories $w^{-1}(\mathbf{A})$ and $w^{-1}(\mathbf{B})$. Then F induces a weak-homotopy equivalence of the associated K-theory spaces, $K(\mathbf{A})$ and $K(\mathbf{B})$.*

Theorem 7.3. *(Localization Theorem :see [TT, 1.8.2] and [Wald, Theorem 1.6.4]) Let $\mathbf{A}$ be a small category with cofibrations and provided with two sub-categories of weak-equivalences $v(\mathbf{A}) \subseteq w(\mathbf{A})$ so that both $(\mathbf{A}, co(\mathbf{A}), v(\mathbf{A}))$ and $(\mathbf{A}, co(\mathbf{A}), w(\mathbf{A}))$ are complicial Waldhausen categories (as in [TT, section 1] .) Let $\mathbf{A}^w$ denote the full sub-category of $\mathbf{A}$ of objects A for which $0 \to A$ is in $w(\mathbf{A})$, i.e. are w-acyclic. This is a Waldhausen category with $co(\mathbf{A}^w) = co(\mathbf{A}) \cap \mathbf{A}^w$ and $v(\mathbf{A}^w) = v(\mathbf{A}) \cap \mathbf{A}^w$. Then one obtains the fibration sequence of K-theory spaces: $K(\mathbf{A}^w, v) \to K(\mathbf{A}, v) \to K(\mathbf{A}, w)$.*

Theorem 7.4. *(Additivity theorem :see [Wald, Proposition 1.3.2, Theorem 1.4.2].) Let $\mathbf{A}$ and $\mathbf{B}$ be small Waldhausen categories and let $F, F', F'' : \mathbf{A} \to \mathbf{B}$ be three exact functors so that there are natural transformations $F' \to F$ and $F \to F''$ so that (i) for all A in $\mathbf{A}$, $F'(A) \to F(A)$ is a cofibration with its cofiber $\cong F''(A)$ and (ii) for any cofibration $A' \to A$ in $\mathbf{A}$, the induced map $F'(A) \underset{F'(A')}{\sqcup} F(A') \to F(A)$ is a cofibration. Then the induced maps KF, KF' and KF'' on K-theory spaces have the property that $KF \simeq KF' + KF''$.*

8 Appendix B: Injective resolutions of dg-modules

Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as before. Let $For : \mathrm{Mod}(\mathcal{S}, \mathcal{A}) \to \mathrm{C}(\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S}))$ denote the forgetful functor sending an $\mathcal{A}$-module M to itself, but viewed as a complex of $\mathcal{O}_\mathcal{S}$-modules. Let $M \varepsilon \mathrm{Mod}(\mathcal{S}, \mathcal{A})$ so that M has bounded cohomology sheaves: without loss of generality, we may in fact assume M is bounded below. Then one may find a complex of injectives $J(0)$ of $\mathcal{O}_\mathcal{S}$-modules together with a quasi-isomorphism $For(M) \to J(0)$. If $M^i = 0$ for all $i < n$, then we may choose $J(0)$ so that $J(0)^i = 0$ for all $i < n$ as well. Now consider $\mathcal{H}om_{\mathcal{O}_\mathcal{S}}(For(\mathcal{A}), J(0))$ where $\mathcal{H}om_{\mathcal{O}_\mathcal{S}}$ denotes the internal hom in the category of $\mathcal{O}_\mathcal{S}$-modules. Using the right $\mathcal{A}$-module structure on $\mathcal{A}$, one sees that this belongs to $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$. Given any $N \varepsilon \mathrm{Mod}(\mathcal{S}, \mathcal{A})$, (where we assume the $\mathcal{A}$-module structure is on the left), one observes the adjunction isomorphism

$$Hom_\mathcal{A}(N, \mathcal{H}om_{\mathcal{O}_\mathcal{S}}(For(\mathcal{A}), J(0))) \cong Hom_{\mathcal{O}_\mathcal{S}}(\mathcal{A} \underset{\mathcal{A}}{\otimes} N, J(0)) \cong Hom_{\mathcal{O}_\mathcal{S}}(N, J(0))$$

where $Hom_{\mathcal{O}_\mathcal{S}}$ denotes the external hom in the category of $\mathcal{O}_\mathcal{S}$-modules and $Hom_\mathcal{A}$ denotes the external hom in the category $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$. Since $J(0)$ is a complex of injectives in the former category, it follows that $Hom_\mathcal{A}(\quad, \mathcal{H}om_{\mathcal{O}_\mathcal{S}}(For(\mathcal{A}), J(0)))$ preserves quasi-isomorphisms in the first argument. i.e. $\mathcal{H}om_{\mathcal{O}_\mathcal{S}}(For(\mathcal{A}), J(0))$ is *an injective object of* $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$.

Next the obvious augmentation $\mathcal{O}_{\mathcal{S}} \to \mathcal{A}$ induces a map $\mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(For(\mathcal{A}), J(0)) \to \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}, J(0)) = J(0)$. Taking $N = M$ above, the above adjunction shows that there is a map $M \to \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(For(\mathcal{A}), J(0))$ of $\mathcal{A}$-modules corresponding to the map $M \cong \mathcal{A} \underset{\mathcal{A}}{\otimes} M \to J(0)$. Composing with the map $\mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(For(\mathcal{A}), J(0)) \to \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}, J(0)) = J(0)$ corresponds to the composition $M = \mathcal{O}_{\mathcal{S}} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} For(M) \to \mathcal{A} \underset{\mathcal{A}}{\otimes} M \to J(0)$ and therefore identifies with the original quasi-isomorphism $M \to J(0)$ of $\mathcal{O}_{\mathcal{S}}$-modules. Therefore, the map $d^{-1} : M \to I(0) = \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(For(\mathcal{A}), J(0))$ induces an injective map on cohomology sheaves. Now one may replace M with $coker(d^{-1})$ and find a map $\bar{d}^0 : coker(d^{-1}) \to I(1)$ of objects in $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$, with $I(1)$ an injective object in $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$, which will induce an injective map on cohomology sheaves. Let $d^0 : I(0) \to I(1)$ denote the composition of $I(0) \to coker(d^{-1})$ and $\bar{d}^0$. One may repeat this construction to define a collection, $\{I(n)|n \geq 0\}$ of injective objects in $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ together with maps $d^i : I(i) \to I(i+1)$ in $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$ so that the compositions $d^i \circ d^{i-1} = 0$ for all $i \geq 1$. Moreover, one may choose these so that $I(n)^i = 0$ for all $i < n$ where n is chosen as in the beginning of this section. This is a resolution of M in the following sense: the cohomology objects, $ker(d^i)/Imd(d^{i-1}) = 0$ for all $i > 0$ and $\cong M$ if $i = 0$. Now $\{I(n) \overset{d^n}{\to} I(n+1)|n\}$ forms a double complex in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, the total complex of which will be quasi-isomorphic to M in view of the last property. The construction shows that the total complex will be bounded below and belongs to $\mathrm{Mod}(\mathcal{S}, \mathcal{A})$.

There are obvious variants of this that apply in the graded situation as in section 5.1, the details of which are skipped.

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