K-Theory and G-Theory of Algebraic Stacks

Outline

- The lax-functor approach to stacks and algebraic stacks

- Review: the smooth site, $\mathcal{O}_{\mathcal{S}}$-modules, quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules, coherent $\mathcal{O}_{\mathcal{S}}$-modules and vector bundles

- Waldhausen categories and their K-theory

- Examples: G-theory and K-theory of algebraic stacks (definitions)

- Basic results on G-theory and K-theory of algebraic stacks

The lax-functor approach to stacks and algebraic stacks

- $(schms/S) \subseteq (alg.spaces/S) \subseteq (spaces/S)$

$\subseteq (functors: (schms/S)^{op} \to (sets))$

- $(schms/S) \subseteq (alg.stacks/S) \subseteq (stacks/S)$

$\subseteq (lax - functors: (schms/S)^{op} \to (groupoids))$

Example: The stack of smooth curves of genus g

Review: the smooth site, $\mathcal{O}_{\mathcal{S}}$-modules, quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules, etc.

- Definition: the smooth site of a given algebraic stack

- $\mathcal{O}_{\mathcal{S}}$-modules, quasi-coherent, coherent $\mathcal{O}_{\mathcal{S}}$-modules and vector bundles

- Quasi-coherent vs. $\mathcal{O}_{\mathcal{S}}$-modules

Waldhausen categories: review

- Categories with cofibrations
- Categories with cofibrations and weak-equivalences
- Functor of Waldhausen categories
- K-theory of Waldhausen categories
- K-theory of exact categories
- The Gillet-Waldhausen theorem: $K(\mathcal{E}) \simeq K(C_b(\mathcal{E}))$

The Waldhuasen approximation Theorem

Given $F : \mathcal{C}' \to \mathcal{C}$ a functor of Waldhausen categories so that

i) $F(f)$ is weak-equivalence in $\mathcal{C}$ if and only if f is a weak-equivalence in $\mathcal{C}'$ and

ii) any map $x : F(C') \to C$ in $\mathcal{C}$ factors as $x' \circ F(c')$ where $c' : C' \to C''$ in $\mathcal{C}'$ and $x' : F(C'') \to C$ is a weak-equivalence in $\mathcal{C}$.

Then $K(F) : K(\mathcal{C}') \simeq K(\mathcal{C})$ is a homotopy equivalence.

Examples of Waldhausen categories (for algebraic stacks)

- G-theory: the K-theory of the category of coherent sheaves, other interpretations ($G(\mathcal{S})$)

- The K-theory of vector bundles ($K_{naive}(\mathcal{S})$)

- K-theory: the K-theory of perfect complexes ($K(\mathcal{S})$)

G-theory:basic properties

• *Localization theorem* $\mathcal{S}' \subseteq \mathcal{S}$ closed with $\mathcal{S}'' = \mathcal{S} - \mathcal{S}'$. Then $G(\mathcal{S}') \to G(\mathcal{S}) \to G(\mathcal{S}'')$ is a fibration sequence and hence one has the long exact sequence:

$$\cdots \to \pi_{n+1}G(\mathcal{S}'') \to \pi_n G(\mathcal{S}') \to \pi_n G(\mathcal{S}) \to \pi_n G(\mathcal{S}'') \to \cdots$$

• contravariant functoriality for flat maps

• covariant functoriality for proper maps fo finite cohomological dimension

K-theory of perfect complexes

- Contravariant functoriality

- **Theorem** (Poincaré duality) $K(\mathcal{S}) \simeq G(\mathcal{S})$ when the stack $\mathcal{S}$ is smooth

- **Theorem** If every coherent sheaf is the quotient of a vector bundle, then $K(\mathcal{S}) \simeq K_{naive}(\mathcal{S})$.

Examples

- Projective space bundle formula

- Chern classes and Higher Chern classes

Proof of Poincaré duality

• $\mathcal{S}$ smooth implies every finitely presented $\mathcal{O}_{\mathcal{S}}$-module has finite tor dimension.

• suffices to show: every pseudo-coherent complex $E^{\bullet}$ with bounded cohomology is perfect

Since this local on $\mathcal{S}_{smt}$, the same proof as for schemes (due to Thomason-Trobaugh) works. Here is an outline:

$U \to \mathcal{S}$ in $\mathcal{S}_{smt}$ with U affine. Consider $E^{\bullet}_{|U}$.

There exist N and K so that $E^{i} = 0$, $i > N$, $\mathcal{H}^{n}(E^{\bullet}_{|U}) = 0$, $n \leq K$.

$Z^{n}(E^{\bullet}_{|U}) = ker(d^{n}) = Im(d^{n-1}) = B^{n-1}(E^{\bullet}_{U})$ and

$E^{n-2}_{|U} \to E^{n-1}_{U} \to Z^n(E^\bullet_U) \to 0$ is exact, $n \leq K$. Hence $Z^n(E^\bullet_{|U})$ is finitely presented.

Suppose the stalk of $Z^K(E^\bullet_{|U})$ at u has tor dimension p. Using $0 \to Z^{n-1}(E^\bullet_{|U}) \to E^{n-1}_{|U} \to Z^n(E^\bullet_{|U}) \to 0$,

$Tor_i(Z^n(E^\bullet_{|U}, M))_u \cong Tor_{i-1}(Z^{n-1}(E^\bullet_{|U}), M)_u$ for all $\mathcal{O}_{\mathcal{S}}$-modules M.

Hence $Z^{K-p}(E^\bullet_{|U})$ is *flat* and finitely presented over $\mathcal{O}_{\mathcal{S},u}$, hence *free*. So $Z^{K-p}(E^\bullet_{|U})$ is free over some smaller $V \to U$ and

$$0 \to Z^{K-p}(E^\bullet_{|U}) \to E^{K-p}_{|U} \to \cdots \to E^N \to 0$$

is strictly perfect over V. But this is $\tau_{\geq K-p-1}(E^\bullet_{|U}) \simeq E^\bullet_{|U}$. So $E^\bullet$ is perfect.$\square$

Proof of the second theorem

($*$) Given any pseudo-coherent complex $F^\bullet$ and a map $p : P^\bullet \to F^\bullet$, with $P^\bullet$ a bounded above complex of vector bundles, there exists a bounded above complex of vector bundles $Q^\bullet$, and maps $p' : P^\bullet \to Q^\bullet$, $q : Q^\bullet \to F^\bullet$ so that $p = q \circ p'$ and q is a quasi-isomorphism.

• Need to show that if $F^\bullet$ is perfect and $P^\bullet$ is a bounded complex of vector bundles, then $Q^\bullet$ can be chosen to be a bounded complex.

• Can assume $F^\bullet$ is bounded.

• Let $Q^\bullet$ be as in (*). It is perfect and let K be so that $\mathcal{H}^n(Q^\bullet) = 0$, $F^n = 0$, $n \leq K$. We will show $B^p(Q^\bullet)$ is a vector bundle for some $p << 0$ so that $\tau_{\geq p}(Q^\bullet)$ is a bounded complex of vector bundles. Clearly $P^\bullet \to Q^\bullet \to \tau_{\geq p}(Q^\bullet) \to \tau_{\geq p}(F^\bullet) = F^\bullet$.

• Assume $B^{K-1}(Q^\bullet)$ is has finite tor dimension N. Then $0 \to B^{n-1}(Q^\bullet) \to Q^n \to B^n(Q^\bullet) \to 0$, $n \leq K$, shows by argument using Tor that $B^{K-N}(Q^\bullet)$ is flat. Now the proof follows from the argument above.

• Proof that $B^{K-1}(Q^\bullet)$ is of finite tor dimension:

Consider $\alpha : \sigma_{\geq K}(Q^\bullet)(= 0 \to Q^K \to Q^{K+1} \to \cdots) \to Q^\bullet$. Then $Cone(\alpha)[-1]$ is perfect since $Q^\bullet$ and $\sigma_{\geq K}(Q^\bullet)$ are perfect.But $\mathcal{H}^i(Cone(\alpha)[-1]) = 0$ for $i \neq K$ and $= Im(d^{K-1})$ if $i = K$. So $Imd(d^{K-1}) = B^{K-1}(Q^\bullet)$ is of finite tor dimension. $\square$

Examples

• On a quotient stack $[X/G]$, X G-quasi-projective, every coherent sheaf is the quotient of a vector bundle

• Converse (Theorem of Edidin, Hassett, Kresch and Vistoli): If on a Deligne-Mumford stack $\mathcal{S}$, every coherent sheaf is the quotient of a vector bundle, then it is a quotient stack.