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Conditional statements

1. \( P \rightarrow Q : \) if \( P \), then \( Q \).
   \[ \equiv \neg P \lor Q \] (by truth table)

2. \( \neg (P \leftrightarrow Q) = \neg (\neg P \lor Q) \)
   \[ = (\neg \neg P) \land \neg Q \]
   \[ = P \land \neg Q \]

3. \( P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P) \)
   \( \) \( P \) and only if \( Q \).

- \( P \) is a sufficient condition for \( Q \)
  \[ \equiv P \rightarrow Q \]

- \( Q \rightarrow P \) is a necessary condition for \( P \)
  \[ \equiv Q \rightarrow P \]

- \( P \) and \( Q \) are necessary and sufficient conditions
  \[ \equiv P \equiv Q \]

Tautology: a proposition that is always true.

\[ P \lor \neg P \]

Contradiction: a proposition that is always false.

\[ \neg P \land P \]

Example 2.16

Let \( x \) be a real number. Show that

\[ x^2 = 1 \iff (x = 1) \lor (x = -1) \]

\[ x^2 = 1 \iff x^2 - 1 = 0 \]

- \( x = 1 \)
- \( x = -1 \)
\[ (x-1)(x+1) = 0 \]
\[ (x-1) = 0 \quad \lor \quad (x+1) = 0 \]
\[ (x = 1) \quad \lor \quad (x = -1) \]

Solve for \( x \)

\[ \frac{3x-15}{x^2-7x+10} \]

\[ 2(3x-15) = \sqrt{4} \]
\[ 6x-30 \]
\[ x^2-7x+10 \]
\[ x^2-13x+40 = 0 \]
\[ (x-5)(x-8) = 0 \]
\[ (x = 5) \quad \lor \quad (x = 8) \]
\[ x^2-7x+10 : \text{and} \ x = 5 \]
\[ 25-35+10 = 0 \]

**How to prove proportions**

1. **Modus Ponens**
   
   - \( P \implies Q \)
   - \( P \) is true
   - \( \therefore Q \) is true.

2. **Modus Tollens**
   
   - \( \neg Q \implies \neg P \implies P \implies Q \)
   - \( P \) is true
   - \( \therefore Q \) is true.

**Quantifiers**

We are going to consider proportions with variables in them.

**Example 1**

For every \( x \), if \( x > 2 \), then \( x^2 > 4 \).

Universal quantifier

\[(\forall x) \quad (P(x) \implies Q(x)) \quad x > 2 \quad x^2 > 4 \]

2. Every human being is mortal.

Write this as a statement involving \( \forall \):
\[(\forall x) \ (x \text{ is a human being} \implies x \text{ is mortal})\]

3. Every integer is a real no.
\[(\exists x) \ (x \text{ is an integer} \implies x \text{ is a real no.})\]
\[(\exists x) \ (x \text{ is a real no.} \land x \text{ is not an integer})\]

4. Negation of quantified statements
\[\neg(\forall x) \ (x \text{ is a human being} \implies x \text{ is mortal})\]
\[\neg(\forall x) \ (x \text{ is a human being} \implies x \text{ is mortal})\]
\[\neg(\exists x) \ (\neg(\forall x) \ (x \text{ is a human being} \implies x \text{ is mortal}))\]
\[\neg(\exists x) \ (\neg(x \text{ is a human being} \implies x \text{ is mortal}))\]
\[\neg(\exists x) \ (x \text{ is a human being} \land (x \text{ is not mortal}))\]

Rule for taking negations:
\[\neg(\forall x) \ (P(x)) \equiv (\exists x) \ (\neg P(x))\]
\[\neg(\exists x) \ (Q(x)) \equiv (\forall x) \ (\neg Q(x))\]

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(a) \((\exists x \in \mathbb{R}) \ (x + 2 = 5)\)
There exist some real no. \(x\) so that
\(x + 2 = 5\) \(\text{ True} \)
\(x = 3\)
(b) \((\forall x \in \mathbb{R}) \ (x + 4 = 5)\)
\(\text{False}\)
(c) \((\exists x \in \mathbb{R}) \ (x^2 + 4x + 2 = 0)\)

So there are real solutions if
\[x^2 + 4x + 2 = 0\]
\(a = 1, \ b = 4, \ c = 2\)
\[x = -b \pm \sqrt{b^2 - 4ac} \quad \text{for} \quad a \neq 0\]
\[x = -4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 2} \quad \text{for} \quad a = 1\]
\[x = -4 \pm \sqrt{16 - 8} = -4 \pm \sqrt{8} \quad \text{for} \quad a = 1\]
\[x = -4 \pm 2 \sqrt{2} \quad \text{for} \quad a = 1\]
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ = \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} \]
\[ = \frac{-4 \pm \sqrt{4}}{2} \]

\[ \text{Excluded if } (e) \]
\[ (\exists x \in \mathbb{R}) \left( x^2 - 4x + 3 > 0 \right) \]
\[ x \left| \begin{array}{c}
0 \quad 3 > 0 \\
1 \quad 1 - 4 + 3 = 0 \\
\frac{1}{2} \quad \frac{1}{4} - 2 + 3 > 0 \\
2 \quad 4 - 8 + 3 < 0
\end{array} \right. \]
\[ \left( \forall x \in \mathbb{R} \right) \left( x^2 - 4x + 3 > 0 \right) \]

\[ \text{(g) } \left( \exists x \geq 7 \right) \left( x^2 - 4x + 3 > 0 \right) \]
\[ x \left| \begin{array}{c}
7 \quad 49 - 28 + 3 > 0
\end{array} \right. \]

\[ \text{(h) } \left( \forall x > 7 \right) \left( x^2 - 4x + 3 > 0 \right) \]
\[ x^2 - 4x + 3 = x^2 - 4x + 4 - 1 \]
\[ = (x - 2)^2 - 1 \]
\[ x^2 - 4x + 3 > 0 \iff (x - 2)^2 > 1 \]
\[ \therefore \left( \forall x > 7 \right) \left( (x - 2)^2 > 1 \right) \]
\[ \left( \forall x > 7 \right) \left( x^2 - 4x + 3 > 0 \right) \]

\[ \text{(i) } \left( \forall x \in \mathbb{R} \right) \left( x^2 - 2x + 2 > 0 \right) \]
\[ x^2 - 2x + 2 = x^2 - 2x + 1 + 1 \]
\[ = (x - 1)^2 + 1 > 0 \]
\[ \forall x, \ (x - 1)^2 > 0 \quad \therefore (x - 1)^2 + 1 > 0 \]
\[ f(x) = x^2 - 2x + 2 \]
\[ f'(x) = 2x - 2 = 2(x - 1) > 0 \quad \forall x > 1 \]
\[ f(x) \text{ is increasing at } x > 1 \]
\[ \left| \begin{array}{c}
x \quad f(x) = x^2 - 2x + 2 \\
1 \quad 1
\end{array} \right. \]
\[ f(x) = x^2 - 2x + 3 \]
\[ x > 1 \]
\[ \frac{4}{f(x)} > 1 \]

4. For every \( p \in \mathbb{Q} \):
\[(\forall x \in \mathbb{N})(\text{x is even}) \lor (\text{x is odd})\]
\[(\forall x \in \mathbb{N})(\text{x is even}) \lor (\exists x \in \mathbb{N})(\text{x is odd})\]

For every natural \( x \),
- \( x \) is either even or \( x \) is odd.

(For every natural \( x \), \( x \) is even) OR
(For every natural \( x \), \( x \) is odd)

The scope of \( \forall x \) in the list
- (almost applies to \( x \) is even) OR
- \( x \) is odd

The scope of \( \exists x \) in the list
- \( x \) is odd part
- the proportion to which it applies

\#5, p 29
\[(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x < y \implies (\exists a \in \mathbb{N})(x + a = y)) \]
\[ x = -3, y = -1 \quad a = -2 \quad (-3) \]
\[ a = -2 \quad (-3) \]
\[ (-3) \]
\[ \text{True} \]

For every integer \( x \) & every integer \( y \),
- if \( x < y \), then exists a natural \( a \) so
- \( \exists a \in \mathbb{N} : a = y - x \)

(b) \[(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y \implies (\exists a \in \mathbb{N})(x + a = y)) \]
False

\[(\exists a \in \mathbb{N})(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x < y \implies (x + a = y)) \]
There exists a natural \( a \) so that
- for every integer \( x \) and every integer \( y \),
of \( x < y \), then \( x + x = y \). \( x = y \).

False because we are saying true is a value for all \( x, y \).

- Every body likes some body
- Some body likes every body

\[
(\forall x) (x \text{ is a person } \Rightarrow (\exists y) (x \text{ likes } y)) \\

\text{Here the choice of } y \text{ varies depending on the choice of } x.

(\exists x) (x \text{ is a person } \Rightarrow (\forall y) (x \text{ likes } y))

- There is some body who everyone likes.

(\exists x) (\forall y) (y \text{ likes } x)

Contraposition

(\forall y) (\exists x) (y \text{ likes } x)

This is the same as the first statement.

\[\text{Definition of limit from Calculus.}\]
\[
\lim_{x \to 0} f(x) = 1 : \text{how to say this}
\]
\[
\text{precisely}
\]
\[
\frac{1 + \varepsilon}{1 - \varepsilon}
\]
\[
\text{choice of } \delta \text{ depends on } \varepsilon
\]

\[
(\forall \varepsilon > 0) (\exists \delta > 0) (-\varepsilon < x < \varepsilon \Rightarrow -\varepsilon < f(x) - 1 < \varepsilon)
\]

\[\text{Generalized De Morgan's Rules}\]
\[
(-1) \cdot (-P(x))
\]
\[ \neg (\forall x) (P(x)) \equiv (\exists x) (\neg P(x)) \]

\[ \neg (\exists x) (P(x)) \equiv (\forall x) (\neg P(x)) \]

**Example**

\[ \neg (\forall x \in \mathbb{R}) \left( x^2 - 4x + 5 > 0 \right) \]

\[ \equiv (\exists x \in \mathbb{R}) \left( x^2 - 4x + 5 \leq 0 \right) \]

\[ x^2 - 4x + 5 = x^2 - 4x + 4 + 1 \]

\[ = (x-2)^2 + 1 \geq 0 \]

---

**Generalized Distributive Rule**

\[ (\forall x \in A) (Q(x)) \]

\[ \equiv \neg (\forall x \in A) (\neg Q(x)) \]

\[ (\forall x \in A) (Q(x)) \equiv \forall (\forall x \in A) (Q(x)) \]

\[ \neg (\forall x \in A) (\neg Q(x)) \equiv \neg (\forall x \in A) (\forall x \in A) (Q(x)) \]

\[ (\forall x \in A) (Q(x)) \]

\[ \neg (\forall x \in A) (\neg Q(x)) \equiv (\forall x \in A) (\forall x \in A) (Q(x)) \]

---

3.15  p. 84 (a)

\[ (\forall y \in \mathbb{R}) (\exists x \in \mathbb{R}) (x \leq y) \quad \text{True} \]

(b) \[ (\exists x \in \mathbb{R}) (\forall y \in \mathbb{R}) (x \leq y) \quad \text{False} \]

(c) \[ (\exists x \in \mathbb{N}) (\forall y \in \mathbb{N}) (x \leq y) \]

\[ \text{Take } x = 1 \quad \text{True} \]

---

10  (c) \[ (\exists y \in \mathbb{R}) (\forall x \in \mathbb{R}) (x + y = 0) \quad \text{False} \]

\[ (\exists y \in \mathbb{R}) (\forall x \in \mathbb{R}) (x + y = x) \]
(e) \( \exists y \in \mathbb{R} \) \( (\forall x \in \mathbb{R}) \) \( (xy = a) \)

Take \( y = 0 \) \( \text{True} \)

Take \( y = 1 \) \( \text{True} \)

\[ \exists x \in \mathbb{R} \] \[ (2x + 7 = 3) \] \( \text{True} \)

There exists a unique real number \( x \) such that \( 2x + 7 = 3 \)

\[ x = \frac{3 - 7}{2} = -2 \]

\[ \exists x \in \mathbb{R} \] \[ (2x + 7 = 3) \] \( \text{True} \)

\[ (\exists x \in \mathbb{R}) \] \[ (x^2 - 4x + 3 < 0) \] \( \text{False} \)

\[ x^2 - 4x + 3 = (x - 3)(x - 1) \]

\[ \begin{array}{c|c|c}
1 & 0 & 1 \\
3 & 0 & 1 \\
\end{array} \]

False because for any \( 1 < x < 3 \),

\[ x^2 - 4x + 3 < 0 \]

\[ \text{Exercises } \quad p.36 \]

(h) \( (\forall x \in \mathbb{R}) \) \( (x \neq 0 \Rightarrow (\text{Take } y = \frac{1}{x}) ) \)

\[ \exists y \in \mathbb{R} \] \[ (xy = 1) \] \( \text{True} \)

\[ (\forall x \in \mathbb{R}) \] \[ (\exists y \in \mathbb{R}) \] \[ (xy = 0) \]

\( H_2 = 0 \), then an \( y \in \mathbb{R} \) will satisfy \( xy = 0 \).
\[
\forall x \in \mathbb{R} \ (x^2 = 4) \text{ False} \quad \text{since } x^2 \neq 4 \text{ has no root.}
\]

\[
\#7 \text{ p. 30} \\
\left( \exists x \in \mathbb{R} \right) \left( x > 0 \land \sqrt{x+2} < \sqrt{x+5} \right)
\]

Write the negation:
\[
\forall x \in \mathbb{R} \left( \neg \left( P \land Q \right) \right)
\]
\[
\equiv \left( \forall x \in \mathbb{R} \right) \left( \left( \neg P \right) \lor \neg Q \right)
\]
\[
\equiv \left( \forall x \in \mathbb{R} \right) \left( (x < 0) \lor (\sqrt{x+2} \geq \sqrt{x+5}) \right)
\]

\[
\sqrt{x+2} < \sqrt{x+5}
\]
\[
(c^2) < (c')^2
\]
\[
x + 2 < a + 2 + 2 \sqrt{a} - \sqrt{a}
\]
\[
0 < 2 \sqrt{a}
\]
\[
0 < \sqrt{a}
\]

Take \( x = 1 \)
\[
\sqrt{1+2} = \sqrt{3}
\]
\[
\sqrt{1+5} = 1+\sqrt{2}
\]

\[
\sqrt{3} < 1+\sqrt{2} \iff 3 < (1+\sqrt{2})^2
\]

\[
1 + 2 + 2 \sqrt{2}
\]

Proof:

\( \text{Modus Ponens} \)

\( \text{Modus Tollens} \)
\[ P \Rightarrow Q \]

\[ \neg \neg Q \Rightarrow \neg P \]

\[ \therefore \neg Q \]

\[ \neg P \]

\[ \therefore \neg Q \]

\[ (\forall x)(x \text{ is a human being} \Rightarrow x \text{ is mortal}) \]

\[ \therefore \text{A is mortal} \]

\[ \text{Substitute } \forall x \text{ for } A \]

\[ \text{Chapter 2} \]

\[ \text{Properties of Numbers} \]

\[ N, Z, Q = \{ \frac{p}{q} | p, q \in Z, q \neq 0 \}, \quad \mathbb{R}, \mathbb{C} \]

\[ a + b \mathbf{i}, \quad \mathbf{i} = \sqrt{-1} \]

\[ \text{complex no.} \]

\[ \text{Home Work 1} \]

\[ \text{Exercise 5} \]

\[ (a) \quad J, S, P, C \]

Has a glass of some beverage in front.

Drunken age is 21.

J drinks beer — check J’s age.

S is over 21 — no need to check.

P drinks Moke — no need to check.

C is under 21 — check what C is drinking.
If $x$ drinks an alcoholic beverage, then $x$ is over 21. $\neg Q$ and $\neg P$ are true if

\[ P \Rightarrow Q \]

is true if $P$ is false.

7. $\neg Q \Rightarrow \neg P$

5(b) Four cards on the table. One side has a letter, the other side has a number.

Rule: If one side has a vowel, then the other side has an even no.

\[
\begin{align*}
\text{A} & \quad \text{2} \\
\uparrow & \quad \uparrow \\
\text{turn this over} & \quad \text{no need to turn it over}
\end{align*}
\]

\[
\begin{align*}
\text{X} & \quad \text{3} \\
\uparrow & \quad \uparrow \\
\text{no need to turn it over} & \quad \text{turn this over}
\end{align*}
\]

7. $\neg Q \Rightarrow \neg P$

10 (b) $1 \times 2 < 2 \lor (x^2 > 4)$
10 (b) \((x > 2) \lor (x^2 > 4)\)

Answer: \[x > 2 \lor x < -2\]

---

**Even and Odd Integers**

Even integers are divisible by 2

\[n = 2k, \quad k \in \mathbb{Z}\]

Odd integers are not divisible by 2

\[n = 2k+1, \quad k \in \mathbb{Z}\]

---

**Basic Properties**

1. If \(a, y\) are even, \(x + y\) is even.

   Proof: \(a = 2n, \quad y = 2m\)

   \[x + y = 2n + 2m = 2(n + m)\]

2. If \(x + y\) is odd, then either \(x\) is even and \(y\) is odd.
(3) If \( x \) and \( y \) are odd,
\[ x \cdot y \text{ is odd} \]

**Pf.** \( x = 2n+1, \ y = 2m+1 \)
\[
\begin{align*}
- & \quad x \cdot y = (2n+1)(2m+1) \\
= & \quad 2n \cdot 2m + 2n + 2m + 1 \\
= & \quad 2(n \cdot m + n + m) + 1 \\
\end{align*}
\]

i.e. If \( x \cdot y \) is even, then either \( x \) or \( y \) is even.

If \( x^2 \) is even, then \( x \) is even.
If \( x \) is odd, then \( x^2 \) is odd.

**Rational numbers:**

1. \( \mathbb{Q} \) is closed under \( +, -, \times, 0 \) and \( 1 \) by nonzero nos. in \( \mathbb{Q} \).

**Proof** \[ \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \]
\[ \frac{a}{1} - \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \]
\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}
\]

Consequences

1. Suppose \( a, y \in \mathbb{R} \) and \( x = y \) is irrational. Then either \( x \) or \( y \) is irrational.

2. Suppose \( x - y \) is irrational. Then either \( x \) or \( y \) is irrational.

\[\therefore \sqrt{2} \text{ is irrational.}\]

Proof (Idea): Assume the negation is true. Show that this leads to a contradiction.

Suppose \( \sqrt{2} = \frac{a}{b}, a, b \in \mathbb{Z} \) and \( b \neq 0 \)

\[\therefore 2 = \frac{a^2}{b^2}\]

\[\therefore a^2 = 2b^2\]

\((1)\) \[a^2 \text{ is even}\]

\[\therefore a = 2k, k \in \mathbb{Z}\]

Substitute in (1)
\[ 4k^2 = 2b^2 \Rightarrow 2k^2 = b^2 \]
\[ b^2 \text{ is even} \Rightarrow b \text{ is even.} \]

\[ \therefore 2 \mid a \text{ and } 2 \mid b \]

This is a contradiction since we assumed \( \text{hcf}(a, b) = \pm 1 \)

We will consider next time how to show \( \sqrt{3} \) is irrational.

Prove \( \sqrt{8} \) is irrational:

\[ \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2} \]